Large Deviations in Quantum Spin Systems

Luc Rey-Bellet

University of Massachusetts, Amherst

Collaborators: Marco Lenci, University of Bologna
Yoshiko Ogata, Kyushu University

Entropy and the Quantum: Arizona, March 2009

Quantify macroscopic fluctuations

Concrete example: Heisenberg model

- For $V \subset \mathbf{Z}^d$ (think of V as very large) the Hamiltonian H_V is

$$H_V = \sum_{x,y \in V} J(x-y)\sigma(x)\cdot\sigma(y) + h\sum_{x\in V}\sigma_x^{(3)}$$

with $\sigma(x)\cdot\sigma(y) = \sum_{i=1}^{3}\sigma^{(i)}(x)\sigma^{(i)}(y)$, $\sigma^{(i)}$ = Pauli matrices.

- Let ω_V^β be the Gibbs state with density matrix $\frac{e^{-\beta H_V}}{\mathrm{tr}(e^{-\beta H_V})}$, i.e.,

$$\omega_V^\beta(A) = \frac{\mathrm{tr}(e^{-\beta H_V}A)}{\mathrm{tr}(e^{-\beta H_V})}$$

- Consider another extensive observable, for example the total magnetization in direction 1, in the domain $\Lambda \subset \mathbf{Z}^d$

$$K_\Lambda = \sum_{x\in\Lambda}\sigma^{(1)},$$

The average magnetization is $m(\beta) \equiv \omega_V^{(\beta)}\left(\frac{K_V}{|V|}\right)$

Question: Take a domain Λ which is very large but much smaller than the size of the system V.

Quantify how likely is it to have a magnetization in the volume $\Lambda \subset V$ which is different from the average magnetization $m(\beta)$?

Consider the probability measure

$$P_\Lambda\{(a,b)\} \equiv \omega_V^\beta \left(\chi_{(a,b)} \left(\frac{K_\Lambda}{|\Lambda|} \right) \right)$$

where $\chi_{(a,b)}(A)$ is the spectral projection onto the eigenspace spanned by the eigenvalues in the interval (a,b).

One expects $\lim_{\Lambda \nearrow \mathbf{Z}^d} P_\Lambda\{(a,b)\} \to 0$ if $m(\beta) \notin (a,b)$.

Large deviation principle: There exists a nonnegative convex function $I(x)$ such that

$$\omega_V^\beta \left(\chi_{(a,b)} \left(\frac{M_\Lambda}{|\Lambda|} \right) \right) \asymp e^{-|\Lambda| I(x)}$$

Mathematically we need to separate scales and so take first $V \nearrow \mathbf{Z}^d$ (thermodynamic limit) and obtain a state ω^β of the infinite system. Then show

$$\lim_{\epsilon \to 0} \frac{1}{|\Lambda|} \log \omega^\beta \left(\chi_{(x-\epsilon, x+\epsilon)} \left(\frac{M_\Lambda}{|\Lambda|} \right) \right) = -I(x) \,.$$

For our example the average magnetization is between 0 and 1 so we expect $I(x)$ to be a convex function which is equal to $+\infty$ outside of $(-1,1)$ and $I(x) = 0$ at the mean value $m(\beta)$.

Note further that if we are in a single phase region then we expect $I(x)$ to be strictly convex while in a phase coexistence regime (first order phase transition) it might happen that $I(x)$ to has a "flat piece" for the right observable.

Quantum Lattice Systems

• Lattice $\mathbf{Z}^d$, write $\Lambda \subset \mathbf{Z}^d$ for finite box (cube), and $\Lambda \nearrow \mathbf{Z}^d$ means limit taken along a increasing sequence of cubes.

• Hilbert space: At each lattice site there is a finite level quantum system (a spin) with finite dimensional Hilbert space $\mathcal{H}_x \cong \mathbf{C}^N$.

For $\Lambda \subset \mathbf{Z}^d$ the Hilbert space is

$$\mathcal{H}_\Lambda = \otimes_{x\in\Lambda}\mathcal{H}_x$$

• Observable algebras: For a finite volume Λ

$$\mathcal{O}_\Lambda = \mathcal{B}(\mathcal{H}_\Lambda) = \{A\,:\,\mathcal{H}_\Lambda \to \mathcal{H}_\Lambda, \text{linear}\}$$

and there is a natural inclusion $\mathcal{O}_\Lambda \subset \mathcal{O}'_\Lambda$ for $\Lambda \subset \Lambda'$.

