

A Unified Treatment of Convexity
of Relative Entropy and Related Trace Functions,
with Conditions for Equality

simple proofs of SSA and Lieb's concavity of $\text{Tr } K^* A^p K B^{1-p}$

Mary Beth Ruskai

marybeth.ruskai@tufts.edu

Department of Mathematics, Tufts University

Supported by U.S. National Science Foundation

Tucson March 2009

joint with Anna Jenčová

arXiv:0903.2895

Outline of Talk

- ▶ Background
- ▶ Proof of convexity
 - Introduce left L_A and right R_B mult ops
 - Integral reps
 - Operator Schwarz inequality with L_A and R_B
 - extend to $q \neq 1 - p$
- ▶ Prove corollaries MPT and SSA
- ▶ Equality conds
- ▶ Revisit Carlen- Lieb inequalities
 - Equality conditions
- ▶ Final comments on SSA and gen

Properties of **Relative Entropy**: $H(A, B) = \text{Tr } A(\log A - \log B)$

Joint Convexity $H\left(\sum_j A_j, \sum_j B_j\right) \leq \sum_j H(A_j, B_j)$

Cor: Monotone under Partial Trace (MPT) $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$
 $H(A_1, B_2) \leq H(A_{12}, B_{12})$ ILW: **that's all folks!**

Cor: Monotone under CPT Map $H[\Phi(A), \Phi(B)] \leq H(A, B)$
(quantum channel)

Special Case of MPT $A_{12} \rightarrow A_{123}, B_{12} \rightarrow I_1 \otimes A_{23}$

$$H(A_{12}, A_2) \leq H(A_{123}, A_{23})$$

$$-S(A_{12}) + S(A_2) \leq -S(A_{123}) + S(A_{23}) \quad S(A) = -\text{Tr } A \log A$$

strong subadditivity (SSA) of quantum entropy

Aside:

G Homogenous of degree one

$$G(\lambda A) = \lambda G(A) \Rightarrow G(A) \text{ convex} \Leftrightarrow \text{subadditive}$$

convex \Rightarrow subadditive

$$\frac{1}{2}G(A + B) = G\left(\frac{1}{2}A + \frac{1}{2}B\right) \leq \frac{1}{2}G(A) + \frac{1}{2}G(B)$$

subadditive \Rightarrow convex

$$\begin{aligned} G[xA + (1 - x)B] &\leq G(xA) + G[(1 - x)B] \\ &= xG(A) + (1 - x)G(B) \end{aligned}$$

Background

Original proof of SSA based on Lieb's result below

WYD skew entropy $\frac{1}{2} \text{Tr} [K, \gamma^p][K, \gamma^{1-p}]$

for $K = K^*$ and γ a density matrix

Wigner-Yanase introduced for $p = \frac{1}{2}$ and proved concave in γ .

Dyson suggested $p \in (0, 1)$ – led to conjecture

Conj: $\gamma \mapsto \text{Tr} K \gamma^p K \gamma^{1-p} - \text{Tr} K \gamma K$ concave

Lieb dropped linear term and proved generalization

$(A, B) \mapsto \text{Tr} K^* A^p K B^{1-p}$ concave for $p \in (0, 1)$

Claim: But advantage to retaining linear term !!

Retaining the linear term

$$J_p(K, A, B) = \frac{1}{p(1-p)} [\text{Tr } K^* A K - \text{Tr } K^* A^p K B^{1-p}]$$

Note: well-def for $p > 0$ and factor $(1-p)$ changes sign at $p = 1$

Thm: $(A, B) \mapsto J_p(K, A, B)$ is convex for $p \in (0, 2)$

$\Rightarrow \text{Tr } K^* A^p K B^{1-p}$ concave for $p \in (0, 1)$

$\Rightarrow \text{Tr } A(\log A - \log B)$ convex $p = 1$ – extend by cont $K = I$

$\Rightarrow \text{Tr } K^* A^p K B^{1-p}$ convex for $p \in (1, 2]$ and $p \in [-1, 0)$

will give proof, which is elementary, short, and sweet

$\text{Tr } A = \text{Tr } B \Rightarrow J_p(K, A, B) \geq 0$ with equality $\Leftrightarrow A = B$

pseudo-metric in same sense as relative entropy $H(A, B)$ gen Klein

Pedestrian modular operator

$d \times d$ matrices form Hilbert space with $\langle A, B \rangle = \text{Tr } A^* B$

