The Annals of Applied Probability
2011, Vol. 21, No. 1, 312–331
DOI: 10.1214/10-AAP697

RANDOM PERMUTATIONS WITH CYCLE WEIGHTS

BY VOLKER BETZ[1], DANIEL UELTSCHI[2] AND YVAN VELENIK[3]

University of Warwick, University of Warwick and Université de Genéve

We study the distribution of cycle lengths in models of nonuniform random permutations with cycle weights. We identify several regimes. Depending on the weights, the length of typical cycles grows like the total number n of elements, or a fraction of n or a logarithmic power of n.

1. Introduction. We study the cycle distributions in models of weighted random permutations. The probability of a permutation π of n elements is defined by

$$P(\pi) = \frac{1}{h_n n!} \prod_{j \geq 1} \theta_j^{r_j(\pi)}, \tag{1.1}$$

where $(\theta_1, \theta_2, \ldots) \equiv \boldsymbol{\theta}$ are real nonnegative numbers, $r_j(\pi)$ denotes the number of j-cycles in π [we always have $\sum_j j r_j(\pi) = n$] and h_n is the normalization. We are mainly interested in the distribution of cycle lengths in the limit $n \to \infty$ and in how these lengths depend on the set of parameters $\boldsymbol{\theta}$.

The probability P is really a probability on sequences $\mathbf{r} = (r_1, r_2, \ldots)$ that satisfy $\sum_j j r_j = n$. It is well known that $\mathbf{r}$ is the sequence of "occupation numbers" of a partition $\boldsymbol{\lambda}$ of n. That is, if $\boldsymbol{\lambda}$ denotes the partition $\lambda_1 \geq \lambda_2 \geq \cdots$ with $\sum_i \lambda_i = n$, then r_j is the number of λ_i that satisfies $\lambda_i = j$. Thus we are really dealing with random partitions. The number of permutations that are compatible with occupation numbers $\mathbf{r}$ is equal to

$$\frac{n!}{\prod_{j \geq 1} j^{r_j} r_j!}.$$

It follows that the marginal of (1.1) on partitions is given by

$$P(\boldsymbol{\lambda}) = \frac{1}{h_n} \prod_{j \geq 1} \frac{1}{r_j!} \left(\frac{1}{j} \theta_j \right)^{r_j}. \tag{1.2}$$

The formulas look simpler and more elegant for permutations than for partitions and this is why we consider the former.

Received August 2009; revised February 2010.
[1]Supported by the EPSRC fellowship EP/D07181X/1.
[2]Supported in part by the Grant DMS-0601075 of the US National Science Foundation.
[3]Supported in part by the Swiss National Science Foundation.
AMS 2000 subject classification. 60K35.
Key words and phrases. Random permutations, cycle weights, cycle lengths, Ewens distribution.

Random permutations with the uniform distribution have a compelling history [1, 6, 7, 13]. They are a special case of the present setting, with $\theta_j \equiv 1$. The uniform distribution of random partitions has been studied, for example, in [8, 12, 16, 18]. They do not fit the present setting because there are no parameters $\boldsymbol{\theta}$ that make the right-hand side of (1.2) constant. Another distribution for random partitions is the Plancherel measure, where the probability of $\boldsymbol{\lambda}$ is proportional to $\frac{1}{n!}(\dim \boldsymbol{\lambda})^2$; the "dimension" $\dim \boldsymbol{\lambda}$ of a partition is defined as the number of Young tableaux in Young diagrams and it does not seem to have an easy expression in terms of $\mathbf{r}$. Here again, we do not know of any direct relation between weighted random permutations and the Plancherel measure.

The present model was introduced in [4] but variants of it have been studied previously. The case of constant $\theta_j \equiv \theta$ is known as the Ewens distribution. It appears in the study of population dynamics in mathematical biology [9]; detailed results about the number of cycles were obtained by Hansen [14] and by Feng and Hoppe [10]. The distribution of cycle lengths was considered by Lugo [15]. Another variant of this model involves parameters $\theta_j \in \{0, 1\}$, with finitely many 1's [2, 17] or with parity dependence [15].

Weighted random permutations also appear in the study of large systems of quantum bosonic particles [3, 5], where the parameters $\boldsymbol{\theta}$ depend on such quantities as the temperature, the density and the particle interactions. The θ_j's are thus forced upon us and they do not necessarily take a simple form. This motivates the present study where we only fix the asymptotic behavior of θ_j as $j \to \infty$.

The relevant random variables in our analysis are the lengths $\ell_i = \ell_i(\pi)$ of the cycle containing the index $i = 1, \dots, n$. These random variables are always identically distributed and obviously not independent. Another relevant random variable is the number of indices belonging to cycles of length between a and b, $N_{a,b}(\pi) = \#\{i = 1, \dots, n : a \le \ell_i(\pi) \le b\}$. It follows from the exchangeability of $\ell_1, \dots, \ell_n$ that

$$\frac{1}{n} E(N_{a,b}) = P(\ell_1 \in [a, b]). \tag{1.3}$$

The properties of the distribution of ℓ_1 that we derive below can then be translated into properties of the expectation of $N_{a,b}$.

From a statistical mechanics point of view it is natural to introduce the sequence $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots)$ of parameters such that $\mathrm{e}^{-\alpha_j} = \theta_j$. The model has an important symmetry which is also a source of confusion, namely, the probability of the permutation π is left invariant under the transformation

$$\alpha_j \mapsto \alpha_j + cj, \qquad h_n \mapsto \mathrm{e}^{-cn} h_n \tag{1.4}$$

for any constant $c \in \mathbb{R}$. In particular, the case $\alpha_j = cj$ is identical to $\alpha_j \equiv 0$, the case of uniform random permutations.

The general results which we prove in this article rely on various technical assumptions. To keep this Introduction simple, we only describe the results in the particular but interesting case $\alpha_j \sim j^\gamma$.

- The case $\gamma < 0$ is a special case of the model studied in [4] which is close to the uniform distribution.
- In the case $\gamma = 0$, that is, when $\theta_j \to \theta$ (the Ewens case, asymptotically), we find that $P(\ell_1 > sn) \to (1-s)^\theta$. Thus, almost all indices belong to cycles whose length is a fraction of n. Precise statements and proofs can be found in Section 2.
- The case $0 < \gamma < 1$ is surprising. At first glance we might expect smaller cycles than in the uniform case $\alpha_j \equiv 0$. However, we find that almost all indices belong to a single giant cycle! The symmetry (1.4) is indeed playing tricks on us. In addition, we prove that the probability of the occurrence of a single cycle of length n is strictly positive and strictly less than 1. This is explained in detail in Section 3.
- The case $\gamma = 1$ corresponds to uniform permutations because of the symmetry (1.4).
- When $\gamma > 1$, the cycles become shorter and ℓ_1 behaves asymptotically as $(\frac{1}{\gamma-1}\log n)^{1/\gamma}$; see Section 4.

Weighted random permutations clearly show a rich behavior and only a little part has been uncovered so far. The case of negative parameters $\alpha_j \asymp -j^\gamma$ remains to be explored and the future will hopefully bring more results regarding concentration properties.

In the case of uniform permutations, it is known that the random variables r_k converge to independent Poisson random variables with parameter $1/k$ in the limit $n \to \infty$ [1, 13]. An open problem is to understand how this generalizes to weighted random permutations.

2. Asymptotic Ewens distribution. In the case of the uniform distribution, it is an easy exercise to show that $P(\ell_1 = a) = 1/n$ for any $a = 1, \dots, n$. It follows that $P(\ell_1 > sn) \to 1 - s$ for any $0 \le s \le 1$. This result was extended to the case of small weights in [4]. We consider here parameters that are close to Ewens weights. A result similar to (a) below has been recently derived by Lugo [15].