The algebra of observable for the infinite system is the C^*-algebra

$$\mathcal{O} = \overline{\cup_\Lambda \mathcal{O}_\Lambda}$$

• Interactions and Hamiltonians The interactions between the spins is specified by the collection

$$\Phi = \{\phi_X \;\; X \subset \mathbf{Z}^d \text{ finite}\}$$

where $\phi_X = \phi_X^*$ describes the multi-body interactions for spins in X and we will always assume that ϕ_X is translation invariant.

Finite-volume Hamiltonians

$$H_\Lambda = \sum_{X \subset \Lambda} \phi_X \qquad \text{free boundary conditions}$$

and one assumes, for example, that $\sum_{X \in x} \|\phi_X\| < \infty$ (i.e., the energy per site is bounded).

For example we can assume finite range interactions, $\phi_X = 0$ if diam$(X) > R$.

• Finite-volume Gibbs states at inverse temperature β

$$\omega^{\beta\Phi}_{\Lambda,can}(A) := \frac{\mathrm{tr}(e^{-\beta H_\Lambda} A)}{\mathrm{tr}(e^{-\beta H_\Lambda})}$$

$\omega^{\beta\Phi}_{\Lambda,can}$ is the state with density matrix

$$\sigma^{\beta\Phi}_{\Lambda,can} = \frac{e^{\beta H_\Lambda}}{Z_\Lambda}$$

and $Z_\Lambda = \mathrm{tr}(e^{-\beta H_\Lambda})$ is called the partition function

The variational principle I : finite volume

For a state ω_Λ on $\mathcal{O}_\Lambda$ with density matrix σ_Λ the entropy of ω_Λ is

$$S(\omega_\Lambda) = -\mathrm{tr}(\sigma_\Lambda \log \sigma_\Lambda)$$

For states ω_Λ and ω'_Λ the relative entropy of ω_Λ with respect to ω'_Λ is

$$S(\omega_\Lambda \,|\, \omega'_\Lambda) = \mathrm{tr}(\sigma_\Lambda(\log \sigma_\Lambda - \log \sigma'_\Lambda))$$

Fact:

- $S(\omega_\Lambda \,|\, \omega'_\Lambda) \geq 0$
- $S(\omega_\Lambda \,|\, \omega'_\Lambda) = 0 \quad \text{iff} \quad \omega_\Lambda = \omega'_\Lambda$

Theorem: The functional

$$\omega \mapsto S(\omega_\Lambda) - \beta\omega(H_\Lambda)$$

is maximized if and only if $\omega_\Lambda = \omega_{\Lambda,can}^{\beta\Phi}$ (finite volume Gibbs state). Moreover

$$\log Z_\Lambda = \max_{\omega_\Lambda} \left(S(\omega_\Lambda) - \beta\omega(H_\Lambda)\right)$$

Proof: For any state ω_Λ we have

$$\begin{aligned} 0 &\leq S(\omega_\Lambda \,|\, \omega_{\Lambda,can}^{\beta\Phi}) \\ &= -\mathrm{tr}(\sigma_\Lambda(\log\sigma_\Lambda - \log\sigma_{\Lambda,can}^{\beta\Phi})) \\ &= -S(\omega) + \beta\omega(H_\Lambda) + \log Z_\Lambda \end{aligned}$$

The variational principle II : Thermodynamic limit

ω state for the infinite system (= positive normalized linear functional on $\mathcal{O}$)

Assume ω is translation invariant.

Write ω_Λ for the restriction of ω to $\mathcal{O}_\Lambda$

Facts: The following limits exist

Specific entropy $s(\omega) = \lim_{\Lambda \nearrow \mathbf{Z}^d} \frac{1}{|\Lambda|} S(\omega_\Lambda)$

Specific energy $e_\Phi(\omega) = \lim_{\Lambda \nearrow \mathbf{Z}^d} \frac{1}{|\Lambda|} \omega(H_\Lambda)$

Pressure $p(\beta\Phi) = \lim_{\Lambda \nearrow \mathbf{Z}^d} \frac{1}{|\Lambda|} \log Z_\Lambda = \lim_{\Lambda \nearrow \mathbf{Z}^d} \frac{1}{|\Lambda|} \log \mathrm{tr}(e^{-\beta H_\Lambda})$