Def. Left and Right mult as linear operators on this vector space

$$L_A(X) = AX \quad \text{and} \quad R_B(X) = XB$$

a) L_A and R_B commute $L_A[R_B(X)] = AXB = R_B[L_A(X)]$

b) $A = A^* \Rightarrow L_A, R_A$ self-adjoint wrt H-S inner prod

For $A, B > 0$ positive definite

c) L_A, R_A pos def $\langle X, R_A(X) \rangle = \text{Tr } X^* XA = \text{Tr } XAX^* \geq 0$

d) $(L_A)^{-1} = L_{A^{-1}}, \quad (R_B)^{-1} = R_{B^{-1}}$

e) $f(L_A) = L_{f(A)} \quad f(R_B) = R_{f(B)}$, e.g., $L_A^p = L_{A^p}, R_A^p = R_{A^p}$

simple form of deep idea: Araki $\Delta_{AB} = L_A R_B^{-1}$ relative modular op

Aside: Functions of operators

For $A = UDU^*$, define $f(A) = Uf(D)U^*$

$$A = U \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_d \end{pmatrix} U^* \quad f(A) = U \begin{pmatrix} f(\lambda_1) & 0 & \dots & 0 \\ 0 & f(\lambda_2) & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & f(\lambda_d) \end{pmatrix} U^*$$

equiv. to any reasonable def using power series, integral rep., etc.

also applies to operators, e.g., L_A acting on M_d space of matrices

$$A|\phi_j\rangle = \alpha_j|\phi_j\rangle \quad \Rightarrow \quad L_A|\phi_j\rangle\langle\phi_k| = \alpha_j|\phi_j\rangle\langle\phi_k|$$

$k = 1, 2, \dots, d$

e-vals of L_A deg

$$f(A)|\phi_j\rangle = \alpha_j|\phi_j\rangle \quad f(L_A)|\phi_j\rangle\langle\phi_k| = f(\alpha_j)|\phi_j\rangle\langle\phi_k|$$

In particular, $\text{Tr } L_{\log Q}(P) = \text{Tr}(\log L_Q)(P)$

Back to $J_p(K, A, B)$

$$g_p(x) = \begin{cases} \frac{1}{p(1-p)}(x - x^p) & p \neq 1 \\ x \log x & p = 1 \end{cases}.$$

well-defined for $x > 0$ and $p \neq 0$, but $p \in [\frac{1}{2}, 2]$ would suffice

$$J_p(K, A, B) \equiv \text{Tr} \sqrt{B} K^* g_p(L_A R_B^{-1})(K \sqrt{B})$$

$$= \begin{cases} \frac{1}{p(1-p)} (\text{Tr} K^* A K - \text{Tr} K^* A^p K B^{1-p}) & p \in (0, 1) \cup (1, 2) \\ \text{Tr} K K^* A \log A - \text{Tr} K^* A K \log B & p = 1 \\ -\frac{1}{2} (\text{Tr} K^* A K - \text{Tr} A K B^{-1} K^* A) & p = 2 \end{cases}$$

$$J_1(I, A, B) = \text{Tr} A (\log A - \log B) = H(A, B)$$

Aside: extend to $[-1, 0)$

not quite symmetric around $p = \frac{1}{2}$ $p \leftrightarrow 1 - p$

$$\tilde{g}_p(x) = wg_{1-p}(w^{-1}) = \begin{cases} \frac{1}{p(1-p)}(1 - x^p) & p \neq 0 \\ -\log x & p = 0 \end{cases} \quad p \in [-1, 1)$$

$$J_p(K, B, A) = \tilde{J}_{1-p}(K^*, A, B)$$

$\tilde{J}_p(K, A, B)$ jointly convex for $p \in [-1, 1)$

$$\tilde{J}_0(I, A, B) = \text{Tr } B(\log B - \log A) = H(B, A)$$

Integral representations

$$\frac{g_p(x)}{x} = \begin{cases} \frac{1}{p(1-p)}(1 - x^{p-1}) & p \neq 1 \\ \log x & p = 1 \end{cases}$$

well-def for $x \in (0, \infty)$ and operator monotone for $p \in (0, 2]$, or,
anal cont to upper half of complex plane and UHP \mapsto UHP