THEOREM 2.1. *Let $\theta \in \mathbb{R}_+$. We suppose that $\sum_{j=1}^\infty \frac{1}{j}|\theta_j - \theta| < \infty$ if $\theta \ge 1$ or that $\sum_{j=1}^\infty |\theta_j - \theta| < \infty$ if $\theta < 1$.*

(a) *The distribution of ℓ_1 satisfies, for $0 \le s \le 1$,*

$$\lim_{n\to\infty} P(\ell_1 > sn) = (1-s)^\theta. \tag{2.1}$$

(b) *The joint distribution of ℓ_1 and ℓ_2 satisfies, for $0 \le s, t \le 1$,*

$$\begin{aligned} &\lim_{n\to\infty} P(\ell_1 > sn, \ell_2 > tn) \\ &\quad = \frac{\theta}{1+\theta}(1-s-t)_+^{\theta+1} + \frac{1+\theta(s \vee t)}{1+\theta}(1 - s\vee t)^\theta, \end{aligned} \tag{2.2}$$

where f_+ denotes the positive part of a function f.

Let us recall a few properties that are satisfied by the normalization factors h_n. Summing over the length j of the cycle that contains 1 we find the useful relation

$$P(\ell_1 \in [a,b]) = \frac{1}{n! h_n} \sum_{j=a}^{b} \frac{(n-1)!}{(n-j)!} \theta_j (n-j)! h_{n-j} = \frac{1}{n} \sum_{j=a}^{b} \theta_j \frac{h_{n-j}}{h_n}. \tag{2.3}$$

Choosing $[a,b] = [1,n]$, we get

$$h_n = \frac{1}{n} \sum_{j=1}^{n} \theta_j h_{n-j}, \qquad h_0 = 1. \tag{2.4}$$

Next, let $G_h(s) = \sum_{n \geq 0} h_n s^n$ be the generating function of the sequence (h_n). One can view a permutation as a combinatorial structure made of cycles. It follows from standard combinatorics results that $G_h(s) = \exp \sum_{j \geq 1} \frac{1}{j} \theta_j s^j$. We also refer to [4] for a direct proof of this formula. The first step in the proof of Theorem 2.1 is to control the normalization h_n. Here, $(\theta)_n = \theta(\theta+1) \cdots (\theta+n-1)$ denotes the ascending factorial.

PROPOSITION 2.2. *Under the assumptions of Theorem* 2.1, *we have*

$$h_n = C(\boldsymbol{\theta}) \frac{(\theta)_n}{n!} \bigl(1 + o(1)\bigr) \qquad \text{with } C(\boldsymbol{\theta}) = \exp \sum_{j \geq 1} \frac{1}{j} (\theta_j - \theta).$$

PROOF. We have

$$G_h(s) = \exp\left\{ \theta \sum_j \frac{1}{j} s^j + \sum_j \frac{1}{j} (\theta_j - \theta) s^j \right\} = (1-s)^{-\theta} \mathrm{e}^{u(s)} \tag{2.5}$$

with

$$u(s) = \sum_{j \geq 1} \frac{1}{j} (\theta_j - \theta) s^j. \tag{2.6}$$

Notice that $u(1) = \lim_{s \nearrow 1} u(s)$ exists. Let c_j be the Taylor coefficients of $\mathrm{e}^{u(s)}$, that is, $\mathrm{e}^{u(s)} = \sum c_j s^j$. Then, by Leibniz' rule,

$$h_n = \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}s^n} G_h(s) \bigg|_{s=0} = \frac{(\theta)_n}{n!} \sum_{k \geq 0} d_{n,k} c_k \tag{2.7}$$

with

$$d_{n,k} = \begin{cases} \dfrac{n(n-1) \cdots (n-k+1)}{(\theta+n-1) \cdots (\theta+n-k)}, & \text{if } k \leq n, \\ 0, & \text{otherwise.} \end{cases} \tag{2.8}$$

It is not hard to check that

$$d_{n,k} \le \begin{cases} 1, & \text{if } \theta \ge 1, \\ \theta^{-1}k + 1, & \text{if } \theta > 0. \end{cases} \tag{2.9}$$

Let $U(s) = \sum \frac{1}{j} |\theta_j - \theta| s^j$ and C_j be the Taylor coefficients of $\mathrm{e}^{U(s)}$. It is clear that $|c_j| \le C_j$ for all j. When $\theta \ge 1$, the first bound of (2.9) and the dominated convergence theorem imply

$$\lim_{n\to\infty} \sum_{k\ge 0} d_{n,k} c_k = \sum_{k \ge 0} c_k = \mathrm{e}^{u(1)} = C(\boldsymbol{\theta}). \tag{2.10}$$

When $\theta < 1$, the second bound of (2.9) gives $d_{n,k}|c_k| \le (\theta^{-1}k + 1)C_k$. The sequence (kC_k) is absolutely convergent:

$$\sum k C_k = \frac{\mathrm{d}}{\mathrm{d}s} \mathrm{e}^{U(s)} \bigg|_{s=1} = \mathrm{e}^{U(1)} U'(1) = \mathrm{e}^{\sum (1/j)|\theta_j - \theta|} \sum |\theta_j - \theta| < \infty. \tag{2.11}$$

We again obtain (2.10) by the dominated convergence theorem. □

PROOF OF THEOREM 2.1. We show that, for any $0 < s < t < 1$, we have

$$\lim_{n\to\infty} P(\ell_1 \in [sn, tn]) = (1-s)^\theta - (1-t)^\theta. \tag{2.12}$$

Using Proposition 2.2, we have

$$P(\ell_1 \in [sn, tn]) = \frac{1}{n} \sum_{j=sn}^{tn} \theta_j \frac{h_{n-j}}{h_n} = \frac{\theta}{n} \sum_{j=sn}^{tn} \frac{(\theta)_{n-j}}{(n-j)!} \frac{n!}{(\theta)_n} \big(1 + o(1)\big). \tag{2.13}$$

Here and throughout this article, when a and b are not integers we use the convention

$$\sum_{j=a}^{b} f(j) = \sum_{j \in [a,b] \cap \mathbb{N}} f(j) = \sum_{j=\lceil a \rceil}^{\lfloor b \rfloor} f(j). \tag{2.14}$$

We now use the identity

$$(\theta)_n = \frac{\Gamma(n+\theta)}{\Gamma(\theta)} \tag{2.15}$$

and the asymptotic

$$\frac{\Gamma(n+\theta)}{n!} = n^{\theta-1}\big(1 + o(1)\big). \tag{2.16}$$

We get

$$P(\ell_1 \in [sn, tn]) = \frac{\theta}{n} \sum_{j=sn}^{tn} \left(1 - \frac{j}{n}\right)^{\theta-1} \big(1 + o(1)\big). \tag{2.17}$$

As $n \to \infty$, the right-hand side converges to the Riemann integral $\theta \int_s^t (1-\xi)^{\theta-1}\,\mathrm{d}\xi$ and we obtain the first claim of Theorem 2.1.