Theorem: The functional $\omega \mapsto s(\omega) - \beta e(\omega)$ is upper-semicontinuous and

$$p(\beta\Phi) = \sup_{\omega \; trans.inv.} (s(\omega) - \beta e_\Phi(\omega))$$

This motivates

Definition: A translation invariant state $\omega^{\beta\Phi}$ is a infinite volume Gibbs state if

$$p(\beta\Phi) = s(\omega) - \beta e_\Phi(\omega)$$

We denote by $\Omega(\beta\Phi)$ the set of Gibbs states. The set $\Omega(\beta\Phi)$ is a non-empty compact and convex set but in general it may contain several states (phase transition).

Equivalent characterizations

One can give other characterizations of Gibbs state

KMS-condition: Define a dynamics α_t on $\mathcal{O}$ by (if you can!)

$$\alpha_t(A) = \lim_{\Lambda \nearrow \mathbf{Z}^d} e^{itH_\Lambda} A e^{-itH_\Lambda}$$

Then ω is a $\beta\Phi$ KMS-state if

$$\omega(B\alpha_t(A)) = \omega(\alpha_{t-i\beta}(A)B)$$

for all A, B in a dense subset of $\mathcal{O}$.

Fact: $\omega^{\beta\Phi} \in \Omega(\beta\Phi)$ iff and only iff $\omega^{\beta\Phi}$ is a translation invariant KMS state.

Araki-Gibbs condition: Technically more involved but useful and similar to DLR equation, related to relative entropy.

Fact: $\omega^{\beta\Phi} \in \Omega(\beta\Phi)$ iff and only iff $\omega^{\beta\Phi}$ is a translation invariant state satisfying Araki-Gibbs condition.

Open problem : Existence of the specific relative entropy

We have seen that if $\omega_{\Lambda,can}^{\beta\Phi}$ is the finite volume Gibbs state (free boundary conditions) then

$$\lim_{\Lambda\nearrow\mathbf{Z}^d} \frac{1}{|\Lambda|} S(\omega_\Lambda \,|\, \omega_{\Lambda,can}^{\beta\Phi})$$

exists.

Let $\omega^{\beta\Phi}$ be an equilibrium state at temperature β and let $\omega_\Lambda^{\beta\Phi}$ its restriction to $\mathcal{O}_\Lambda$.

Problem: Prove that for any translation invariant state ω the limit

$$s(\omega \,|\, \omega^\beta) = \lim_{\lambda\nearrow\mathbf{Z}^d} \frac{1}{|\Lambda|} S(\omega_\Lambda \,|\, \omega_\Lambda^\beta)$$

exists and that

$$s(\omega \,|\, \omega^\beta) = -s(\omega) + \beta e(\omega) + p(\beta)\,.$$

Equivalent reformulation of the variational principle:

Let $\omega^{\beta\Phi}$ be Gibbs state. Then we have

$$s(\omega\,|\omega^{\beta\Phi}) = 0 \quad \text{iff} \quad \omega \in \Omega^{\beta}$$

It is not known if the the specific relative entropy $s(\omega\,|\,\omega^{\beta\Phi})$ exists for a general Gibbs state $\omega^{\beta\Phi}$!

Known for

- Classical case
- Quantum case, β sufficently small (high-temperature)
- Quantum case, $d = 1$, finite range interactions.

Difficulty: poor control of the boundary terms for the quantum case, at low temperatures

The proof of existence relies on the following estimates, due to Araki. They show that if $\omega^{\beta\Phi}$ is a Gibbs state for the infinite system then its restriction to $\mathcal{O}_\Lambda$ is close to the finite-volume Gibbs state $\omega^{\beta\Phi}_{\Lambda,can}$ up to boundary terms.

Asymptotic decoupling property:

If $\omega^{\beta\Phi} \in \Omega^{\beta\Phi}$ is a Gibbs state then there exist constants $C(\Lambda)$ with

$$\lim_{\Lambda \nearrow \mathbf{Z}^d} \frac{\log c(\Lambda)}{|\Lambda|} = 0$$

such that

$$c(\Lambda)^{-1}\, \omega^{\beta\Phi}_\Lambda \leq \frac{e^{-\beta H_\Lambda}}{\mathrm{tr}(e^{-\beta H_\Lambda})} \leq c(\Lambda)\, \omega^{\beta\Phi}_\Lambda$$

The proof in the classical case is very easy, at high temperature not too difficult, in dimension 1 quite hard.