$\Rightarrow g_p(x)$ has integral rep of form

$$\begin{aligned} g_p(x) &= ax + \int_0^\infty \frac{x^2 t - x}{x + t} \nu(t) dt \\ &= ax + \int_0^\infty \left[\frac{x^2}{x + t} - \frac{1}{t} + \frac{1}{x + t} \right] t \nu(t) dt \end{aligned}$$

with $\nu(t) \geq 0$

Specific integrals – elementary

$$\int_0^{\infty} \frac{x^{p-1}}{x+1} = \frac{\pi}{\sin p\pi} \quad 0 < p < 1 \quad c_p = \frac{\sin p\pi}{\pi}$$

allows us to give the following explicit representations

$$g(x) = \begin{cases} \frac{1}{p(1-p)} \left[x + c_p \int_0^{\infty} \left(\frac{t}{x+t} - 1 \right) t^{p-1} dt \right] & p \in (0, 1) \\ \int_0^{\infty} \left(\frac{x^2}{x+t} - 1 + \frac{t}{x+t} \right) \frac{1}{1+t} dt & p = 1 \\ \frac{1}{p(1-p)} \left[x + c_{p-1} \int_0^{\infty} \frac{x^2}{x+t} t^{p-2} dt \right] & p \in (1, 2) \\ \frac{1}{2}(-x + x^2) & p = 2 \end{cases}$$

Important: For $p \in (0, 2)$ integrand supported on $(0, \infty)$.

Integral representation using L_A and R_B

Recall $J_p(K, A, B) = \text{Tr} \sqrt{BK}^* g_p(L_A R_B^{-1})(K \sqrt{B})$

$$g_p(x) = ax + \int_0^\infty \left[\frac{x^2}{x+t} - \frac{1}{t} + \frac{1}{x+t} \right] t \nu(t) dt$$

$$\begin{aligned} \text{Tr} \sqrt{BK}^* \frac{1}{L_A R_B^{-1} + tI} (K \sqrt{B}) &= \text{Tr} \sqrt{BK}^* \frac{R_B}{L_A + tR_B} (K \sqrt{B}) \\ &= \text{Tr} BK^* \frac{1}{L_A + tR_B} (KB) \end{aligned}$$

$$\begin{aligned} J_p(K, A, B) &= \text{Tr} K^* AK - \text{Tr} KBK^* \int_0^\infty \nu(t) dt \\ &\quad + \int_0^\infty \left[\text{Tr} K^* A \frac{1}{L_A + tR_B} (AK) + \text{Tr} BK^* \frac{1}{L_A + tR_B} (KB) \right] t \nu(t) dt \end{aligned}$$

Suffices to show $(A, B, X) \mapsto \text{Tr} X^* \frac{1}{L_B + tR_A} (X)$ jointly convex

Proof:

$$\text{Note: } \text{Tr}(\lambda X)^* \frac{1}{L_{\lambda B} + tR_{\lambda A}}(\lambda X) = \lambda \text{Tr} X^* \frac{1}{L_B + tR_A}(X)$$

Homo of degree 1 \Rightarrow suffices to prove subadditivity

Let: $M = ()^{-1/2}(X) - ()^{1/2}(\Lambda)$

$$\begin{aligned} \text{Tr } M^* M &= \langle M, M \rangle \\ &= \langle [()^{-1/2}(X) - ()^{1/2}(\Lambda)], [()^{-1/2}(X) - ()^{1/2}(\Lambda)] \rangle \\ &= \langle X, ()^{-1}(X) \rangle - \langle X, \Lambda \rangle - \langle \Lambda, X \rangle + \langle \Lambda, ()(\Lambda) \rangle \end{aligned}$$

Choose $M = (L_A + tR_B)^{-1/2}(X) - (L_A + tR_B)^{1/2}(\Lambda)$

$$\begin{aligned} \text{Tr } M^* M &= \\ &\text{Tr } X^*(L_A + tR_B)^{-1}(X) - \text{Tr } X^* \Lambda - \text{Tr } \Lambda^* X + \text{Tr } \Lambda^*(L_A + tR_B)(\Lambda) \end{aligned}$$

Let $M_j = (L_{A_j} + tR_{B_j})^{-1/2}(X_j) - (L_{A_j} + tR_{B_j})^{1/2}(\Lambda)$. Then

$$0 \leq \sum_j \text{Tr } M_j^* M_j = \sum_j \text{Tr } X_j^* (L_{A_j} + tR_{B_j})^{-1} (X_j) \\ - \text{Tr} (\sum_j X_j^*) \Lambda - \text{Tr } \Lambda^* (\sum_j X_j) + \text{Tr } \Lambda^* \sum_j (L_{A_j} + tR_{B_j}) \Lambda$$

