Phase transitions in classical and quantum Heisenberg models

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References
"Reflection positivity, infrared bound, continuous symmetry breaking", Balint Toth, Prague lectures '96,
http://www.math.bme.hu/~balint/oktatas/okbatrasok/iszik_jegyzet/


"Phase transitions & continuous symmetry breaking", Jurg Fröhlich, Vienna lectures '11
http://www.math.phys.univie.ac.at/~chha/speakers.html


1. Classical Heisenberg model

1.1. Setting

Finite lattice \( \Lambda = \{ \frac{1}{2} \} \times \cdots \times \{ -\frac{1}{2} \} \)

\( E \) = set of edges of \( \Lambda \), i.e. nearest-neighbors with per. b.c.

At each site \( x \) there is a "classical spin" \( \sigma_x \in S^2 \subset \mathbb{R}^3 \)

(N.b. dimension of space is \( d=1,2,3,\ldots \), but dimension of spin is always 3.)

State space \( \Omega_\Lambda = \times_{x \in \Lambda} S^2 = (S^2)^\Lambda \)

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Hamiltonian: \[ \mathcal{H}(\sigma) = -2 \sum_{j \neq i} \sigma_j \cdot \sigma_i - h \sum_{i \in A} \sigma_i \]
with \( h \in \mathbb{R}^3 \): external magnetic field

Partition function: \[ Z(\beta, \Lambda, h) = \sum_{\sigma \in \Lambda} e^{-\beta \mathcal{H}(\sigma)} \]

where \( \beta = k_B T \)

Order parameter for spontaneous magnetization:
\[ \langle (\frac{1}{4!1} \sum_{x \in \Lambda} \sigma_x)^2 \rangle = \frac{1}{4!1} \sum_{x \in \Lambda} \sum_{y \in \Lambda} \sum_{z \in \Lambda} \langle \sigma_x \sigma_y \sigma_z \rangle \]

where \( \langle \cdot \rangle = \frac{1}{Z(\beta, \Lambda, h)} \sum_{\sigma \in \Lambda} \cdot e^{-\beta \mathcal{H}(\sigma)} \)

Main question: occurrence of ferromagnetism? That is, is the above expression positive, uniformly in \( \Lambda \), when \( h = 0 \)?

- \( \beta \) small: no, as can be proved by high temperature expansion (+ cluster expansion method)
- \( d = 1,2 \), all \( \beta \): no (Mermin-Wagner theorem)
- \( d \geq 3 \): yes. Amazing result of Fröhlich, Simon, Spencer (1976), reflection positivity, \( \delta \)-infrared bounds, to be explained below.

1.2 Occurrence of Spontaneous Magnetization

We look at correlation functions \( \langle \sigma_x \sigma_y \rangle \). They do not depend on \( i = 1, 2, 3 \) (rotation invariance of \( \mathbb{R}^3 \)). They only depend on \( x \cdot y \) (translation invariance).

Notation: \[ N(x) = \langle \sigma_x \sigma_x \rangle = \langle \sigma_y \sigma_y \rangle \]

Let us introduce the dual lattice for Fourier theory, \( \Lambda^* \), and the lattice dispersion relation \( E(k) \), \( k \in \Lambda^* \):
\[ \Lambda^* = \{ \frac{2\pi}{L_1}, \frac{2\pi}{L_2}, \ldots, \frac{2\pi}{L_d}, \ldots \} \]
\[ E(k) = 2 \sum_{i=1}^d \left( 1 - \cos k_i \right) \]

Theorem (Long-range order, Fröhlich-Simon-Spencer '76)
Assume \( \bar{h} = 0 \), and \( L_1, \ldots, L_d \) are even. Then we have the lower bound
\[ \frac{1}{4!1} \sum_{x \in \Lambda} \chi(x) \geq \frac{1}{3} - \frac{1}{\beta^4 \Lambda} \sum_{k \in \Lambda^*} \frac{1}{E(k)} \]

Consequences: In thermodynamic limit \( L_1, \ldots, L_d \to \infty \),
\[ \lim_{L_1, \ldots, L_d \to \infty} \left| \left( \frac{1}{4!1} \sum_{x \in \Lambda} \sigma_x \right)^2 \right| = 1 - \frac{3}{\log^4 \beta} \sum_{k \in \Lambda^*} \frac{1}{E(k)} \]
small when \( \beta \) large
In words, the classical Heisenberg model exhibits spontaneous magnetization at sufficiently low temperatures!

Structure of proof:

LRO (Theorem above)

\[ \tilde{\mathbf{X}}(k) \leq \frac{1}{\beta E(k)} \]

\[ \mathbf{X} = \frac{1}{n} \cos(kd) \]

GD \[ Z(x) \leq Z(0) \]

\[ \text{real, nice!} \]

RP \[ Z(x, y) \leq Z(x, y, 0) \cdot Z(0, y) \]

These steps are explained in Sections 1.3 - 1.6 below.

1.3 IRREVERENT BOUND (IRB) IMPLIES LRO

Definition of Fourier transform: If \( f \in L^2(\Lambda) \),

\[ \hat{f}(k) = \sum_{x \in \Lambda} e^{-ikx} f(x), \quad k \in \Lambda^* \]

\[ f(x) = \frac{1}{|\Lambda|} \sum_{k \in \Lambda^*} e^{ikx} \hat{f}(k), \quad x \in \Lambda \]

Lemma. (IRB \( \implies \) LRO)

Assume that \( \tilde{\mathbf{X}}(k) \leq \frac{1}{\beta E(k)} \). Then we get the lower bound in the Theorem above.