Large deviations for the classical case

For comparison let us recall classical results due to Ruelle, Lanford(!), Varadhan, Föllmer-Orey, Comets, Olla, Georgii, Lewis-Pfister-Sullivan, and others

Set-up:

- Interaction $\Phi = \{\phi_X\}$ with Hamiltonians $H_\Lambda = \sum_{X \subset \Lambda} \phi_X$.
- Pick $\omega^{\beta\Phi} \in \Omega(\beta\Phi)$ a Gibbs state at inverse temperature β.
- Pick another interaction Ψ with observables $K_\Lambda = \sum_{X \subset \Lambda} \psi_X$.

Theorem: We have

$$\omega^{\beta\Phi}\left(\chi_{(x-\epsilon,x+\epsilon)}\left(\frac{K_\Lambda}{|\Lambda|}\right)\right) \asymp e^{-|\Lambda| I(x)}$$

with

$$\begin{aligned} I(x) &= \inf\{s(\omega \,|\, \omega^{\beta\Phi})\,,\, e_\Psi(\omega) = x\} \\ &= \inf\{P(\beta\Phi) - s(\omega) + \beta e_\Phi(\omega)\,,\, e_\Psi(\omega) = x\} \end{aligned}$$

By general arguments (Varadhan Lemma, Gartner-Ellis Theorem...) the rate function is obtained as follows

1) Take the moment generating function $\omega^{\beta\Phi}(e^{\alpha K_\Lambda})$ and compute

$$e(\alpha) = \lim_{\Lambda \nearrow \mathbf{Z}^d} \frac{1}{|\Lambda|} \log \omega^{\beta\Phi}(e^{\alpha K_\Lambda})$$

If one has good control of the boundary terms (easy in the classical case using DLR equation) then

$$\begin{aligned} e(\alpha) &= \lim_{\Lambda \nearrow \mathbf{Z}^d} \frac{1}{|\Lambda|} \log \frac{\mathrm{tr}(e^{-\beta H_\Lambda} e^{\alpha K_\Lambda})}{\mathrm{tr}(e^{-\beta H_\Lambda})}, \\ &= \lim_{\Lambda \nearrow \mathbf{Z}^d} \frac{1}{|\Lambda|} \log \mathrm{tr}(e^{-\beta H_\Lambda + \alpha K_\Lambda}) - \log \mathrm{tr}(e^{-\beta H_\Lambda}) \\ &= P(\beta\Phi - \alpha\Psi) - P(\beta\Phi) \end{aligned}$$

2) The rate function $I(x)$ is obtained by taking the Legendre transform i.e.,

$$I(x) = \sup_{\alpha \in \mathbf{R}} \left[\alpha x - (P(\beta\Phi - \alpha\Psi) - P(\beta\Phi))\right] .$$

Large deviations for the tracial state – equivalence of ensembles

We consider the tracial state ω^0, i.e. if $A \in \mathcal{O}_\Lambda$ is a local observable then $\omega^0(A) = \mathrm{tr}(A)$ (tr is the normalized trace $\mathrm{tr}(1) = 1$).

Note that

$$\omega^0\left(\chi_{(a,b)}\left(\frac{H_\Lambda}{|\Lambda|}\right)\right) = \frac{\#\{\mu \text{ e.v of } \frac{H_\Lambda}{|\Lambda|}, \mu \in (a,b)\}}{N^{|\Lambda|}}$$

and so the large deviation rate function is

$$\begin{aligned} I(x) &= -\lim_{\epsilon\to 0}\frac{1}{|\Lambda|}\log\left\{\text{Proportion. of e.v. of } \frac{H_\Lambda}{|\Lambda|} \text{ in } (x-\epsilon, x+\epsilon)\right\} \\ &\equiv -s(x) \end{aligned}$$

The function $s(x)$ is the microcanonical entropy density (Boltzman entropy) which counts the number of microstates (=eigenvalues) consistent with a macrostate (energy density x)

Theorem: Boltzman et. al. The tracial state ω_0 satisfies a large deviation principle

$$\omega^0\left(\chi_{(x-\epsilon,x+\epsilon)}\left(\frac{K_\Lambda}{|\Lambda|}\right)\right) \asymp e^{-|\Lambda| I(x)}$$

with a convex rate function

$$\begin{aligned}
I(x) &= -s(x) \\
&= -\max\{s(\omega)\,,\, e_\psi(\omega) = x\} \\
&= -s(\omega^{\beta(x)\Psi}) \\
&= \min_\beta (p(\beta\Psi) + x\beta)
\end{aligned}$$

This is a remarkable formula! It says that

(i) The entropy is the Legendre transform of the pressure (free energy), that is thermodynamics!