Choose $\Lambda = \frac{1}{L_{\sum_j A_j} + tR_{\sum_j B_j}} (\sum_j X_j)$. Use $\sum_j L_{A_j} = L_{\sum_j A_j}$

$$\text{Tr } \Lambda^* \sum_j (L_{A_j} + tR_{B_j}) \Lambda = \text{Tr} (\sum_j X_j^*) \frac{1}{L_{\sum_j A_j} + tR_{\sum_j B_j}} (\sum_j X_j) \\ = \text{Tr} (\sum_j X_j^*) \Lambda = \text{Tr } \Lambda^* (\sum_j X_j)$$

$$0 \leq \sum_j \text{Tr } X_j^* \frac{1}{L_{A_j} + tR_{B_j}} (X_j) - \text{Tr} (\sum_j X_j^*) \frac{1}{L_{\sum_j A_j} + tR_{\sum_j B_j}} (\sum_j X_j)$$

compare elementary C-S ineq:

$$\left| \sum_k \bar{v}_k w_k \right|^2 \leq \sum_k |v_k|^2 \sum_k |w_k|^2$$

For $a_k > 0$ let $v_k = a_k^{1/2}$, $w_k = a_k^{-1/2} x_k$

$$\left| \sum_k x_k \right|^2 \leq \sum_k a_k \sum_k \bar{x}_k \frac{1}{a_k} x_k$$

Rewrite $\left(\sum_k \bar{x}_k \right) \frac{1}{\sum_k a_k} \left(\sum_k x_k \right) \leq \sum_k \bar{x}_k \frac{1}{a_k} x_k$

Lieb and Ruskai (1973) proved operator version

$$\left(\sum_k X_k^* \right) \frac{1}{\sum_k A_k} \left(\sum_k X_k \right) \leq \sum_k X_k^* \frac{1}{A_k} X_k$$

Not suff. for SSA — need Araki rel mod op hidden in L_A and R_B .

Compare proof: $\left| \sum_k v_k + t w_k \right|^2 \geq 0 \quad \forall t$ choose t to minimize

Remarks on $q \neq 1 - p$

$p, q > 0, p + q < 1$ $\text{Tr } K^* A^p K B^{1-p}$ concave

Write $\text{Tr } K^* A^p K B^q = \text{Tr } K^* A^p K (B^s)^{1-p}$ $0 < s = \frac{1}{1-p} < 1$

B^s is op monotone and op concave for $s \in (0, 1)$

$$(\lambda B_1 + (1 - \lambda) B_2)^s > \lambda B_1^s + (1 - \lambda) B_2^s$$

Note: $f(x)$ strictly concave and op concave \Rightarrow strict op ineq

$$\begin{aligned} \text{Tr } K^* A^p K B^q &= \text{Tr } K^* A^p K [(\lambda B_1 + (1 - \lambda) B_2)^s]^{1-p} \\ &> \text{Tr } K^* A^p K (\lambda B_1^s + (1 - \lambda) B_2^s)^{1-p} \\ &\geq \lambda \text{Tr } K^* A_1^p K (B_1^s)^{1-p} + (1 - \lambda) \text{Tr } K^* A_2^p K (B_2^s)^{1-p} \\ &= \lambda \text{Tr } K^* A_1^p K B_1^q + (1 - \lambda) \text{Tr } K^* A_2^p K B_2^q \end{aligned}$$

get equal only for trivial cases, $B_1 = B_2$ or $\lambda = 0, 1$.

Monotonicity under partial traces

Prove MPT: Recall gen Pauli ops,

$$Z|e_n\rangle = e^{2\pi in/d}|e_n\rangle \qquad X|e_n\rangle = |e_{n+1}\rangle$$

$$\sum_j Z^j A Z^{-j} = d A_{\text{diag}} \qquad \sum_j X^j A_{\text{diag}} X^{-j} = (\text{Tr } A) I$$

$$\frac{1}{d} \sum_j \sum_k X^j Z^k A (X^j Z^k)^* = (\text{Tr } A) I$$

$W_n = X^j Z^k$ in some ordering $n = 1, 2, \dots, d^2$, e.g., $n = j + d(k - 1)$

$$\frac{1}{d_2} \sum_n (I_1 \otimes W_n) A_{12} (I_1 \otimes W_n)^* = A_1 \otimes I_2$$