Proof: \[ \frac{1}{3} = \mathbf{X}(0) = \frac{1}{|\Lambda|} \tilde{\mathbf{X}}(0) + \frac{1}{|\Lambda|} \sum_{k \in \Lambda^*} \mathbf{X}(k) \]

\[ \mathbf{X} = \sum_{k \in \Lambda^*} \hat{X}(k) \cdot e^{ikx} \]

\[ \text{Then } \frac{1}{|\Lambda|} \sum_{x \in \Lambda} X(x) = \frac{1}{|\Lambda|} \tilde{\mathbf{X}}(0) \leq \frac{1}{\beta E(0)} \sum_{x \in \Lambda} \frac{1}{|\Lambda|} \sum_{x \in \Lambda} X(x) \]

\[ \square \]
1.4. **GAUSSIAN DOMINATION (GD) IMPLIES 1RB**

We introduce a position function \( Z(x) \) that depends on an external real field \( \nu = (\nu_x)_{x \in \Lambda} \). At this stage, we need the discrete Laplacian \( \Delta \):

\[
(\Delta v)_x = \sum_{y : x \neq y} \left( v_y - v_x \right)
\]

As a matrix in \( \ell^2(\Lambda) \):

\[
\Delta \varphi = \begin{cases} -2d & \text{if } x = y \\ 1 & \text{if } x \neq y \in E \\ 0 & \text{if } x \in V \end{cases}
\]

Exercise: check the useful identity

\[
(\varphi, -\Delta \varphi) = \sum_{x \in V} \left( \varphi_x - \varphi_y \right)^2
\]

Inner product in \( \ell^2(\Lambda) \)

This identity is the discrete analogue of \( -\int f \partial^2 g + \int g \partial f \).

The Heisenberg Hamiltonian \( H(\varphi, \psi) \) with \( \beta = 0 \):

\[
H(\varphi) = \frac{1}{2} \sum_{x \in \Lambda} \alpha \varphi_x^2 - 2d \| \nabla \|_2^2
\]

Definition: \( Z(\nu) = \sum_{x \in \Lambda} \exp \left\{ -\beta \sum_{y : x \neq y} \left( \nu_y - \nu_x \right)^2 \right\} \)

We have \( Z(0) = Z(\beta, A, 0) e^{2\beta \| \nabla \|_2^2} \)

Lemma \( (GD \Rightarrow 1RB) \)

Assume \( \hat{B}(x) \leq Z(0) \forall x \). Then \( \hat{X}(x) \leq \frac{1}{\beta \| \nabla \|_2^2} \).

Proof:

\[
Z(\nu) = \sum_{x \in \Lambda} \exp \left\{ -\beta \sum_{y : x \neq y} \left( \nu_y - \nu_x \right)^2 \right\}
\]

Let \( b = -\Delta \nu \). Then

\[
\mathbb{E}(\nu) \leq Z(0) \Rightarrow \langle e^{-\frac{\beta}{2} (\varphi, b)} \rangle \leq e^{-\beta \langle \nu, \Delta \nu \rangle}
\]

We use the inequality with \( \nu = \varphi(x) \), \( b \) is \( \Delta \).

\[
(-\Delta \nu)_x = -\Re \langle \Delta e^{i k x} \rangle = -\Re \left( \frac{1}{2} e^{i k x} + i e^{i k x} + e^{-i k x} - 2 \nu_x \right)
\]

\[
= -\Re e^{i k x} \left( \frac{1}{2} e^{i k x} + i e^{i k x} + e^{-i k x} - 2 \nu_x \right)
\]

We see that \( b = \nu \) is a possible choice of \( b \). Consider the field \( \eta \), \( \eta \) small.

\[
\langle e^{-\beta \eta \cdot \eta} \rangle \leq e^{\beta \eta \cdot \eta} \]

Since

\[
1 - 2 \beta \eta \cdot \eta \leq \left( \langle \eta, \eta \rangle \right) + 2 \beta \eta \cdot \eta = \left( \langle \eta, \eta \rangle \right) + 2 \beta \eta \cdot \eta = \left( \langle \eta, \eta \rangle \right) + 2 \beta \langle \eta, \eta \rangle = \langle \eta, \eta \rangle + O(\eta^2)
\]

We get

\[
1 + 2 \nu \cdot \eta \leq \nu \cdot \eta + O(\eta^2)
\]

So that

\[
2 \beta \varphi(0) \hat{X}(0) \leq 1
\]

\( \blacksquare \)
1.5. REFLECTION POSITIVITY (RP) IMPLIES GD

Let \( R : \Lambda \to \Lambda \) denote the following reflection map:

\[
\begin{array}{c}
\bullet \quad \bullet \\
\bigcirc \quad \bigcirc \\
\end{array}
\]

Let \( \Lambda = \Lambda_1 \cup \Lambda_2 \) with \( \Lambda_1 \cap \Lambda_2 = \emptyset \), \( \Lambda_1 = R\Lambda_2 \).

Denote \( \nu = (\nu_1, \nu_2) \), where \( \nu_1 \) is a field on \( \Lambda_1 \) and \( \nu_2 \) is a field on \( \Lambda_2 \).

Lemma (RP \Rightarrow GD)

Assume that \( Z(\nu, \nu)^* \leq Z(\nu_1, R\nu_2) Z(R\nu_1, \nu_2) \) for any reflection \( R \) across edges, in all lattice directions. Then

\[
Z(\nu) \leq Z(\nu) \quad \forall \nu \text{ real field.}
\]

Proof: Observe that \( Z(\text{const}) = Z(\nu) \), so we need to show

that there exists a constant maximizer.