(ii) The entropy can be computed either with microcanonical means (Boltzman) or via canonical means (Gibbs).

Equivalence of ensembles I:

Define $e_{\mathrm{min}} = \lim_{\Lambda \nearrow \mathbf{Z}^d} \inf \mathrm{spec} \frac{1}{|\Lambda|} H_\Lambda$ and e_{max} similarly.

If Ψ is non trivial then $p(\beta\Psi)$ is strictly convex and $s(x)$ is concave and finite on the interval $(e_{\mathrm{min}}, e_{\mathrm{max}})$. Thus there exists a unique $\beta(x)$ such that

$$s(x) = P(\beta(x)\psi) + x\beta(x)$$

.

Moreover the maximum is attained if and only if $\omega \in \Omega(\beta(x)\Psi)$, i.e., if ω is a Gibbs state at temperature $\beta = \beta(x)$. and we have

microcanonical entropy $s(x) = s(\omega^{\beta(x)\Psi})$ canonical entropy

Equivalence of ensembles II: One can use the theorem to prove the following form of equivalence of ensembles at the level of states.

Define a microcanonical state $\omega^0_{\Lambda,x,\epsilon}$ by

$$\omega^0_{\Lambda,x,\epsilon}(A) = \frac{\mathrm{tr}\left(\chi_{(x-\epsilon,x+\epsilon)}\left(\frac{K_\Lambda}{|\Lambda|}\right)A\right)}{\mathrm{tr}\left(\chi_{(x-\epsilon,x+\epsilon)}\left(\frac{K_\Lambda}{|\Lambda|}\right)\right)}.$$

One would like to show that if one take first $\Lambda \nearrow \mathbf{Z}^d$ and then $\epsilon \to 0$ then every limit point of the sequence $\omega^0_{\Lambda,x,\epsilon}$ is a linear combination of Gibbs states $\omega^{\beta(x)\Phi}$

But only a weaker version is known by considering a periodized version of $\omega^0_{\Lambda,x,\epsilon}$

Large deviations for quantum Gibbs states

Bad news : For the quantum case the rate function $I(x)$ for the large deviations of

$$\omega^{\beta\Phi}\left(\chi_{(x-\epsilon,x+\epsilon)}\left(\frac{K_\Lambda}{|\Lambda|}\right)\right)$$

cannot, in general, be expressed in terms of the specific quantum relative entropy, unlike the classical case:

$$I(x) \neq \inf\{s(\omega\,|\,\omega^{\beta\Phi})\,,\, e_\Psi(\omega) = x\}$$

The reason is that, by abstract nonsense, (Varadhan's lemma), if $I(x)$ exists then $I(x)$ is the Legendre transform of

$$e(\alpha) = \lim_{\Lambda \nearrow \mathbf{Z}^d} \frac{1}{|\Lambda|} \log \omega^{\beta\Phi}(e^{\alpha K_\Lambda})$$

If we have a good control of the boundary terms then we should have

$$\omega^{\beta\Phi}(e^{\alpha K_\Lambda}) \cong \frac{\mathrm{tr}(e^{-\beta H_\Lambda} e^{\alpha K_\Lambda})}{\mathrm{tr}(e^{-\beta H_\Lambda})} \quad \text{modulo boundary terms}$$

But

$$\mathrm{tr}(e^{-\beta H_\Lambda + \alpha K_\Lambda}) \neq \mathrm{tr}(e^{-\beta H_\Lambda} e^{\alpha K_\Lambda})$$

unless K_Λ and H_Λ commute!

So $I(x)$ is not the Legendre transform of $P(\beta\Phi - \alpha\Psi) - P(\beta\Phi)$

Large deviations in classical subalgebras

Results obtained in various versions by Lebowitz-Lenci-Spohn, Gallavotti-Lebowitz-Mastropietro, Netocny-Redig, Lenci-R.B., Petz-Hiai-Mosonyi, Ogata-R.B, Petz-Hiai-Mosonyi.