Let us now turn to the second claim. Let $1 \le a \le b \le n$ and $1 \le c \le d \le n$. We get an expression for the joint probability of ℓ_1 and ℓ_2 in a similar fashion as for (2.3). When both indices belong to different cycles (noted $1 \not\sim 2$), we have

$$P(\ell_1 \in [a,b], \ell_2 \in [c,d], 1 \not\sim 2) = \frac{1}{n! h_n} \sum_{\substack{j\in[a,b]\\ k\in[c,d]\\ j+k\le n}} \sum_{\substack{|c_1|=j\\ |c_2|=k}} \theta_j \theta_k \sum_{\pi'} \prod_{\ell\ge 1} \theta_\ell^{r_\ell(\pi')}. \tag{2.18}$$

Here c_1 and c_2 denote the cycles that contain 1 and 2, respectively, and π' denotes a permutation of the $n-j-k$ indices that do not belong to c_1 or c_2. The number of cycles of length j that contain 1 but not 2 is $\frac{(n-2)!}{(n-1-j)!}$; given c_1, the number of cycles of length k that contain 2 is $\frac{(n-j-1)!}{(n-j-k)!}$. Since the sum over π' gives $(n-j-k)! h_{n-j-k}$, we get

$$P(\ell_1 \in [a,b], \ell_2 \in [c,d], 1 \not\sim 2) = \frac{1}{n(n-1)} \sum_{\substack{j\in[a,b]\\ k\in[c,d]\\ j+k\le n}} \theta_j \theta_k \frac{h_{n-j-k}}{h_n}. \tag{2.19}$$

When both indices belong to the same cycle one can first sum over the length j of the common cycle, then over $j-2$ indices other than 1, 2 and then over $j-1$ locations for 2. This gives $\frac{(n-2)!}{(n-j)!}(j-1)$ possibilities. The sum over permutations on remaining indices gives $(n-j)! h_{n-j}$. The result is

$$\begin{aligned} P(\ell_1 \in [a,b], \ell_2 \in [c,d]) &= \frac{1}{n(n-1)} \sum_{\substack{j\in[a,b]\\ k\in[c,d]\\ j+k\le n}} \theta_j \theta_k \frac{h_{n-j-k}}{h_n} \\ &\quad + \frac{1}{n(n-1)} \sum_{j\in[a,b]\cap[c,d]} (j-1)\theta_j \frac{h_{n-j}}{h_n}. \end{aligned} \tag{2.20}$$

Let $\varepsilon > 0$ and set $a = sn$, $c = tn$ and $b = d = n$. We assume, without loss of generality, that $1 \ge s \ge t \ge 0$. Using the above expression, Proposition 2.2 and equations (2.15) and (2.16), we deduce that, for n large,

$$\begin{aligned} &P(\ell_1 \ge sn, \ell_2 \ge tn) \\ &\quad = \frac{\theta^2}{n^2} \sum_{\substack{j\ge sn, k\ge tn\\ j+k\le(1-\varepsilon)n}} \left(1 - \frac{j+k}{n}\right)^{\theta-1} \bigl(1 + o_\varepsilon(1)\bigr) \\ &\qquad + \frac{\theta}{n^2} \sum_{sn\le j\le(1-\varepsilon)n} (j-1)\left(1-\frac{j}{n}\right)^{\theta-1} \bigl(1+o_\varepsilon(1)\bigr) + O(\varepsilon). \end{aligned} \tag{2.21}$$

Taking first the limit $n \to \infty$ and then the limit $\varepsilon \to 0$, the right-hand side of the latter expression is seen to converge to

$$1_{\{s+t\le 1\}}\theta^2 \int_{s+t}^1 (\xi - s - t)(1-\xi)^{\theta-1}\,\mathrm{d}\xi + \theta \int_s^1 \xi(1-\xi)^{\theta-1}\,\mathrm{d}\xi \tag{2.22}$$

and the second claim of Theorem 2.1 follows. □

3. Slowly diverging parameters. This section is devoted to parameters α_j that grow slowly to $+\infty$. The typical case is $\alpha_j = j^\gamma$ with $0 < \gamma < 1$ but our conditions allow more general sequences. As mentioned in the Introduction, the system displays a surprising behavior: almost all indices belong to a single giant cycle.

THEOREM 3.1. *We assume that* $0 < \frac{\theta_{n-j}\theta_j}{\theta_n} \le c_j$ *for all* n *and for* $j = 1, \dots, \frac{n}{2}$, *with constants* c_j *that satisfy* $\sum_{j\ge 1} \frac{c_j}{j} < \infty$. *Then*

$$\lim_{m\to\infty} \lim_{n\to\infty} P(\ell_1 > n - m) = 1.$$

It may be worth recalling that in this article n always denotes the number of elements and that P depends on n. The proof of this theorem can be found later in this section. In the case $\alpha_j = j^\gamma$ we have

$$\frac{\theta_{n-j}\theta_j}{\theta_n} = \mathrm{e}^{-n^\gamma[(1-j/n)^\gamma + (j/n)^\gamma - 1]} \approx \begin{cases} \mathrm{e}^{-j^\gamma}, & \text{if } j \ll n, \\ \mathrm{e}^{-cn^\gamma}, & \text{if } j = sn, \end{cases} \tag{3.1}$$

where the constant in the last equation is $c = (1-s)^\gamma + s^\gamma - 1$. It is positive for $0 < \gamma < 1$ and the condition of the theorem is fulfilled. Another interesting example is $\theta_j = j^{-\gamma}$ with $\gamma > 0$, where we can choose $c_j = 2j^{-\gamma}$.

Let us understand why parameters $\alpha_j = j^\gamma$ favor longer and longer cycles when $\gamma < 1$. The heuristics are actually provided by statistical mechanics, namely, we can write the probability $P(\pi)$ as a Gibbs distribution $\frac{1}{Z}\mathrm{e}^{-H(\pi)}$ with "Hamiltonian" $H(\pi) = \sum_{i=1}^n \frac{\alpha_{\ell_i(\pi)}}{\ell_i(\pi)}$. Thus, an "energy" $\frac{\alpha_j}{j} = j^{\gamma-1}$ is associated with each index i that belongs to a cycle of length j. Indices in longer cycles have lower energy so they are favored. This discussion also provides an illustration for the symmetry (1.4); it amounts to shifting the Hamiltonian by a constant and this does not affect the Gibbs distribution.

We can state a more precise result than Theorem 3.1 if we make the additional assumption that $\frac{\theta_{n+1}}{\theta_n}$ converges to 1 as $n \to \infty$. This condition is easy to check when $\alpha_j = j^\gamma$, $0 < \gamma < 1$ or when $\alpha_j = \gamma \log j$, $\gamma > 0$.

THEOREM 3.2. *Suppose that the assumptions of Theorem* 3.1 *hold true. In addition, we suppose that* $\frac{\theta_{n+1}}{\theta_n}$ *converges to* 1 *as* $n \to \infty$. *Then* $\sum_j h_j < \infty$, *and* *for any fixed* $m \geq 0$,

$$\lim_{n\to\infty} P(\ell_1 = n - m) = \frac{h_m}{\sum_{j\geq 0} h_j}.$$

Theorem 3.2 shows in particular that a single cycle of length n occurs with probability $1/\sum_j h_j$, but that finite cycles may be present as well.

This theorem is proved at the end of the section. We first obtain estimates for h_n.