The existence of maximizers follows from the fact \( Z_\nu = 0 \)

for any \( \nu \), and any fixed \( \nu_1 \).
Proof: The partition function \( Z(v_i, v_{i'}) \) has the form

\[
Z(v_i, v_{i'}) = \sum_{\mathbf{A}} \left( \sum_{v_i, v_{i'}} e^{-F(v_i, v_{i'})} \right) \exp \left( -\beta \sum_{\mathbf{A}} \frac{1}{2} \sum_{v_i, v_{i'}} \left( v_i^2 + v_{i'}^2 - v_i - v_{i'} \right) \right)
\]

The Ising model, \( A \) and \( A' \), are coupled by the last exponential. The key insight is to introduce fields to decouple them. Recall that

\[
ed^2 = \int_{-\infty}^{\infty} \frac{d\phi}{2\pi} e^{i\phi}
\]

Introducing a field for each term in the exponential, we get

\[
Z(v_i, v_{i'}) = \left( \prod_{\mathbf{A}} \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{d\phi_1}{2\pi} \right)^N e^{-\frac{1}{2} \sum_{\mathbf{A}} d^2_1 \phi_1^2} \exp \left( \frac{1}{2} \sum_{\mathbf{A}} \sum_{v_i, v_{i'}} \left( v_i^2 + v_{i'}^2 \right) \phi_i \phi_{i'} \right)
\]

Clearly, the Schwartz inequality:

\[
Z(v_i, v_{i'})^2 \leq \left( \prod_{\mathbf{A}} \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{d\phi_1}{2\pi} \right)^N \exp \left( \frac{1}{2} \sum_{\mathbf{A}} \sum_{v_i, v_{i'}} \left( v_i^2 + v_{i'}^2 \right) \phi_i \phi_{i'} \right)
\]

The Schwartz inequality:

\[
Z(v_i, v_{i'})^2 \leq Z(v_i, R v_{i'}) Z(R v_i, v_{i'})
\]

\[
= Z(v_i, R v_{i'}) Z(R v_i, v_{i'})
\]

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2. GENERAL SETTING OF QUANTUM SPIN SYSTEMS

R. B. Israel, "Convexity in the Theory of Lattice Gases", 1979
B. Simon, "The Statistical Mechanics of Lattice Gases", 1993
J. Fröhlich's Vienna lectures

Statistical mechanics is about macroscopic behavior of systems that satisfy known microscopic laws. A standard approach is to start with finite systems and let the system size tend to infinity. Another approach is to introduce the notion of infinite-volume states, and formulate a property in order to characterize equilibrium states (this is the KMS condition). We choose a fixed infinite lattice, \( \mathbb{Z}^d \) for concreteness. Let \( \mathcal{H}_0 \) be the local Hilbert space. We suppose dim \( \mathcal{H}_0 \) finite. (The relevant space for spin \( \frac{1}{2} \) is \( \mathbb{C}^2 \).) For each \( x \in \mathbb{Z}^d \), let \( \mathcal{H}_x \) be a copy of \( \mathcal{H}_0 \).

Let \( \mathcal{A} \), the algebra of operators on \( \mathcal{H}_0 \) (\( \mathcal{A} \) is \( \mathbb{C}^2 \)-algebra) and \( \mathcal{A}_x \) the copy of \( \mathcal{A} \).
For each finite subset \( \Lambda \subset \mathbb{Z}^d \), let

\[
\mathcal{R}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x
\]

\[
\mathcal{R}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{R}_x
\]

(To be precise, \( \mathcal{R}_\Lambda \) is the algebra of all linear combinations of operators of the form \( A_x \otimes A_{x_2} \otimes \cdots \otimes A_{x_n} \), if \( \Lambda = \{ x_1, \ldots, x_n \} \).

There is a natural inclusion relation between these algebras: if \( \Lambda' \subset \Lambda \) and \( a \in \mathcal{R}_\Lambda \), we can construct the operator \( a \otimes \text{Id}_{\mathcal{R}_{\Lambda'}} \) in \( \mathcal{R}_{\Lambda'} \). Notice that \( \|a\| = \|a \otimes \text{Id}_{\mathcal{R}_{\Lambda'}}\| \).

With this contraction in mind, we have \( \mathcal{R}_{\Lambda'} \subset \mathcal{R}_\Lambda \) if \( \Lambda' \subset \Lambda \).

In order to describe infinite-volume systems, it may be tempting to consider the Hilbert space \( \bigotimes_{x \in \mathbb{Z}^d} \mathcal{H}_x \). But this is not appropriate — this would be a non-separable Hilbert space, to be avoided at all costs!

Instead, we consider the \( \mathbb{C}^* \)-algebra of all local operators, i.e., \( \mathcal{R}_\Lambda \). This is a normed space. Its completion is the algebra of quasi-local observables, denoted \( \mathcal{R}_\Lambda \).

An (infinite-volume) state is a positive normalized linear functional \( \rho \) on \( \mathcal{R}_\Lambda \). That is, it satisfies

\[
\rho(\text{Id}) = 1 \quad \text{(normalized)}
\]

\[
\rho(A^*A) \geq 0 \quad \text{(positive)}
\]

A (translation-invariant) interaction is a family \( \{ \phi_x \}_{x \in \mathbb{Z}^d} \) of operators defined for each finite, \( \mathbb{Z}^d \)-invariant subset \( X \) of \( \mathbb{Z}^d \), that satisfies

\[
\phi_x \in \mathcal{R}_X \quad \text{and} \quad \phi_{x+x_0} = \phi_x \quad \forall x_0 \in \mathbb{Z}^d \quad (R_x \text{ is a translation operator } R_x \phi_x = \phi_{x+x_0})
\]

We introduce the following norm on interactions:

\[
\|\phi\|_r = \sum_{x \in \mathbb{Z}^d} \|\phi_x\|_r e^{r|x|} \quad , \quad r > 0
\]

This yields Banach spaces of interactions.

