Ising Euclidean Fields and Cluster Area Measures *

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Two Dimensional Ising Model

Hamiltonian:

$$\mathbf{H} = -\sum_{\{x,y\}} S_x S_y$$

Sites: $x, y \in \mathbb{Z}^2$
Spins: $S_x, S_y = \pm 1$
Bonds: $b = \{x, y\}$ $(||x - y|| = 1)$
Boltzmann weights: $\sim e^{-\beta H}$ with $\beta = 1/T$

$\exists \beta_c$ such that nonuniqueness of infinite volume Gibbs measure happens only for $\beta > \beta_c$.

Rest of talk: unique Gibbs measure
Expectation: $\langle \cdot \rangle_\beta$ and $\langle \cdot \rangle_c \equiv \langle \cdot \rangle_{\beta_c}$ or E_c

FK Representation

Given $\{S_x\}$, let $p(\beta) = 1 - e^{-2\beta}$; assign bond variables: $n_b = 0$ (closed) or $n_b = 1$ (open):

If $S_x \neq S_y$, $n_b = 0$.

If $S_x = S_y$, $n_b = \begin{cases} 1 & \text{with probability } p(\beta)\,. \\ 0 & \text{with probability } 1 - p(\beta)\,. \end{cases}$

Given $\{n_b\}$ and its clusters $\{C_i\}$, let $\eta_i = \pm 1$ be i.i.d., symmetric; assign $S_x = \eta_i \; \forall x \in C_i$. Then

Two-point correlation:

$$\langle S_x S_y \rangle_\beta = P_{p(\beta)}(x \overset{FK}{\longleftrightarrow} y) \equiv P_{p(\beta)}(x, y \in \text{same}\, C_i)$$

At criticality:

$$\tau_c(y - x) \equiv \langle S_x S_y \rangle_c$$

Scaling Limit: $\mathbb{Z}^2$ replaced by $a\mathbb{Z}^2$; $a \to 0$

Approach 1: Boundaries of $\left\{ \begin{array}{l} \text{Spin} \\ \text{FK} \end{array} \right\}$ clusters

as loop process in plane related to SLE_κ, CLE_κ

with $\kappa = \left\{ \begin{array}{l} 3 \\ 16/3 \end{array} \right\}$ (Schramm, Smirnov)

Approach 2 (*today*): Random (Euclidean) field,

$$\Phi^a(z) = \Theta_a \sum_{x \in \mathbb{Z}^2} S_x \delta(z - ax) \,,$$

with Θ_a chosen s.t. as $a \to 0$,

$$\Phi^a(f) \equiv \int_{\mathbb{R}^2} f(z) \Phi^a(z) dz \overset{dist.}{\to} \text{ some } \Phi^0(f)$$

Main Result: Φ^0 can be represented using a "measure process" in plane — limit of rescaled area measures of FK clusters C_i^a in $a\mathbb{Z}^2$.

Euclidean Field and Block Magnetization

Notation:

$$\begin{aligned} \chi_L &= \text{indicator of } [0,L]^2 \subset \mathbb{R}^2 \\ \Lambda_{L,a} &= [0,L]^2 \cap a\mathbb{Z}^2 \\ \Lambda_L &\equiv \Lambda_{L,1} = [0,L]^2 \cap \mathbb{Z}^2 \end{aligned}$$

Rescaled block magnetization:

$$M_{1,a} \equiv \Phi^a(\chi_L) = \Theta_a \sum_{z \in \Lambda_{1,a}} S_{z/a} = \Theta_a \sum_{x \in \Lambda_{1/a}} S_x$$

How to choose Θ_a so that $M_{1,a}$ has nontrivial limit in distribution as $a \to 0$?

Geometric representation of the limit?

Heuristics: Three Regimes

$$\begin{aligned}\langle S_x S_y \rangle_\beta &\sim \frac{1}{||x-y||^\eta} e^{-||x-y||/\xi(\beta)} \\ &= \frac{1}{||z/a - w/a||^\eta} e^{-||z-w||/a\xi(\beta)}\end{aligned}$$

1. $\beta < \beta_c$, $a\xi(\beta) \to 0$ as $a \to 0$: Φ^0 trivial (i.e., Gaussian white noise) by a CLT.