Discrete version of Uhlmann's observation that partial trace can be obtained by integrating over $SU(n)$ using Haar measure.

$$\begin{aligned}
J_p(K_2, A_2, B_2) &= J_p(I_1 \otimes K_2, \frac{1}{d_1} I_1 \otimes A_2, \frac{1}{d_1} I_1 \otimes B_2) \\
&= \frac{1}{d_1^2} J_p\left(K_{12}, \sum_n (W_n \otimes I_2) A_{12} (W_n \otimes I_2)^*, \sum_n (W_n \otimes I_2) B_{12} (W_n \otimes I_2)^*\right) \\
&\leq \frac{1}{d_1^2} \sum_n J_p(I_1 \otimes K_2, (W_n \otimes I_2) A_{12} (W_n \otimes I_2)^*, (W_n \otimes I_2) B_{12} (W_n \otimes I_2)^*) \\
&= \frac{1}{d_1^2} \sum_n J_p(I_1 \otimes K_2, A_{12}, B_{12}) = J_p(I_1 \otimes K_2, A_{12}, B_{12})
\end{aligned}$$

used $J_p(I_1 \otimes K_2, A_{12}, B_{12})$ wrote $K_{12} = I_1 \otimes K_2$

$$= J_p(I_1 \otimes K_2, (W_n \otimes I_2) A_{12} (W_n \otimes I_2)^*, (W_n \otimes I_2) B_{12} (W_n \otimes I_2)^*)$$

$$J_1(I, A_2, B_2) \leq J_1(I, A_{12}, B_{12}) \text{ gives } H(A_2, B_2) \leq H(A_{12}, B_{12})$$

Cor: SSA $H(A_{23}, A_2) \leq H(A_{123}, A_{13})$

“no transparent proof of SSA is known”

p. 645 of *Quantum Computation and Quantum Information*

Michael A. Nielsen and Isaac L. Chuang (Cambridge Press, 2000)

based on B. Simon's version adapted from Uhlmann (1977) of

“elementary” proof of $(A, B) \mapsto \text{Tr } K^* A^p K B^{1-p}$ concave
similar argument in Wehrl *Rev. Mod. Phys* (1978). **BUT**

- MBR, “Lieb's simple proof of concavity . . .” quant-ph/0404126
Int. J. Quant Info. **3**, 579–590 (2005) **Schwarz + max mod**
- Ando's argument described in **Carlen's talk**
- Petz – uses Δ_{AB} in book; elem version in quant-ph/0408130
- Proof here based on Schwarz ineq. using L_A, R_B really elem.
based on Lesniewski and Ruskai, JMP; and MBR quant-ph/0604206

Equality conditions in $J_p(K, A, X)$ convex

$$\begin{aligned} \int_0^\infty \text{Tr} K^* A \frac{1}{L_A + tR_B} (AK) \nu(t) dt &\leq \int_0^\infty \sum_j \text{Tr} (A_j K)^* \frac{1}{L_{A_j} + tR_{B_j}} (A_j K) \nu(t) dt \\ &= \sum_j \int_0^\infty \text{Tr} (A_j K)^* \frac{1}{L_{A_j} + tR_{B_j}} (A_j K) \nu(t) dt \end{aligned}$$

Equal \Leftrightarrow equal for each term in integ , i.e., $M_j = 0 \quad \forall j, \forall t$

$$(L_{A_j} + tR_{B_j})^{-1}(X_j) = (L_A + tR_B)^{-1}(X) \quad \forall j, \forall t$$

equality conditions independent of $p \in (0, 2)$

$$X = AK \quad (I + t\Delta_{A_j B_j}^{-1})^{-1}(K) = (I + t\Delta_{AB}^{-1})^{-1}(K) \quad \forall j, \forall t$$

$$X = BK \quad (\Delta_{A_j B_j} + tI)^{-1}(K) = (\Delta_{AB} + tI)^{-1}(K) \quad \forall j, \forall t$$

Recall $\Delta_{AB} = L_A R_B^{-1} > 0$ prod of commuting pos def ops

$$(\Delta_{A_j B_j} + tI)^{-1}(K) = (\Delta_{AB} + tI)^{-1}(K) \quad \forall j, \forall t$$