A given interaction give rise to a family of Hamiltonians indexed by domains. For a finite subset of \( \mathbb{Z}^d \), let

\[
H^d_\Lambda = \sum_{x \in \Lambda} \phi_x
\]

(Remark: the case of domains with periodic boundary conditions does not fit this framework; it needs a separate treatment.)
Now that we have Hamiltonians, we can define finite-volume Gibbs states. Let
\[ Z^{\beta, \phi}_\Lambda = \frac{1}{Z^{\beta}_\Lambda} \text{Tr} \, e^{-\beta H^\phi_\Lambda} \]
and for \( a \in \mathcal{R}_\Lambda \), let
\[ Z^{\beta, \phi}_\Lambda(a) = \frac{1}{Z^{\beta}_\Lambda} \text{Tr} \, a \, e^{-\beta H^\phi_\Lambda} \]
We get infinite-volume equilibrium states by considering cluster points of \( (\rho^{\beta, \phi}_\Lambda)_{\Lambda \supset \Lambda_n} \), with \( \Lambda \supset \mathbb{Z}^d \) (we do not work with cluster points where \( \Lambda \) tend to \( \mathbb{Z}^d \) or to a line ...)

A better way to describe the infinite-volume limit is to observe that the set of finite subsets of \( \mathbb{Z}^d \) is a “directed set” where the partial order is the inclusion. Then \( (\rho^{\beta, \phi}_\Lambda) \) is a net, and there exist converging subnets by a compactness argument. The limits of converging subnets, with \( \Lambda \supset \mathbb{Z}^d \), are infinite-volume states.

In practice, it is enough to consider the sequence \( (\rho^{\beta, \phi}_\Lambda) \) where \( \Lambda_n \) is the box of size \( n \) centered at the origin \( (0, ..., 0) \) and to consider the cluster points of this sequence.

Let us now look at the KMS condition that characterizes infinite-volume equilibrium states. We need to introduce the operator that describes the time evolution of observables:
\[ \alpha^{\alpha, \beta}_t(a) = e^{itH^\phi_\Lambda} a e^{-itH^\phi_\Lambda}, \quad a \in \mathcal{R}_\Lambda \]
Using cyclicity of the trace, and using the definition above with \( \Lambda \supset \mathbb{C} \), we get the following identity: if \( a, b \in \mathcal{R}_\Lambda \),
\[ \alpha^{\alpha, \beta}_t(a \, \alpha^{\alpha, \beta}_t(b)) = \frac{1}{Z^{\beta}_\Lambda} \text{Tr} \, a \, e^{itH^\phi_\Lambda} b \, e^{-itH^\phi_\Lambda} \]
\[ = \frac{1}{Z^{\beta}_\Lambda} \text{Tr} \, a \, e^{itH^\phi_\Lambda - itH^\phi_\Lambda} b \, e^{-itH^\phi_\Lambda - itH^\phi_\Lambda} \]
\[ = \alpha^{\alpha, \beta}_t(a \, \alpha^{\alpha, \beta}_t(b)) \, \alpha^{\alpha, \beta}_t(a) \]

The idea behind KMS is to use the infinite-volume version of this identity as the definition of equilibrium. We first need to extend \( \alpha^{\alpha, \beta}_t \) to the whole of \( \mathcal{R} \).

**lemma.** Let \( \alpha^{\alpha, \beta}_t(B) = [A, B] \). Then
\[ e^{\Lambda B} e^{-\Lambda} = \sum_n \frac{1}{n!} \alpha^{\alpha, \beta}_n(B) \]

**Proof:** The left side is equal to
\[ \sum_{k, m \geq 0} \frac{(\Lambda)^m}{k! \, m!} \, A^k B A^{-k} = \sum_{n \geq 0} \sum_{k \geq 0} \frac{(-1)^{m-k}}{k! \, (n-k)!} \, A^k B A^{-k} \]
We prove by induction that
\[
\frac{1}{n!} \omega^n_A (B) = \frac{1}{n!} \left[ \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} A^k B A^{-k} \right]
\]
This is clear for \( n = 0 \). For \( n \geq 1 \),
\[
\frac{1}{n} \omega^n_A (B) = \frac{1}{n} \left[ A \omega^n_A (B) - \omega^n_A (B) A \right]
= \frac{1}{n} A^n B + \frac{1}{n} \sum_{k=1}^{n} A^k B A^{-k} \left( \frac{1}{n!} \binom{n}{k} (-1)^{n-k} \right) + \frac{1}{n} \sum_{k=1}^{n} A^k B A^{-k} \left( \frac{1}{n!} \binom{n}{k} (-1)^{n-k} \right)
= \frac{1}{n!} \left[ \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} A^k B A^{-k} \right]
\]
\( \square \)

We now construct the infinite-volume time-evolution operator.

For all \( t \in \mathbb{R} \).

**Theorem:** Assume \( t \in \mathbb{R} \) and \( \lambda \in \mathbb{R} \). \( \forall s > 0 \).

There exists a unique bounded operator \( \alpha^t_{\lambda, s} : A \to A \) such that
\[
\alpha^t_{\lambda, s} \lim_{n \to \infty} \alpha_{\lambda}^n (a) = \alpha_{\lambda}^t (a) \quad \text{in } A
\]
\( \alpha_{\lambda}^t (a) = \alpha_{\lambda} \left( \alpha_{\lambda}^s (a) \right) \quad \forall s \in \mathbb{R} \).