2. $\beta = \beta_c$, $\xi(\beta_c) = \infty$ (*today*): Φ^0 massless;

   related to either $\left\{ \begin{array}{ll} SLE_3 & \& \, CLE_3 \\ SLE_{16/3} & \& \, CLE_{16/3} \end{array} \right\}$

   but seems only FK (not Spin) clusters work.

3. $\beta = \beta(a) \uparrow \beta_c$ s. t. $a\xi(\beta(a)) \to 1/m \in (0, \infty)$ as $a \to 0$: "near-critical" Φ^0 is massive.

Back to Magnetization

$$M_{1,a} = \Theta_a \sum_{z \in \Lambda_{1,a}} S_{z/a} \overset{dist.}{=} \Theta_a \begin{cases} \sum_k \psi_k |\tilde{C}_k^a| \\ \sum_i \eta_i |\hat{C}_i^a| \end{cases}$$

$|\tilde{C}_k^a|$: no. of sites in Spin cluster $\cap ([0,1]^2 \cap a\mathbb{Z}^2)$

$|\hat{C}_i^a|$: no. of sites in FK cluster $\cap ([0,1]^2 \cap a\mathbb{Z}^2)$

$$\psi_k = \begin{cases} +1 & \text{for plus Spin clusters} \\ -1 & \text{for minus Spin clusters} \end{cases}$$

$$\eta_i = \begin{cases} +1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$

Rescaled Area Measures

$M_{1,a}$: mean zero random variable

Choose Θ_a s.t. $E_c(M_{1,a}^2) \equiv \langle M_{1,a}^2 \rangle_c = 1 \ \forall a$:

$$\Theta_a^{-1} = \sqrt{\sum_{z,w \in \Lambda_{1,a}} \langle S_{z/a} S_{w/a} \rangle_c} = \sqrt{\sum_{z,w \in \Lambda_{1,a}} \tau_c^a(w-z)}$$

$$\begin{aligned} \Theta_a^{-2} &= \sum_{z,w \in \Lambda_{1,a}} \tau_c^a(w-z) = \sum_{z,w \in \Lambda_{1,a}} P_c(z \overset{FK}{\longleftrightarrow} w) \\ &= \sum_{z,w \in \Lambda_{1,a}} E_c(\sum_i 1_{z,w \in \widehat{C}_i^a}) = E_c(\sum_i |\widehat{C}_i^a|^2) \end{aligned}$$

Rescaled areas: Let $W_i^a = \Theta_a |\widehat{C}_i^a|$;

$$\langle M_{1,a}^2 \rangle_c = E_c[\sum_i (W_i^a)^2] = 1, \ \ \forall a$$

Euclidean Field

$f(z)$: test function of bounded support on $\mathbb{R}^2$

$$\begin{aligned}
\Phi^a(f) &= \int_{\mathbb{R}^2} f(z)\Phi^a(z)dz \\
&= \int_{\mathbb{R}^2} f(z)[\Theta_a \sum_{x\in\mathbb{Z}^2} S_x\delta(z-ax)]dz \\
&= \Theta_a \sum_{z\in a\mathbb{Z}^2} f(z)S_{z/a} \\
&\overset{dist.}{=} \sum_i \eta_i \mu_i^a(f),
\end{aligned}$$

where $\mu_i^a = \Theta_a \sum_{x\in C_i} \delta(z-ax)$.

In the limit $a \to 0$, get fractal μ_j^0's :

$$\begin{aligned}
\Phi^0(f) &\overset{dist.}{=} \sum_j \eta_j \mu_j^0(f) \\
&= \int_{\mathbb{R}^2} f(z) \sum_j \eta_j \mu_j^0(dz).
\end{aligned}$$

Does this make sense?

Back Again to Magnetization

$$M_{1,a} \overset{dist.}{=} \sum_i \eta_i W_i^a = \sum_i \eta_i \mu_i^a([0,1]^2)$$

(without η_i's, $\sum_i W_i^a$ should diverge as $a \to 0$)

Does the collection $\{W_i^a\}$ have a nontrivial limit in distribution as $a \to 0$?

Since $E_c[\sum_i (W_i^a)^2] = 1 \ \forall a$, no W_i^a can diverge.

But what prevents them all from tending to zero? (*In fact, expect this in dimension* $d \geq 4$.)