PROPOSITION 3.3. *Under the assumptions of Theorem* 3.1 *there exists a constant* B *such that, for all* $n \geq 1$,

$$1 \leq \frac{nh_n}{\theta_n} \leq B.$$

The constant B *depends on* $\{c_j\}$ *only.*

PROOF. The lower bound follows obviously from (2.4) but the upper bound requires some work. Let $a_n = \frac{nh_n}{\theta_n}$. The relation (2.4) can be written as

$$a_n = 1 + \sum_{j=1}^{n-1} \frac{1}{j} \frac{\theta_{n-j}\theta_j}{\theta_n} a_j. \tag{3.2}$$

We can rewrite this relation as

$$a_n = \begin{cases} 1 + \displaystyle\sum_{j=1}^{(n-1)/2} \frac{\theta_{n-j}\theta_j}{\theta_n}\left(\frac{a_j}{j} + \frac{a_{n-j}}{n-j}\right), & \text{if } n \text{ is odd,} \\ 1 + \displaystyle\sum_{j=1}^{n/2-1} \frac{\theta_{n-j}\theta_j}{\theta_n}\left(\frac{a_j}{j} + \frac{a_{n-j}}{n-j}\right) + \frac{2\theta_{n/2}^2}{n\theta_n} a_{n/2}, & \text{if } n \text{ is even.} \end{cases} \tag{3.3}$$

We define the sequence (b_n) by the recursion equation

$$b_n = 1 + \sum_{j=1}^{n/2} c_j \left(\frac{b_j}{j} + \frac{b_{n-j}}{n-j}\right). \tag{3.4}$$

It is clear that $a_n \leq b_n$ for all n. Next, let m be a number such that

$$\frac{2}{n}\sum_{j=1}^{n/2} c_j + \sum_{j>m/2} \frac{c_j}{j} \leq \frac{1}{2} \tag{3.5}$$

for all $n \geq m$. Such an m exists because (c_j/j) is summable and the first term of the above equation is less than $\frac{2}{\sqrt{n}}\sum_{j=1}^{\sqrt{n}} \frac{c_j}{j} + \sum_{j>\sqrt{n}} \frac{c_j}{j} = o(1)$. We set

$$B = 2 \max_{1\leq j\leq m} b_j. \tag{3.6}$$

Notice that B depends on the c_js but not on the θ_js. Finally, we introduce another sequence (b'_n) defined by

$$b'_n = \begin{cases} b_n, & \text{if } n \le m, \\ 1 + \sum_{j=1}^{n/2} c_j \Big(\frac{b'_j}{j} + \frac{2B}{n} \Big), & \text{if } n > m. \end{cases} \tag{3.7}$$

It is clear that $b'_n \le \frac{1}{2} B$ for $n \le m$; we now show by induction that $b'_n \le B$ for all n. We have

$$\begin{aligned} b'_n - b'_m &= \frac{2B}{n} \sum_{j=1}^{n/2} c_j - \sum_{j=1}^{m/2} c_j \frac{b'_{m-j}}{m-j} + \sum_{j=m/2+1}^{n/2} c_j \frac{b'_j}{j} \\ &\le \Big(\frac{2}{n} \sum_{j=1}^{n/2} c_j + \sum_{j>m/2} \frac{c_j}{j} \Big) B. \end{aligned} \tag{3.8}$$

This is less than $\frac{1}{2} B$ by definition (3.5) of m. Since $b'_m \le \frac{1}{2} B$, we find that $b'_n \le B$ for all n. The final step is to see that $b_n \le b'_n$. This is clear when $n \le m$ and we get it by induction when $n > m$:

$$\begin{aligned} b_{n+1} &= 1 + \sum_{j=1}^{n/2} c_j \Big(\frac{b_j}{j} + \frac{b_{n-j+1}}{n-j+1} \Big) \\ &\le 1 + \sum_{j=1}^{n/2} c_j \Big(\frac{b'_j}{j} + \frac{2B}{n+1} \Big) = b'_{n+1}. \end{aligned} \tag{3.9}$$

We have shown that $a_n \le b_n \le b'_n \le B$ for all n. □

PROOF OF THEOREM 3.1. Using Proposition 3.3 we get

$$\begin{aligned} P(\ell_1 \le n-m) &= \frac{1}{n} \sum_{j=m}^{n-1} \theta_{n-j} \frac{h_j}{h_n} \le B \sum_{j=m}^{n-1} \frac{1}{j} \frac{\theta_{n-j}\theta_j}{\theta_n} \\ &\le B \sum_{j=m}^{n/2} \frac{c_j}{j} + B \sum_{j=n/2}^{n-1} \frac{c_{n-j}}{j}. \end{aligned} \tag{3.10}$$

The last term goes to zero as $n \to \infty$. The first term goes to zero as $n \to \infty$ and $m \to \infty$. □

PROOF OF THEOREM 3.2. From equation (2.3)

$$P(\ell_1 = n-m) = \frac{1}{n} \theta_{n-m} \frac{h_m}{h_n} = \frac{\theta_{n-m}}{\theta_n} \frac{\theta_n}{n h_n} h_m. \tag{3.11}$$

Further, (2.4) can be written as

$$\frac{nh_n}{\theta_n} = \sum_{j=0}^{n/2}\Big(\frac{\theta_{n-j}h_j}{\theta_n} + \frac{\theta_j h_{n-j}}{\theta_n}\Big). \tag{3.12}$$

This is actually correct for odd n only; there is an unimportant correction for even n coming from $j = n/2$. Since $h_j \leq B\frac{\theta_j}{j}$ (Proposition 3.3), the summand is less than $Bc_j(\frac{1}{j} + \frac{1}{n-j}) \leq 2B\frac{c_j}{j}$. For each j, and as $n \to \infty$, we have $\frac{\theta_{n-j}}{\theta_n} \to 1$ and $\frac{\theta_j h_{n-j}}{\theta_n} \leq B\frac{c_j}{n-j} \to 0$. The right-hand side of (3.12) then converges to $\sum_j h_j$ by dominated convergence. We can now take the limit $n \to \infty$ in (3.11) and we indeed obtain the claim. □

4. Quickly diverging parameters. Here we treat parameters $\theta_j = \mathrm{e}^{-\alpha_j}$ with α_j diverging quickly, or equivalently θ_j decaying quickly. More precisely, we shall make the following two assumptions: for some $M > 0$, all $k \geq 1$ and two coprime numbers $j_1, j_2 \geq 4$,

$$0 \leq \theta_k \leq \frac{\mathrm{e}^{Mk}}{k!}, \qquad \theta_{j_1} > 0, \qquad \theta_{j_2} > 0. \tag{4.1}$$

It is necessary to impose some kind of aperiodicity condition on the set of indices corresponding to nonvanishing coefficients θ_j. This prevents us from prescribing, for example, permutations with only even lengths of cycles. In this case we have $h_n = 0$ for all odd n, as can be easily seen from the recursion (2.4); Proposition 4.5 below would fail.

Our assumptions allow us to get the asymptotics of h_n using the saddle point method. We write down the steps explicitly in order to keep the article self-contained. A slightly shorter path would be to prove that our assumptions imply that e^f, with $f(z) = \sum_{j=0}^{\infty}\theta_j z^j$, is "Hayman admissible" and to use standard results [11]. Hayman admissibility is implicitly derived in our proof.

We describe general results in Section 4.1, relegating proofs to Section 4.2. The general results turn out to be somewhat abstract so we use them to study the particularly interesting class $\alpha_j = j^\gamma$, $\gamma > 1$, in Section 4.3.

4.1. *Main properties.* We now describe three general theorems about cycle lengths. In all theorems conditions (4.1) are silently assumed. The first statement concerns the absence of macroscopic cycles.

THEOREM 4.1. *For arbitrarily small $\delta > 0$ and arbitrarily large $k > 0$, there exists $n_0 = n_0(\delta, k)$ such that*

$$P\Big(\max_{1\leq i\leq n} \ell_i \geq \delta n\Big) \leq n^{-k}$$

for all $n \geq n_0$.

More precise information about typical cycle lengths can be extracted from the following result. Let $r_n > 0$ be defined by the equation

$$\sum_{j\geq 1} \theta_j r_n^j = n. \tag{4.2}$$

That such r_n exists uniquely is immediate.