$\Delta_{AB} > 0 \Rightarrow (\Delta_{AB} + tI)^{-1}$ anal cont to $\mathbf{C} \setminus (-\infty, 0]$

can apply Cauchy integral Thm. to get

$$\Rightarrow G(\Delta_{A_j B_j})(K) = G(\Delta_{AB})(K) \quad \forall j \quad G \text{ anal on } \mathbf{C} \setminus (-\infty, 0]$$

allows several useful formulations

$$\Rightarrow (\Delta_{AB} + tI) \text{ and } (I + \Delta_{AB}^{-1}t) \text{ forms equiv.}$$

Equivalent equality conditions

Thm: For fixed K , and $A = \sum_j A_j, B = \sum_j B_j$ **TFAE**

- a) $J_p(K, A, B) = \sum_j J_p(K, A_j, B_j)$ for **all** $p \in (0, 2)$.
- b) $J_p(K, A, B) = \sum_j J_p(K, A_j, B_j)$ for **some** $p \in (0, 2)$.
- c) $(\Delta_{A_j B_j} + tI)^{-1}(K) = (\Delta_{AB} + tI)^{-1}(K) \quad \forall j$ and $\forall t > 0$.
- d) $A_j^{it} K B_j^{-it} = A^{it} K B^{-it} \quad \forall j$ and $\forall t > 0$.
- e) $(\log A - \log A_j)K = K(\log B - \log B_j) \quad \forall j$.

In addition when $K = I$, equiv to

- f) There are $D_j > 0$ such that $[A_j, D_j] = [B_j, D_j] = 0$, and
 $A_j = A D^{-1} D_j, \quad B_j = B D^{-1} D_j$ with $D = \sum_j D_j$

necessity of (f) uses sufficient subalgebra – developed by Petz
formulation here from Jenčová and Petz, CMP, **263**, 259–276 (2006).

Equality conditions for SSA

use form $\log A_{123} - \log A_{12} - \log A_{23} + \log A_2 = 0$

Easy to see $A_{123} = A_1 \otimes A_{23}$ or $A_{12} \otimes A_3$ will suffice

If $\mathcal{H}_2 = \mathcal{H}_{2_L} \otimes \mathcal{H}_{2_R}$ then $A_{123} = A_{12_L} \otimes A_{2_R3}$ will suffice

Thm: Equality holds in SSA if and only if

$$\mathcal{H}_2 = \bigoplus_n \mathcal{H}_n^L \otimes \mathcal{H}_n^R \quad \text{and} \quad A_{123} = \bigoplus_n A_n^L \otimes A_n^R$$

with $A_n^L \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_n^L)$, $A_n^R \in \mathcal{B}(\mathcal{H}_n^R \otimes \mathcal{H}_3)$

Cor: Equality in $\text{Tr} A_{23}^p A_2^{1-p} \leq \text{Tr} A_{123}^p A_{12}^{1-p}$ iff same cond

$$p \in (0, 1) \leq \quad \quad \quad p \in (1, 2) \geq$$

Carlen Lieb Inequalities

$$\begin{aligned}\widehat{\Upsilon}_{p,1}(K, A) &= \frac{1}{(p-1)} [\text{Tr } K^* A^p K]^{1/p} - \frac{1}{p} \text{Tr } K^* A K \\ &= \inf \left\{ J_p(K, A, X) + \frac{1}{p} \text{Tr } X : X > 0 \right\}\end{aligned}$$

$$\widehat{\Phi}_{p,1}\left(\sum_k A_k\right) = \widehat{\Phi}_{p,1}(\mathcal{A}) = \frac{1}{(p-1)} \left[\text{Tr} \left(\sum_k A_k^p \right)^{1/p} - \frac{1}{p} \text{Tr} \sum_k A_k \right]$$

$$\widehat{\Psi}_{p,1}(\mathcal{A}_{12}) = \frac{1}{(p-1)} \left[\text{Tr}_1 (\text{Tr}_2 \mathcal{A}_{12}^p)^{1/p} - \frac{1}{p} \text{Tr}_{12} \mathcal{A}_{12} \right]$$