Two-Point Correlation

Hypothesis $(\star)$**:** For some $\theta < 1$, $\exists K_1, K_2$ s.t. for small ε and any $x \in \mathbb{Z}^2$ with large $||x||$,

$$K_2 \tau_c(x_\varepsilon) \;\geq\; \tau_c(x) \;\geq\; K_1\, \varepsilon^{2\theta} \tau_c(x_\varepsilon)$$

for any $x_\varepsilon \in \mathbb{Z}^2$ with $||x_\varepsilon - \varepsilon x|| \leq 1/\sqrt{2}$.

Claim: Hypothesis $(\star)$ can be verified for the $d = 2$ critical Ising model (using certain bounds of Russo-Seymour-Welsh type (RSW)).

Prop. 1: Hypothesis $(\star)$ implies

$$\lim_{\varepsilon \to 0} \limsup_{a \to 0} \Theta_a^2 E_c\Big(\sum_{i : diam(\widehat{C}_i^a) \leq \varepsilon} |\widehat{C}_i^a|^2 \Big) = 0.$$

(*Prop. 1 should extend to all* $d \geq 2$.)

Macroscopic FK Clusters

Prop. 2: For $z \in \mathbb{R}^2$, $0 < r_1 < r_2$, let $N^a(z, r_1, r_2)$ be the no. of distinct clusters C_i^a that touch both $\{z' : ||z' - z|| < r_1\}$ and $\{z' : ||z' - z|| > r_2\}$. Assuming RSW, $\exists \lambda \in (0, 1)$ s.t. for small a and any k

$$P_c(N^a(z, r_1, r_2) \geq k) \leq \lambda^k.$$

Thus for bounded $\Lambda \subset \mathbb{R}^2$ and $\varepsilon > 0$, the no. of distinct C_i^a of diameter $> \varepsilon$ touching Λ is bounded in prob. as $a \to 0$.

(RSW & Prop. 2 should not extend to $d \geq 4$.)

Conclusions and Perspectives

Props. 1 and 2 imply that Φ^a has subsequential scaling limits expressible as

$$\Phi^0 = \sum_j \eta_j \mu_j^0(dz)$$

with $\{\mu_j^0\}$ a measure process in the plane.

Work in progress (with Camia, Garban) to prove:

- the limit Φ^0 is unique,

- Φ^0 has the expected conformal covariance properties: e.g., $\forall \alpha > 0$,

$$\alpha^{1/8}\Phi^0(\alpha z) \stackrel{dist.}{=} \Phi^0(z) \quad \text{and}$$

$$\{\alpha^{-15/8}\mu_j^0(d(\alpha z))\} \stackrel{dist.}{=} \{\mu_j^0(dz)\}.$$

Future Hopes

- near-critical limits as $\beta(a) \to \beta_c$: massive Euclidean fields,

- near-critical limits as external field $h \to 0$ (with $\beta = \beta_c$) : more massive fields,

- $d = 2$ Potts models (for $q = 3, 4$),

- $d = 3$ Ising model (without SLE/CLE).

Vanishing External Field

$\beta = \beta_c, h \to 0, a \to 0, \beta_c h/\Theta_a \to \lambda \in (0, \infty)$

Heuristics: multiply measure describing continuum system by $\exp(\lambda \int_{\mathbb{R}^2} \Phi^0(z) dz)$

ν : (marginal) distribution of $\{\eta_j \mu_j\}$
$Z_L = \int \exp[\lambda \Phi^0(\chi_L)] d\nu$
$d\nu_L^\lambda = (Z_L)^{-1} \exp[\lambda \Phi^0(\chi_L)] d\nu$

$\nu_L^\lambda \to \nu^\lambda$ as $L \to \infty$? If so, is the field obtained from ν^λ the physically correct near-critical Euclidean field?

Potts Models

Scaling limits at second order phase transition:

- $q = 3, 4$ for $d = 2$
- $q = 2$ for $d = 3$ ($d \geq 4$: massless Gaussian)

Consider $d = 2$ and $q = 3$ or 4.

Magnetization field in color-k direction:

$$\sum_j \eta_j^k \mu_j^q$$

$$\eta_j^k = \begin{cases} +1 & \text{with probability } 1/q \\ -1/(q-1) & \text{with probability } (q-1)/q \end{cases}$$

independently for different j's and fixed k, with the condition $\sum_{k=1}^q \eta_j^k = 0$.