The proof is done as follows:

1. If \( \lambda \leq \epsilon \) and \( \epsilon \) is small enough, \( \alpha_{\lambda}^n (a) \to \alpha_{\lambda}^0 (a) \) as \( n \to \infty \).

2. By the uniform boundedness theorem, there exists \( \epsilon > 0 \) with the property above, i.e., \( \| \alpha_{\lambda}^s \| < \epsilon \) for all \( s \in \mathbb{R} \).

3. We use the group property to extend \( \Phi \) to all \( t \in \mathbb{R} \).

Proof: We show that for any \( n \geq 0 \), there exists \( \lambda \) such that
\[
\| \alpha^t_{\lambda, s} (a) - \alpha^t_{\lambda, s} (a) \| \leq \epsilon
\]
for all \( M \leq \lambda \).

By the lemma above,
\[
\alpha^t_{\lambda, s} (a) = \sum_{n=0}^{\infty} \frac{(\lambda s)^n}{n!} \omega^n_A (a)
= \sum_{n=0}^{\infty} \frac{(\lambda s)^n}{n!} \sum_{|x|=n} [x_{K_1}, [x_{K_2}, ..., [x_{K_n}, a]...]]
\]

In order to form the terms to differ from 0, it is necessary that \( x_i \in K_i \setminus X_i \cap X_{i+1} \subseteq X_{i+1} \setminus X_i \) holds. In other words, where \( \supp a_i \) is the support of \( a_i \), i.e., the smallest set \( Y \) such that \( a_i \in L_Y \). IF \( \lambda > \lambda \), we have
\[
\alpha^t_{\lambda, s} (a) - \alpha^t_{\lambda, s} (a) = \sum_{n=0}^{\infty} \frac{(\lambda s)^n}{n!} \sum_{|x|=n} \sum_{|x|=n} \sum_{|x|=n} \sum_{|x|=n} ...
= \sum_{n=0}^{\infty} \frac{(\lambda s)^n}{n!} \sum_{|x|=n} \sum_{|x|=n} \sum_{|x|=n} \sum_{|x|=n} ...
\]

We have \( \lambda^t_{\lambda, s} (a) = \lambda^t_{\lambda, s} (a) \). For \( s = \infty \), \( \lambda^t_{\lambda, s} (a) = \lambda^t_{\lambda, s} \).
It is not too hard to check that $K_\alpha = 1 + \sum_{i=1}^{\infty} \beta_i^2 K_{i\alpha}$, so $K_{i\alpha} \to 0$ for $\beta_i \to 0$.

If $\alpha$ and $\beta$ are both less than $E$ (but not necessarily strict), we have

$$\phi_{\alpha \beta}^\alpha (\nu) = \phi_{\beta \alpha}^\beta (\phi_{\beta \nu} \phi_{\alpha \beta} \phi_{\alpha \beta}) + \phi_{\beta \alpha}^\beta (\phi_{\beta \nu} \phi_{\alpha \beta} \phi_{\alpha \beta})$$

This proves that $\phi_{\alpha \beta}^\alpha (\nu)$ converges for all $\alpha \in E$, and the limit satisfies the group properties.

We can now formulate the KMS condition. A state $\rho$ on $A$ satisfies the KMS condition for the interaction $\phi$ (i.e., it is an infinite-volume equilibrium state) if for all $a, b \in A$, we have

$$\rho(a, \phi_{\beta \alpha}^\alpha (b)) = \rho(\phi_{\beta \alpha}^\alpha (b), a).$$

(Compare with finite-volume expression above!) It is not hard to see that the limit $A/F$ of finite-volume Gibbs states is KMS.

Also, while it may seem very abstract and general, the set of KMS states does not contain undesirable pathologies.

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Thermodynamic Functions & Gibby variational principle

We introduce the entropy $s_{\alpha}$, which is a function of states, and the free energy $f_{\alpha}$, which is a function of interactions.

If $\rho$ is a state on $A$, then for any finite $\Lambda \neq 0$ there exists a unique $\rho_{\Lambda}$ on $A_{\Lambda}$ such that

$$\rho_{\Lambda} = \rho_{\Lambda} a , \quad \forall a \in A_{\Lambda}.$$  

We define

$$s_{\alpha} (\rho) = - \frac{1}{\beta} \text{Tr}_{\Lambda} (\rho_{\Lambda} \log \rho_{\Lambda})$$

$$f_{\alpha} (\beta, \phi) = - \frac{1}{\beta} \frac{1}{\beta} \text{Tr} \phi e^{-\beta \phi}$$

$$w_{\alpha} (\rho, \phi) = - \frac{1}{\beta} \frac{1}{\beta} \rho (e^{-\beta \phi})$$  

(average energy)


Theorem (Ruelle)

Assume $\alpha, \rho$ are translation-invariant, i.e., $\rho \in \mathcal{P}_{\alpha}$ for suitable $r$. Then the limit of $s_{\alpha}, f_{\alpha}, w_{\alpha}$ as $\Lambda \to \infty$ (in the sense of Fisher) exist, and the limiting functions enjoy convexity/concavity properties.

(See a good reference for precise statements.)
Theorem (Gibbs variational principle)

\[ s(p) \leq \beta u(p, \Phi) = \beta f(\beta, \Phi), \]

with equality iff \( p \) satisfies the KMS condition.