(1) Assumptions on $\omega^{\beta\Phi}$: Asymptotic decoupling property:

There exist constants $C(\Lambda)$ with $\lim_{\Lambda \nearrow \mathbf{Z}^d} \frac{\log c(\Lambda)}{|\Lambda|} = 0$ such that

$$c(\Lambda)^{-1}\, \omega_\Lambda^{\beta\Phi} \le \frac{e^{-\beta H_\Lambda}}{\mathrm{tr}(e^{-\beta H_\Lambda})} \le c(\Lambda)\, \omega_\Lambda^{\beta\Phi}$$

(2) Assumptions on the observables $K_\Lambda = \sum_{X \subset \Lambda} \psi_X$: Classical observables

There exists an basis of $\mathcal{H} = \mathcal{H}_x$ such that all ψ_X are diagonal in the product basis of $\mathcal{H}_X$.

Let $\mathcal{O}_{cl}$ denote the commutative subalgebra of all observables diagonal in that basis.

Example: Assumptions are satisfied for example

(1)Φ consists of a two body interaction and one body interaction (like in the Heisenberg model) and $\beta \sum_x \phi_{0x} \leq C_0 = 1$

(2) $K_\Lambda = \sum_{x \in \Lambda} \psi_x$, i.e. only one-site observable.

Theorem: Under assumptions (1) and (2)

$$\omega^{\beta\Phi} \left(\chi_{(x-\epsilon, x+\epsilon)} \left(\frac{K_\Lambda}{|\Lambda|} \right) \right) \asymp e^{-|\Lambda| I(x)}$$

with

$$I(x) = \inf \left\{ s(\omega \,|\, \omega^{\beta\Phi} \circ E_{cl}) \,,\, \omega \text{ state on } \mathcal{O}_{cl} \right\}$$

Remark:

(a) $I(x) \neq \inf S(\omega|\omega^{\beta\Phi})$ and I is given by a classical relative entropy!

(b) The proof is as in the classical case. Use Lanford techniques (see Ogata-R.B.)

Large deviations in dimension 1

Partial results in Lenci-R.-B.; full large deviation result is due to Ogata; formula for the rate function is in the forthcoming Ogata-R.-B. Relies heavily on Araki's results and also Matsui's

Assumptions

(1) State: $d = 1$ and Φ finite range

(2) Observable: finite range.

Theorem: Under assumptions (1) and (2)

$$\omega^{\beta\Phi}\left(\chi_{(x-\epsilon,x+\epsilon)}\left(\frac{K_\Lambda}{|\Lambda|}\right)\right) \asymp e^{-|\Lambda| I(x)}$$

with

$$I(x) = \inf\left\{ s_\Psi(\omega \,|\, \omega^{\beta\Phi})\,,\right\}$$

with

$$s_\Psi(\omega \,|\, \omega') = \lim_{|\Lambda| \nearrow \mathbf{Z}^d} S(\omega_\Lambda|_{\mathcal{O}_{\Lambda,cl}} \,|\, \omega'_\Lambda|_{\mathcal{O}_{\Lambda,cl}})$$

and $\mathcal{O}_\Lambda$ is the classical subalgebra generated by K_Λ.

Remark: The proof use Araki-Ruelle quantum transfer operator which provides analyticity of the rate function.

Related problems

(1) Show the existence of the specific relative entropy. For translation invariant ω the limit

$$s(\omega \,|\, \omega^{\beta\Phi}) = \lim_{\Lambda \nearrow \mathbf{Z}^d} \frac{1}{|\Lambda|} S(\omega_\Lambda \,|\, \omega_\Lambda^{\beta\Phi})$$

exist.

(2) Consider two families of Hamiltonians H_Λ and K_Λ. Prove the existence of the limit

$$\lim_{\Lambda \nearrow \mathbf{Z}^d} \frac{1}{|\Lambda|} \log \mathrm{tr} \left(e^{H_\Lambda} e^{K_\Lambda} \right)$$

Also useful in quantum information theory (Hypothesis testing)

(3) Obtain bounds on imaginary time-evolution

$$e^{izH_\Lambda} A e^{-izH_\Lambda}$$

uniformly in $|\Lambda|$. In 1-dimension it is an entire-analytic function (Araki) for local A and $\Lambda \nearrow \mathbf{Z}^d$.