All convex for $0 < p \leq 2$. $\widehat{\Phi}(\mathcal{A})$ is block diag case of $\widehat{\Psi}(\mathcal{A}_{12})$

conditional entropy $\widehat{\Psi}_{(1,1)}(\mathcal{A}_{12}) = S(\mathcal{A}_1) - S(\mathcal{A}_{12})$

$$|\mathbb{1}\rangle = (1, 1, \dots, 1) \quad |e_1\rangle = (1, 0, \dots, 0)$$

$$\mathcal{K} = \frac{1}{d} I \otimes |\mathbb{1}\rangle\langle e_1| = \begin{pmatrix} I & 0 & \dots & 0 \\ I & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ I & 0 & \dots & 0 \end{pmatrix}$$

$$\mathcal{A} = \sum_k A_k \otimes |e_k\rangle\langle e_k| = \begin{pmatrix} A_1 & 0 & 0 & \dots & 0 \\ 0 & A_2 & 0 & \dots & 0 \\ 0 & 0 & A_1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \end{pmatrix}$$

$$\mathcal{K}^* \mathcal{A}^p \mathcal{K} = (\sum_k A_k^p) \otimes |e_1\rangle\langle e_1| \quad \Rightarrow \quad \hat{\Phi}_{p,1}(\mathcal{A}) = \hat{\Upsilon}_{p,1}(K, A)$$

MPT in Carlen-Lieb

monotonicity: $\widehat{\Psi}_{(p,1)}(\mathcal{A}_{23}) \leq \widehat{\Psi}_{(p,1)}(\mathcal{A}_{123})$

conditional entropy: $\widehat{\Psi}_{(1,1)}(\mathcal{A}_{23}) = S(\mathcal{A}_2) - S(\mathcal{A}_{23})$

$p = 1$ $S(\mathcal{A}_2) - S(\mathcal{A}_{23}) \leq S(\mathcal{A}_{12}) - S(\mathcal{A}_{123})$ SSA

Carlen-Lieb Minkowski:

$$\begin{aligned} \text{Tr}_3 [\text{Tr}_2 (\text{Tr}_1 \mathcal{A}_{123})^p]^{1/p} &= \Psi_{(p,1)}(\mathcal{A}_{32}) \\ &\leq \Psi_{(p,1)}(\mathcal{A}_{132}) = \text{Tr}_3 \text{Tr}_1 (\text{Tr}_2 \mathcal{A}_{123}^p)^{1/p} \end{aligned}$$

for $1 < p \leq 2$ and reverse ineq \geq for $0 < p < 1$.

$p = 1$ equality conditions same as for SSA

Equality conditions in Carlen-Lieb

Extend equal conds to all $p \in (0, 2)$ by Cor to SSA equal conds

Rough idea: Equal in SSA \Rightarrow equal in MPT proof

\Rightarrow equal in convexity for $J_p(I, \mathcal{A}_{12}, I_1 \otimes \mathcal{A}_2)$

can actually use SSA equal conds to improve this

equal in $\widehat{\Phi}(\mathcal{A})$ convex \Leftrightarrow equal in $J_p(I, \mathcal{A}, \text{Tr}_2(\mathcal{A}) \otimes I_2)$

equal in $\widehat{\Psi}(\mathcal{A}_{12})$ convex \Leftrightarrow equal in $J_p(I, \mathcal{A}_{123}, \mathcal{A}_1 \otimes I_{23})$

$$\mathcal{A}_{123} = \sum_n (I \otimes W_n) \mathcal{A}_{12} (I \otimes W_n)^* \otimes |e_n\rangle\langle e_n|$$

equal in “new” SSA from $\widehat{\Psi}_{(p,1)}(\mathcal{A}_{23}) \leq \widehat{\Psi}_{(p,1)}(\mathcal{A}_{123})$ indep of p

Generalizations of SSA

Uniform treatment led to two distinct generalizations of SSA

pseudo p -metric based on MPT of $J_p(1, A_{123}, A_{23})$

$$\begin{aligned} \text{Tr } A_{23}^p A_2^{1-p} &\leq \text{Tr } A_{123}^p A_{12}^{1-p} & p \in (0, 1) \\ &\geq & p \in (1, 2) \end{aligned}$$

pseudo p -norm based on MPT of $\hat{\Psi}(\mathcal{A}_{123})$

$$\begin{aligned} \text{Tr}_2(\text{Tr}_3 \mathcal{A}_{23}^p)^{1/p} &\geq \text{Tr}_{12}(\text{Tr}_3 \mathcal{A}_{123}^p)^{1/p} & p \in (0, 1) \\ &\leq & p \in (1, 2) \end{aligned}$$

Compare Renyi $\frac{1}{1-p} \log \text{Tr } A^p$ and Tsallis $\frac{1}{p-1}(1 - \text{Tr } A^p)$ entropy