THEOREM 4.2. *Let $a(n), b(n)$ be such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{a(n)} \theta_j r_n^{j+1/2} = 0, \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{j=b(n)}^{n} \theta_j r_n^{j+1/2} = 0.$$

Then

$$\lim_{n\to\infty} P\big(\ell_1 \in [a(n), b(n)]\big) = 1.$$

When the information about the coefficients θ_j is sufficiently detailed, some control on r_n is possible and Theorem 4.2 can be used to obtain sharp results. This is exemplified in Section 4.3 for the special case $\alpha_j = \alpha(j) = j^\gamma$ with $\gamma > 1$. In such cases, the sum $\sum_{j=1}^{\infty} \theta_j r_n^{j+1/2}$ (whose value is $r_n^{1/2} n$) is dominated by the terms corresponding to indices j close to the solution $j_{\max}$ of the equation $\alpha'(j) = \log r_n$.

Finally, it is also possible to extract from Theorem 4.2 a general result proving absence of small cycles.

THEOREM 4.3.

$$\lim_{n\to\infty} P\left(\ell_1 \leq \frac{\log n}{\log r_n} - \frac{3}{4}\right) = 0.$$

We shall see below that the proof of Theorem 4.3 is straightforward; nonetheless, the result is quite strong. In the case where only finitely many θ_j are nonzero, we find $r_n \sim n^{1/j_0}$, where j_0 is the last index with nonzero θ_j. Thus $\log n / \log r_n \approx j_0$ and we obtain the probability that $\ell_1 \leq j_0 - 1$ is zero. It follows that almost all cycles have length j_0, a fact already observed in [2, 17]. On the other hand, if infinitely many θ_j are nonzero, it is easy to see that $\log n / \log r_n$ diverges. Thus ℓ_1 goes to infinity in probability. To summarize, the only way to force a positive fraction of indices to lie in finite cycles is to forbid infinite cycles altogether, in which case typical cycles have the maximal length that is allowed.

4.2. *Proofs of the main properties.* We now prove Theorems 4.1–4.3. We use the following elementary result, which is a consequence of the first assumption in (4.1).

LEMMA 4.4. *Let* $f(x) = \sum_{k=0}^{\infty} c_k x^k$ *with Taylor coefficients that satisfy* $0 \le c_k \le \mathrm{e}^{Mk} k^{-k}$ *for some* $M > 0$ *and all* $k \ge 1$. *Then for all* $\delta > 0$ *and all* $x \ge 0$, *we have*

$$f'(x) \le (1+\delta)\mathrm{e}^M f(x) + \mathrm{e}^M/\delta.$$

PROOF. Let $k_0 = k_0(x) = \lfloor (1+\delta)\mathrm{e}^M x \rfloor$. We decompose

$$f'(x) = \sum_{k=1}^{\infty} c_k k x^{k-1} = \sum_{k=1}^{k_0} c_k k x^{k-1} + R(x).$$

By our assumptions,

$$\begin{aligned} R(x) = \sum_{k=k_0+1}^{\infty} c_k k x^{k-1} &\le \mathrm{e}^M \sum_{k=k_0+1}^{\infty} \left(\frac{x\mathrm{e}^M}{k}\right)^{k-1} \\ &\le \mathrm{e}^M \sum_{k=k_0+1}^{\infty} \left(\frac{1}{1+\delta}\right)^k \le \mathrm{e}^M/\delta. \end{aligned}$$

On the other hand, for the terms up to k_0, we have $k \le k_0 \le (1+\delta)x\mathrm{e}^M$ and thus

$$\sum_{k=1}^{k_0} c_k k x^{k-1} \le (1+\delta)\mathrm{e}^M \sum_{k=0}^{k_0} c_k x^k \le (1+\delta)\mathrm{e}^M f(x).$$

This completes the proof. □

Let us define the functions

$$I_\beta(z) = \sum_{j=1}^{\infty} j^\beta \theta_j z^j$$

for $\beta \in \mathbb{R}$. $\phi(z) := I_{-1}(z)$ plays a special role since the generating function of (h_n) is given by $G_h(z) = \exp(\phi(z))$. All I_β are analytic by the first assumption in (4.1), monotone increasing and positive on $\{z > 0\}$ together with all their derivatives and $I_{\beta+1}(z) = zI'_\beta(z)$. Lemma 4.4 implies that for each $\beta > 0$ there exists C such that for all $z \ge 0$ we have

$$I'_\beta(z) \le C I_\beta(z). \tag{4.3}$$

Recall that $r_n = I_0^{-1}(n)$, where I_0^{-1} denote the inverse function.

PROPOSITION 4.5. *We have*

$$h_n = \frac{r_n^{-n}}{\sqrt{2\pi I_1(r_n)}} \mathrm{e}^{\phi(r_n)} \big(1 + o(1)\big).$$

PROOF. Condition (4.1) on Taylor coefficients implies that $I_0(z) < \tilde{D} \times \exp(Cz)$. Then

$$r_n \geq c \log n \tag{4.4}$$

for some $c > 0$. On the other hand, r_n diverges more slowly than $n^{1/4}$ since $I_0(x)$ diverges faster than x^4 by (4.1).

For the saddle point method, we use Cauchy's formula and we obtain

$$\begin{aligned} h_n &= \frac{1}{2\pi r^n} \int_{-\pi}^{\pi} \mathrm{e}^{\phi(r\mathrm{e}^{\mathrm{i}\gamma}) - n\mathrm{i}\gamma} \,\mathrm{d}\gamma \\ &= \frac{\mathrm{e}^{\phi(r)}}{2\pi r^n} \biggl[\int_{-\gamma_0}^{\gamma_0} \mathrm{e}^{\phi(r\mathrm{e}^{\mathrm{i}\gamma}) - \phi(r) - n\mathrm{i}\gamma} \,\mathrm{d}\gamma + 2 \int_{\gamma_0}^{\pi} \mathrm{e}^{\phi(r\mathrm{e}^{\mathrm{i}\gamma}) - \phi(r) - n\mathrm{i}\gamma} \,\mathrm{d}\gamma \biggr] \end{aligned} \tag{4.5}$$

for any $r > 0$ and any $0 < \gamma_0 < \pi$. We choose the $r = r_n$ defined by equation (4.2) since it is the minimum point of $r^{-n}\mathrm{e}^{\phi(r)}$ and $\gamma_0 = \gamma_0(n) = r_n^{-(1+\delta)}$ for some $0 < \delta < 1/2$. The leading order of the first term above can be found by expanding $\phi(z) - n \log z$ around $\gamma = 0$. We have

$$\phi(r_n\mathrm{e}^{\mathrm{i}\gamma}) - \phi(r_n) - n\mathrm{i}\gamma = \sum_{j \geq 1} \frac{\theta_j}{j} r_n^j (\mathrm{e}^{\mathrm{i}j\gamma} - 1 - \mathrm{i}j\gamma). \tag{4.6}$$