This is a much more attractive characterization of infinite-volume equilibrium states. It works only for translation-invariant states and interactions. However, unlike the KMS condition,

Finally, let us remark that translation-invariant states can also be defined as functionals that are tangent to the free energy: if the domain \( \Lambda \) is finite, the Gibbs state can be written as

\[ \rho_\beta(a) = \left. e^{\frac{1}{\beta h} \frac{1}{\beta h} \log \text{Tr} e^{-\beta H_0 - \beta h \sum_i \phi_i a_i} \right|_{h=0} \]

\[ = \left. e^{\frac{1}{\beta h} \phi_\beta(\beta, \Phi + h \phi_0)} \right|_{h=0} \]

where \( \phi_0 \) is the interaction from \( a \):

\[ \phi_0, x = \begin{cases} \phi_0 x & \text{if } x = \text{supp} a + x \\ 0 & \text{otherwise} \end{cases} \]

In infinite-volume limit, one can look at the linear functionals that are tangent to \( f(\beta, \Phi) \).

Finally, it is not hard to check that the set of equilibrium states has the structure of a simplex. It is actually a Choquet simplex, i.e. every state can be written as an average (with suitable measure) over external states.

One can define phase transitions as the change of the number of extremal states. There are typically just one equilibrium state at high temperatures, but there may be many at low temperatures. See Fröhlich's notes for more details.
3. QUANTUM HEISENBERG MODEL

3.1. DEFINITION OF THE MODEL

Let $S \in \mathbb{H}$. The spin-$S$ model involves local Hilbert spaces $\mathbb{C}^{2S+1} = \mathbb{H}_s$.

On $\mathbb{C}^{2S+1}$, let $S^1, S^2, S^3$ be operators that satisfy the relations:

$$(S^1)^2 + (S^2)^2 + (S^3)^2 = S(S+1) \text{ Id}$$

$[S^i, S^j] = i \varepsilon^{ijk} S^k, \quad [S^1, S^2] = iS^3, \quad [S^2, S^1] = iS^3, \quad [S^3, S^1] = iS^2.$

Let $\vec{S} = (S^1, S^2, S^3)$ and, with $\vec{a} \in \mathbb{R}^3$, $S^\vec{a} = \vec{a} \cdot \vec{S}$.

Lemma

(a) $[S^\vec{a}, S^\vec{b}] = i S^\vec{a} \times \vec{b}$

(b) $e^{-iS^\vec{a}} S^\vec{b} e^{iS^\vec{a}} = S^\vec{b}$, \text{ where } $\vec{b}$ is rotated around $\vec{a}$ by angle $\pi/2$.

Proof: (a) follows from linearity & commutation relations.

For (b), replace $\vec{a}$ by $5\vec{a}$, and check that both sides satisfy the same differential equation.

\[ \frac{d}{dt} e^{-iS^\vec{a}} S^\vec{b} e^{iS^\vec{a}} = -i [S^\vec{a}, e^{-iS^\vec{a}} S^\vec{b} e^{iS^\vec{a}}] \]

\[ \frac{d}{dt} S^\vec{b} e^{iS^\vec{a}} \cdot \vec{b} = (\frac{d}{dt} R_{\vec{a}} \cdot \vec{b}) \cdot S = (\vec{a} \times R_{\vec{a}} \cdot \vec{b}) \cdot S = -i [S^\vec{a}, S^\vec{b}] \cdot \vec{a} \]

Case $S = \frac{1}{2}$: Pauli matrices

$S^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $S^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $S^3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Case $S = 1$:

$S^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $S^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, $S^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

The most natural Hamiltonians are:

$H_{\text{ferm}} = - \sum_{\langle xy \rangle \in E} \frac{1}{\sqrt{2}} (S_x^x + S_y^x) - h \sum_{x \in \mathbb{Z}} S_x^3$

$H_{\text{anh}} = \sum_{\langle xy \rangle \in E} \frac{1}{\sqrt{2}} (S_x^x + S_y^x) + h \sum_{x \in \mathbb{Z}} S_x^3$

Here, the notation $S_x^x$ means the operator in $\mathbb{H}_x \otimes \mathbb{H}_y$ given by $S_x^x S^x_x + S_y^x S^y_x + S_z^x S^z_x$. If $S = \frac{1}{2}$, one can check that its eigenvalues are $\pm \frac{1}{2}$ (multiplicity 1) and $\mp \frac{3}{2}$ (multiplicity 3).
Motivation for the Heisenberg Hamiltonian with spin $S= \frac{1}{2}$:
In $\mathbb{H}_S \otimes \mathbb{H}_S$, consider the symmetry group of rotations, represented by unitary operators $U^x = e^{i \frac{\pi}{2} \sigma^x}$, $z \in \mathbb{R}^3$.
One can check that the group yields irreducible decomposition $\mathbb{H}_S \otimes \mathbb{H}_S = \text{singlet} \oplus \text{triplet}$, where singlet is the subspace spanned by $\frac{1}{\sqrt{2}} (|\uparrow, \uparrow\rangle - |\downarrow, \downarrow\rangle)$, and triplet is the subspace spanned by $|\uparrow, \uparrow\rangle, |\uparrow, \downarrow\rangle, |\downarrow, \uparrow\rangle$, and $\frac{1}{\sqrt{2}} (|\downarrow, \downarrow\rangle + |\uparrow, \uparrow\rangle)$.