Expanding $\mathrm{e}^{\mathrm{i}j\gamma} - 1 - \mathrm{i}j\gamma = -\frac{1}{2} j^2\gamma^2 + R(j\gamma)$ with $|R(j\gamma)| \leq \frac{1}{3!}(j\gamma)^3$ we get

$$\begin{aligned} \phi(r_n\mathrm{e}^{\mathrm{i}\gamma}) - \phi(r_n) - n\mathrm{i}\gamma &= -\frac{1}{2}\gamma^2 \sum_{j \geq 1} j\theta_j r_n^j + A(\gamma) \\ &= -\frac{1}{2}\gamma^2 I_1(r_n) + A(\gamma) \end{aligned} \tag{4.7}$$

with

$$|A(\gamma)| \leq \frac{\gamma_0^3}{3!} \sum_{j \geq 1} j^2 \theta_j r_n^j = \frac{\gamma^2}{r_n^{1+\delta} 3!} I_2(r_n) \tag{4.8}$$

for all $\gamma \leq \gamma_0$. Now, by (4.3), we have $I_2(r_n) \leq C r_n I_1(r_n)$. Thus, as $n \to \infty$, the term $A(\gamma)$ is negligible compared to $\gamma^2 I_1(r_n)$ in the first integral, which is therefore given by

$$\begin{aligned} \int_{-\gamma_0}^{\gamma_0} \mathrm{e}^{-1/2\gamma^2 I_1(r_n)(1+o(1))} \,\mathrm{d}\gamma &= \frac{1}{\sqrt{I_1(r_n)}} \int_{-\gamma_0\sqrt{I_1(r_n)}}^{\gamma_0\sqrt{I_1(r_n)}} \mathrm{e}^{-(1/2)\xi^2(1+o(1))} \,\mathrm{d}\xi \\ &= \sqrt{\frac{2\pi}{I_1(r_n)}} \bigl(1 + o(1)\bigr). \end{aligned} \tag{4.9}$$

The last equality is justified by the fact that $\gamma_0(n) I_1(r_n) \geq r_n^{-1-\delta} I_0(r_n) \geq r_n^{-2} n$, which diverges as $n \to \infty$.

We now turn to the second term in (4.5). We want to show that it is negligible and we estimate it by replacing the integral by π times the maximum of the integrand. In view of (4.9) it is enough to show that

$$\lim_{n\to\infty} \frac{1}{2}\log I_1(r_n) - \operatorname{Re}\big(\phi(r_n) - \phi(r_n e^{i\gamma})\big) = -\infty \tag{4.10}$$

for all $\gamma \in [\gamma_0, \pi]$. For the first term we have $\log I_1(r_n) \le \log(C r_n I_0(r_n)) \le \tilde{C}\log n$. For the second term we have

$$\begin{aligned}\operatorname{Re}\big(\phi(r_n) - \phi(r_n e^{i\gamma})\big) &= \sum_{j\ge 1} \frac{1}{j}\theta_j r_n^j \big(1-\cos(\gamma j)\big)\\ &\ge \frac{\theta_{j_1}}{j_1} r_n^{j_1}\big(1-\cos(\gamma j_1)\big) + \frac{\theta_{j_2}}{j_2} r_n^{j_2}\big(1-\cos(\gamma j_2)\big),\end{aligned} \tag{4.11}$$

where j_1 and j_2 are picked according to (4.1). The right-hand side is zero at $\gamma = 0$ and it is strictly positive when $\gamma \in (0, \pi]$ (j_1 and j_2 are coprime); so its minimum is attained at γ_0 when n is sufficiently large (recall that $\gamma_0 \to 0$ when $n \to \infty$). Expanding the cosine, we get

$$\operatorname{Re}\big(\phi(r_n) - \phi(r_n e^{i\gamma})\big) \ge c' r_n^4 \gamma_0^2 = c' r_n^{2-2\delta} \ge cc'(\log n)^{2-2\delta}. \tag{4.12}$$

This dominates the first term of (4.10) since $\delta < 1/2$ and the proof is complete. □

PROOF OF THEOREM 4.1. Clearly,

$$P\Big(\max_i \ell_i > \delta n\Big) \le n P(\ell_1 > \delta n). \tag{4.13}$$

We have $I_1(r_n) \le C^2 r_n^2 \phi(r_n)$ by (4.3) and thus Proposition 4.5 gives $h_n \ge C' r_n^{-n-1}$ for n large enough. Since all the h_{n-j}'s are clearly bounded by some $D > 0$, we have by (2.3)

$$\begin{aligned} nP(\ell_1 > \delta n) &\le D r_n^{n+1} \sum_{j=\delta n}^{n} \left(\frac{e^M}{j}\right)^j \le D r_n^{n+1} n \left(\frac{e^M}{\delta n}\right)^{\delta n}\\ &\le Dn\left(\frac{e^M r_n^{2/\delta}}{\delta n}\right)^{\delta n}.\end{aligned} \tag{4.14}$$

The statement is trivial [and seen directly from (2.3)] if only finitely many θ_j are nonzero; thus we may assume there are infinitely many nonzero θ_j. Then $I_0(z)$ grows faster at infinity than any power of z and r_n diverges more slowly than any power of n. The last bracket is less than 1 for n large enough so that the right-hand side vanishes in the limit $n \to \infty$. □

In order to make more precise statements about the length of typical cycles we need a better control over the terms appearing in (2.3). By the previous result it suffices to consider the case where j is not too close to n.

PROPOSITION 4.6. *For each $\delta > 0$ there exists C_δ such that, for all $n \in \mathbb{N}$ and all $j < (1-\delta)n$, we have*

$$\frac{h_{n-j}}{h_n} \le C_\delta r_n^{j+1/2}.$$

PROOF. By Proposition 4.5 we have

$$\begin{aligned}\frac{h_{n-j}}{h_n} &\approx r_n^j \left(\frac{r_n}{r_{n-j}}\right)^{n-j} \left(\frac{I_1(r_n)}{I_1(r_{n-j})}\right)^{1/2} \mathrm{e}^{\phi(r_{n-j})-\phi(r_n)} \\ &= r_n^j \exp\bigl(-\bigl(\phi(r_n) - \phi(r_{n-j}) \\ &\qquad - (n-j)\bigl(\ln(r_n) - \ln(r_{n-j})\bigr)\bigr)\bigr)\left(\frac{I_1(r_n)}{I_1(r_{n-j})}\right)^{1/2} \qquad (4.15)\\ &= r_n^j \exp\biggl(-\biggl(\phi(r_n) - \phi(r_{n-j}) \\ &\qquad - \phi'(r_{n-j})r_{n-j} \ln\left(\frac{r_n}{r_{n-j}}\right)\biggr)\biggr)\left(\frac{I_1(r_n)}{I_1(r_{n-j})}\right)^{1/2}\end{aligned}$$

when both n and $n-j$ are large. Put $r_{n-j} = x$ and $r_n = x+u$. Since $n \mapsto r_n$ is increasing, we have $u > 0$. The exponent above then has the form

$$\begin{aligned}&\phi(x+u) - \phi(x) - x\phi'(x)\ln\left(\frac{x+u}{x}\right) \\ &\quad = \bigl(\phi(x+u) - \phi(x) - \phi'(x)u\bigr) + \phi'(x)\left(u - x\ln\left(\frac{x+u}{x}\right)\right).\end{aligned} \qquad (4.16)$$

The first bracket in the right-hand side is greater than $\frac{1}{2}u^2\phi''(x)$ since all derivatives of ϕ are positive on $\mathbb{R}^+$. The second bracket is always positive. Thus, for all $n \in \mathbb{N}$ and all $j \le (1-\delta)n$, there exists $C'_\delta > 0$ such that

$$\frac{h_{n-j}}{h_n} \le C'_\delta r_n^j \mathrm{e}^{-(1/2)(r_n - r_{n-j})^2\phi''(r_{n-j})}\left(\frac{I_1(r_n)}{I_1(r_{n-j})}\right)^{1/2}. \qquad (4.17)$$

By (4.3), $I_1(x) = xI_0'(x) \le CxI_0(x)$. We also have $I_0(x) \le I_1(x)$. Since $I_0(r_n) = n$, we get

$$\frac{I_1(r_n)}{I_1(r_{n-j})} \le Cr_n \frac{n}{n-j} \le \frac{C}{\delta} r_n. \qquad (4.18)$$

This proves the claim. □

PROOF OF THEOREM 4.2. The claims follows immediately from (2.3) and Proposition 4.6. □

PROOF OF THEOREM 4.3. Let $m = \log n / \log r_n - \frac{3}{4}$. We use equation (2.3), bounding θ_j by a constant and using Proposition 4.6 for the ratio of normalization factors. Since r_n diverges, we have

$$P(\ell_1 \le m) \le \frac{C}{n} \sum_{j=1}^{m} r_n^{j+1/2} = \frac{C}{n} r_n^{3/2} \frac{r_n^m - 1}{r_n - 1} \le \frac{C'}{n} r_n^{m+1/2}, \tag{4.19}$$

if n is large enough. The right-hand side is equal to $C' r_n^{-1/4}$ and it vanishes in the limit $n \to \infty$. □

4.3. *An explicit example.* In this subsection we treat explicitly the case $\alpha_j = \alpha(j) = j^\gamma$ with $\gamma > 1$ as an example of application of the previous general results. We first observe that the assumptions (4.1) are trivially satisfied so that the general results in this section apply.

The main result of this subsection is that typical cycles are of size $(\frac{1}{\gamma-1} \log n)^{1/\gamma}$ to leading order.

THEOREM 4.7. *Let $\alpha_j = j^\gamma$, with $\gamma > 1$. Then*

$$\frac{\ell_1}{((1/(\gamma-1)) \log_n)^{1/\gamma}} \to 1 \tag{4.20}$$

in probability.

Let us define

$$\Delta(j) = \alpha(j) - \alpha(j_{\max}) - (j - j_{\max}) \log r_n. \tag{4.21}$$

The proof of Theorem 4.7 follows from two simple technical estimates.

LEMMA 4.8. *Let $j_{\max} \in \mathbb{R}$ be such that $\alpha'(j_{\max}) = \log r_n$.*

(a) *Assume that $\gamma \ge 2$. Then for all $j \ge 1$, there exists $c = c(\gamma) > 0$ such that*

$$\Delta(j) \ge c\alpha''(j_{\max})(j - j_{\max})^2. \tag{4.22}$$

(When $j \ge j_{\max}$, one can choose $c = \frac{1}{2}$.)

(b) *Assume that $\gamma \in (1, 2)$. Then, for all $1 \le j \le 2j_{\max}$, there exists $c = c(\gamma) > 0$ such that*

$$\Delta(j) \ge c\alpha''(j_{\max})(j - j_{\max})^2. \tag{4.23}$$

(When $j \le j_{\max}$, one can choose $c = \frac{1}{2}$.) Moreover, for all $j > 2j_{\max}$, there exists $c = c(\gamma) > 0$ such that

$$\Delta(j) \ge cj^\gamma. \tag{4.24}$$

PROOF. We start with the case $\gamma \geq 2$. First of all, since $j_{\max} = (\alpha')^{-1}(\log r_n)$, we have for any $j > j_{\max}$

$$\begin{aligned}\Delta(j) &= \alpha(j) - \alpha(j_{\max}) - (j - j_{\max}) \log r_n \\ &= \alpha(j) - \alpha(j_{\max}) - (j - j_{\max})\alpha'(j_{\max}) \\ &= \int_{j_{\max}}^{j} \mathrm{d}s \int_{j_{\max}}^{s} \alpha''(t)\,\mathrm{d}t \geq \frac{1}{2}\alpha''(j_{\max})(j - j_{\max})^2,\end{aligned} \tag{4.25}$$

since α'' is an increasing function. Similarly, we have for any $\frac{1}{2} j_{\max} \leq j < j_{\max}$

$$\begin{aligned}\Delta(j) &= \int_{j}^{j_{\max}} \mathrm{d}s \int_{s}^{j_{\max}} \alpha''(t)\,\mathrm{d}t \\ &\geq \frac{1}{2}\alpha''\left(\frac{1}{2} j_{\max}\right)(j - j_{\max})^2 \\ &= 2^{1-\gamma}\alpha''(j_{\max})(j - j_{\max})^2.\end{aligned} \tag{4.26}$$

Finally, for $0 \leq j < \frac{1}{2} j_{\max}$ we use

$$\begin{aligned}\Delta(j) &= \int_{j}^{j_{\max}} \mathrm{d}s \int_{s}^{j_{\max}} \alpha''(t)\,\mathrm{d}t \geq \int_{j_{\max}/2}^{j_{\max}} \mathrm{d}s \int_{s}^{j_{\max}} \alpha''(t)\,\mathrm{d}t \\ &\geq \frac{1}{2}\alpha''\left(\frac{1}{2} j_{\max}\right)\frac{1}{4} j_{\max}^2 \geq 2^{-\gamma-1}\alpha''(j_{\max})(j - j_{\max})^2.\end{aligned} \tag{4.27}$$

Let us now turn to the case $\gamma \in (1, 2)$. The proof is completely similar. When $j \leq j_{\max}$ we use (observe that α'' is a decreasing function now)

$$\Delta(j) = \int_{j}^{j_{\max}} \mathrm{d}s \int_{s}^{j_{\max}} \alpha''(t)\,\mathrm{d}t \geq \frac{1}{2}\alpha''(j_{\max})(j - j_{\max})^2. \tag{4.28}$$

When $j_{\max} < j \leq 2 j_{\max}$ we use

$$\begin{aligned}\Delta(j) &= \int_{j_{\max}}^{j} \mathrm{d}s \int_{j_{\max}}^{s} \alpha''(t)\,\mathrm{d}t \geq \frac{1}{2}\alpha''(2 j_{\max})(j - j_{\max})^2 \\ &= 2^{\gamma-3}\alpha''(j_{\max})(j - j_{\max})^2.\end{aligned} \tag{4.29}$$

Finally, when $j > 2 j_{\max}$ we have

$$\begin{aligned}\Delta(j) &= \int_{j_{\max}}^{j} \mathrm{d}s \int_{j_{\max}}^{s} \alpha''(t)\,\mathrm{d}t \geq \frac{1}{2}\alpha''(j)(j - j_{\max})^2 \\ &\geq \frac{1}{8}\alpha''(j) j^2 = \frac{1}{8}\gamma(\gamma - 1) j^{\gamma}.\end{aligned} \tag{4.30}$$

□

COROLLARY 4.9. *For any $\gamma > 1$, we have, as $n \to \infty$,*

$$j_{\max} = \left(\frac{1}{\gamma - 1}\log n\right)^{1/\gamma}(1 + o(1)), \tag{4.31}$$

$$\log r_n = \alpha'(j_{\max}) = \gamma\left(\frac{1}{\gamma - 1}\log n\right)^{(\gamma-1)/\gamma}(1 + o(1)), \tag{4.32}$$

$$e^{-\alpha(j_{\max})} r_n^{j_{\max}} = n^{1+o(1)}. \tag{4.33}$$

PROOF. We start with the case $\gamma \geq 2$. Using the previous lemma, it immediately follows that

$$\begin{aligned} I_0(r_n) &= \sum_{j\geq 1} e^{-\alpha(j)} r_n^j \leq e^{-\alpha(j_{\max})} r_n^{j_{\max}} \sum_{j\geq 1} e^{-c\alpha''(j_{\max})(j-j_{\max})^2} \\ &\leq C_1 e^{-\alpha(j_{\max})} r_n^{j_{\max}}. \end{aligned} \tag{4.34}$$

Since for $j < j_{\max}$, $\Delta(j) \leq \frac{1}{2}\alpha''(j_{\max})(j - j_{\max})^2$, we also have

$$I_0(r_n) \geq e^{-\alpha(\lfloor j_{\max}\rfloor)} r_n^{\lfloor j_{\max}\rfloor} \geq e^{-(1/2)\alpha''(j_{\max})} e^{-\alpha(j_{\max})} r_n^{j_{\max}}. \tag{4.35}$$

Using the relation $I_0(r_n) = n$, (4.34) and (4.35) immediately imply the claimed asymptotics.