The interaction operator should be rotation invariant, so it must be of the form $c_1 \text{Singlet} + c_2 \text{Triplet}$. If $b$ constants and a shift by the identity operator, we get $\pm \sum_{x \in \Lambda} \vec{S}_x \cdot \vec{S}_y$. This suggests $\pm \sum_{x \in \Lambda} \vec{S}_x \cdot \vec{S}_y$.

In these notes, we consider the following Hamiltonian:
$$H_{\Lambda, \beta} = -2 \sum_{x, y \in \Lambda} \left( S^+_x S^-_y + u S^x_y S^x_y + S^x_y S^x_y \right) - \hbar \sum_{x \in \Lambda} S_i^x$$

Most interesting cases are:
* $u = +1$: Heisenberg Ferromagnet
* $u = 0$: XY model
* $u = -1$: (b=0) If $\Lambda$ is bipartite, $\Lambda = \Lambda_A \uplus \Lambda_B$, then $U^1 H_A^0 U = 2 \sum_{x \in \Lambda_A} (S^+_x S^-_x + u S^x_x S^x_x + S^x_x S^x_x) = H_A^0$, with $U = \prod_{x \in \Lambda_B} e^{i \frac{\pi}{2} \sigma^x}$.

3.1. Occurrence of Spontaneous Magnetization

The order parameter for spontaneous magnetization in the quantum Heisenberg model is
$$\langle \left( \frac{1}{1} \sum_{x \in \Lambda} \vec{S}_x \right)^2 \rangle = \frac{1}{1} \sum_{x \in \Lambda} \sum_{y \in \Lambda} \langle S^+_x S^-_y \rangle$$

As in the classical case, we show that correlations do not decay. It is enough to prove it for $i = 3$. The beautiful result of Dyson, Lieb, Simon (1978) can be formulated as follows:

**Theorem (Long-range order, Dyson-Lieb-Simon)**
Assume that $b=0$, $l_1, \ldots, l_d$ even, and $u \in [-1, 0]$.
Then we have the lower bound
$$\frac{1}{l_1} \sum_{x \in \Lambda} \langle S^+_x S^-_x \rangle > \frac{1}{2} \sum_{x \in \Lambda} \frac{\varepsilon_u(x)}{\varepsilon_u(x)} - \frac{1}{(2 l_1)^d} \sum_{x \in \Lambda} \frac{1}{\varepsilon_u(x)} \sum_{y \in \Lambda} \frac{1}{\varepsilon_u(y)}$$

Here, $\varepsilon_u(x) = \frac{1}{l_1} \left( (1-u \varepsilon_u(x)) \langle S^+_x S^-_x \rangle + (u \varepsilon_u(x)) \langle S^+_x S^-_x \rangle \right)$.

It is easy to check that $\varepsilon_u(x) \approx S^+_x S^-_x$, so the right side is positive for $d \geq 3$ and $\varepsilon_u(x)$ large enough, uniform in $x$. Real $\Lambda$ can be picked by $d \geq 3$ and all $S \leq \frac{1}{2}$. (Kennedy, Lieb, Shultz, idea of Nauenberg).
Also, $b=0$, $d=2$, and $S \geq 1$. 

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The proof mirrors the one of FSS for classical systems, but several extra difficulties had to be solved. One difficulty is that the IRB does not hold for the usual correlation function. The reason is that
\[
\langle \bar{S}_x \cdot S_x \rangle - \langle \bar{S}_0 \cdot S_x \rangle \geq \frac{1}{2} \rightarrow \frac{1}{2} \text{ for } x \rightarrow 0.
\]
On the other hand, assuming IRB,
\[
|\langle \bar{S}_x \cdot S_x \rangle - \langle \bar{S}_0 \cdot S_x \rangle| = \sum_{k=1}^{\infty} \frac{|1-e^{-ikx}|}{|1-2ikx|} \langle \bar{S}_k \cdot S_x \rangle(1)
\]
This tends to 0 when \(x \to \infty\), contradicton.

Structure of the proof

1.0 A = IRB for corr. for

3.3 REFLECTION POSITIVITY FOR QUANTUM SYSTEMS

It is enough for our purpose to consider the claim of

Dyson, Lieb, Simon. A more general approach can be

Found in Fröhlich, Israel, Lieb, Simon.

Lemma (Reflection positivity)

\(\mathcal{H} = \mathbb{H} \otimes \mathbb{H}, \dim \mathbb{H} < \infty\). Matrices \(A, B, C, D;\)

\(i = 1, \ldots, k\), in the small space \(\mathbb{H}\), Then

\[
\| \prod_{k=1}^{k-1} \mathcal{T}_{\mathcal{E}} (A \otimes I + i \mathcal{E} \otimes B - \frac{1}{2} (C \otimes I + I \otimes D) )^2 \| \leq \prod_{k=1}^{k-1} \mathcal{T}_{\mathcal{E}} (A \otimes I + i \mathcal{E} \otimes B - \frac{1}{2} (C \otimes I + I \otimes D) )^2
\]

where \(\mathcal{A}\) denotes the complex conjugate of \(A\) (not the adjoint) with respect to a fixed basis (the inequality holds for all basis).