Let us now turn to the case $\gamma \in (1, 2)$. The lemma implies that

$$\begin{aligned} I_0(r_n) &= e^{-\alpha(j_{\max})} r_n^{j_{\max}} \sum_{j\geq 1} e^{-\Delta(j)} \\ &\leq C_2 e^{-\alpha(j_{\max})} r_n^{j_{\max}} \left\{\alpha''(j_{\max})^{-1/2} + \sum_{j>2j_{\max}} e^{-cj^\gamma}\right\}. \end{aligned} \tag{4.36}$$

Since $j_{\max} \nearrow \infty$ as $n \to \infty$, we see that $\sum_{j>2j_{\max}} e^{-cj^\gamma} \ll \alpha''(j_{\max})^{-1/2}$ and thus that, for large n,

$$I_0(r_n) \leq C_3 \alpha''(j_{\max})^{-1/2} e^{-\alpha(j_{\max})} r_n^{j_{\max}}. \tag{4.37}$$

As above, we also have

$$\begin{aligned} I_0(r_n) &\geq e^{-\alpha(\lceil j_{\max}\rceil)} r_n^{\lceil j_{\max}\rceil} \geq e^{-(1/2)\alpha''(j_{\max})} e^{-\alpha(j_{\max})} r_n^{j_{\max}} \\ &\geq C_4 e^{-\alpha(j_{\max})} r_n^{j_{\max}}. \end{aligned} \tag{4.38}$$

The claimed asymptotics follow as before. □

PROOF OF THEOREM 4.7. Let $\varepsilon > 0$. It is sufficient to check that Theorem 4.2 applies with $a(n) = (1 - \varepsilon)j_{\max}$ and $b(n) = (1 + \varepsilon)j_{\max}$. It follows from Lemma 4.8 and Corollary 4.9 that

$$\frac{1}{n}\sum_{j=b(n)}^{\infty} e^{-\alpha(j)} r_n^{j+1/2} \leq n^{o(1)} \sum_{j=b(n)}^{\infty} e^{-c\alpha''(j_{\max})(j-j_{\max})^2}, \tag{4.39}$$

which goes to 0 as $n \to \infty$, since

$$\mathrm{e}^{-c\alpha''(j_{\max})(b(n)-j_{\max})^2} = n^{-c\varepsilon^2\gamma(1+o(1))}. \tag{4.40}$$

Similarly,

$$\begin{aligned} \frac{1}{n}\sum_{j=1}^{a(n)} \mathrm{e}^{-\alpha(j)} r_n^{j+1/2} &\le n^{o(1)} \sum_{j=1}^{a(n)} \mathrm{e}^{-c\alpha''(j_{\max})(j-j_{\max})^2} \\ &\le n^{o(1)} \mathrm{e}^{-c\alpha''(j_{\max}) j_{\max}^2 \varepsilon^2}, \end{aligned} \tag{4.41}$$

which again goes to 0 as $n \to \infty$. □

Acknowledgments. We are indebted to the referee for several useful comments and especially for suggesting the claim of Theorem 3.2. D. Ueltschi is grateful to Nick Ercolani and several members of the University of Arizona for many discussions about the Plancherel measure. D. Ueltschi also acknowledges the hospitality of the University of Geneva, ETH Zürich, the Center of Theoretical Studies of Prague and the University of Arizona where parts of this project were carried forward.

REFERENCES

[1] ARRATIA, R. and TAVARÉ, S. (1992). The cycle structure of random permutations. *Ann. Probab.* **20** 1567–1591. MR1175278

[2] BENAYCH-GEORGES, F. (2007). *Cycles of Random Permutations with Restricted Cycle Lengths*. Available at arxiv:0712.1903.

[3] BETZ, V. and UELTSCHI, D. (2009). Spatial random permutations and infinite cycles. *Comm. Math. Phys.* **285** 469–501. MR2461985

[4] BETZ, V. and UELTSCHI, D. (2010). Spatial permutations with small cycle weights. *Probab. Theory Related Fields* DOI: 10.1007/s00440-009-0248-0.

[5] BETZ, V. and UELTSCHI, D. (2010). Critical temperature of dilute Bose gases. *Phys. Rev. A* **81** 023611.

[6] BOLLOBÁS, B. and PITTEL, B. (2009). The distribution of the root degree of a random permutation. *Combinatorica* **29** 131–151. MR2520278

[7] DELAURENTIS, J. M. and PITTEL, B. G. (1985). Random permutations and Brownian motion. *Pacific J. Math.* **119** 287–301. MR803120

[8] ERDÖS, P. and LEHNER, J. (1941). The distribution of the number of summands in the partitions of a positive integer. *Duke Math. J.* **8** 335–345. MR0004841

[9] EWENS, W. J. (1972). The sampling theory of selectively neutral alleles. *Theoret. Population Biology* **3** 87–112. MR0325177

[10] FENG, S. and HOPPE, F. M. (1998). Large deviation principles for some random combinatorial structures in population genetics and Brownian motion. *Ann. Appl. Probab.* **8** 975–994. MR1661315

[11] FLAJOLET, P. and SEDGEWICK, R. (2009). *Analytic Combinatorics*. Cambridge Univ. Press, Cambridge. MR2483235

[12] FRISTEDT, B. (1993). The structure of random partitions of large integers. *Trans. Amer. Math. Soc.* **337** 703–735. MR1094553

[13] GONCHAROV, V. L. (1944). Some facts from combinatorics. *Izv. Akad. Nauk SSRS Ser. Mat.* **8** 3–48.

[14] HANSEN, J. C. (1990). A functional central limit theorem for the Ewens sampling formula. *J. Appl. Probab.* **27** 28–43. MR1039182

[15] LUGO, M. (2009). Profiles of permutations. *Electron. J. Combin.* **16** Research Paper 99, 20. MR2529808

[16] SZALAY, M. and TURÁN, P. (1977). On some problems of the statistical theory of partitions with application to characters of the symmetric group. I. *Acta Math. Acad. Sci. Hungar.* **29** 361–379. MR0506108

[17] TIMASHEV, A. N. (2008). Random permutations with cycle lengths in a given finite set. *Diskret. Mat.* **20** 25–37. MR2420494

[18] VERSHIK, A. and YAKUBOVICH, Y. (2001). The limit shape and fluctuations of random partitions of naturals with fixed number of summands. *Mosc. Math. J.* **1** 457–468. MR1877604

V. BETZ
D. UELTSCHI
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WARWICK
COVENTRY, CV4 7AL
UK
E-MAIL: v.m.betz@warwick.ac.uk
daniel@ueltschi.org
URL: http://www.maths.warwick.ac.uk/~betz/
http://www.ueltschi.org

Y. VELENIK
SECTION DE MATHÉMATIQUES
UNIVERSITÉ DE GENÈVE
1211 GENÈVE 4
SWITZERLAND
E-MAIL: Yvan.Velenik@unige.ch
URL: http://www.unige.ch/math/folks/velenik