Proof: Recall Trotter's formula \(e^{A+B} = \lim_{n \to \infty} (e^{A/n} e^{B/n})^n\)

As before, we introduce fields to decouple the two Hilbert spaces. Here, proof for \(k=1\) (otherwise, use more fields).
3.4. \textbf{Permutation Partition Function with Fields and Gain}

Given a field $\mathbf{v} = (v_x)_{x \in A}$ with $v_x \in \mathbb{R}$, let

$$H(\mathbf{v}) = H_{0,0}^{(m)} - \sum_{x \in A} h_x S_x$$

where $h_x = \Delta x$. The partition function with field $\mathbf{v}$ is

$$\mathcal{Z}(\mathbf{v}) = \text{Tr}_{\mathcal{H}} e^{-\beta H(\mathbf{v})}$$

Let us also define

$$\mathcal{Z}'(\mathbf{v}) = \mathcal{Z}(\mathbf{v}) e^{\beta \Delta S(\mathbf{v})}$$

The property of “Gaussian domination” is

$$\mathcal{Z}(\mathbf{v}) \leq \mathcal{Z}(\mathbf{0}) e^{\frac{1}{2} \beta \Delta S(\mathbf{v})} \quad \Rightarrow \quad \mathcal{Z}'(\mathbf{v}) \leq \mathcal{Z}'(\mathbf{0})$$

As in the classical case, it is a consequence of the following lemma:

\textbf{Lemma} (RP of quantum Heisenberg model)

\textbf{If} $u \leq 0$, \textbf{then} for any reflection $R$ across edges,

$$\mathcal{Z}'(v, v') \leq \mathcal{Z}(v, v') \mathcal{Z}(Rv', v')$$
3.5. **GD IMPLIES IRB FOR DUHAMEL FUNCTION**

**Definition:** Duhamel correlation function:

\[
S_0^+ = \frac{1}{2\pi} \int_{\alpha} \nu \frac{\partial^2 Z(\nu, \omega)}{\partial \nu^2} \left( S_0^+, S_0^- \right) e^{-i \omega \nu} d\nu
\]

**Lemma:** (GD \Rightarrow IRB)

\[
\left( S_0^+, S_0^- \right) (k) \leq \frac{1}{2\beta |k|}
\]

**Proof:** Choose \( \nu = \eta \cos(kx) \) in GD inequality. We have

\[Z(\nu) = Z(0) + \frac{i}{2} \int_{\alpha} \left( \frac{\partial^2 Z(\nu)}{\partial \nu^2} \right) \left( S_{0-1}^+, S_{0-1}^- \right) e^{-i \omega \nu} d\nu
\]

(Recall that \( h = \Delta \nu = -\nu \cos(kx) \))

Also, \( Z(\nu) = Z(0, \omega, h) \underline{\omega} = \text{Tr} e^{-\beta \mathbf{H}_0} \).

We can check (using Duhamel's formula) that

\[
\frac{1}{Z(0, \omega, h)} \frac{\partial^2}{\partial \nu_x \partial \nu_y} \frac{Z(0, \omega, h)}{Z(0, \omega, h)} \left( S_{0-1}^+, S_{0-1}^- \right) = \beta^2 \left( S_{0-1}^+, S_{0-1}^- \right)
\]

Then

\[Z(\nu) = Z(0) + Z(0) \frac{i}{2} \eta^2 \epsilon^2 (k^2) \sum_{x, y} \beta^2 e^{i \omega \nu} e^{i (kx)} e^{i (ky)} \left( S_{0-1}^+, S_{0-1}^- \right) + O(\eta^2)
\]

Also, \( e^{-\beta \mathbf{H}_0} \underline{\omega} = e^{-\beta \mathbf{H}_0} \left( S_{0-1}^+, S_{0-1}^- \right) e^{i \omega \nu} \).

Looking at order \( \eta^2 \), we get the lemma. \( \Box \)
3.6 Duhamel Function and Falk-Brock Inequality

We need to transfer the IRE for the Duhamel correlation function to the usual corr. function. This can be done using Falk-Brock's inequality.

Let us consider a more general setting, which has the advantage to elucidate the notation.

Let if a finite-dimensional Hilbert space, and $H$ a Hermitian operator. The set of linear maps $H \to H$ is also a Hilbert space. On this space, we consider the Duhamel inner product

$$(A, B) = \frac{1}{2} \int_0^1 \text{Tr} e^{-\alpha H} A e^{\alpha H} B d\alpha$$

where $\mathcal{Z} = \text{Tr} e^{H}$. Exercise: check that it is inner product.

Useful identity:

$$\frac{d}{d\alpha} \text{Tr} e^{-\alpha H} A e^{\alpha H} B = \text{Tr} e^{-\alpha H} \left[ H, A e^{\alpha H} B \right]$$

Then, $(\Sigma A, H, B) = (A, \Sigma B, H)$

and $(A, \Sigma B, H) = \frac{1}{2} \int_0^1 \text{Tr} e^{-\alpha H} A e^{\alpha H} B d\alpha = \langle [B, A^\alpha] \rangle$

Here, $\langle \cdot \rangle = \frac{1}{2} \text{Tr} e^H$.

Let $F(\alpha) = \text{Tr} e^{-\alpha H} A e^{-\alpha H} A$,

$$\frac{d^2}{d\alpha^2} F(\alpha) = \text{Tr} e^{-\alpha H} [A, H^2] e^{-\alpha H} [A, H] \geq 0,$$

so $F$ is convex (it is actually log-convex, i.e., $F^{\alpha} F - (F')^2 \geq 0$).

$$\frac{1}{2} \langle A^2 A + A A^2 \rangle = \frac{1}{2} \left( F(0) + F(1) \right) \geq \frac{1}{2} \int_0^1 F(\alpha) d\alpha = (A, A)$$

(Identity, iff $[A, H] = 0$).

We will not use it, but let us notice that we get BogoIouBOB inequality from Cauchy-Schwarz:

$$1 \langle A, [B, H] \rangle^2 \leq \langle (A, A) \rangle \langle [B, H], [B, H] \rangle$$

$$\frac{1}{2} [B, A^2] \rangle^2 \leq \frac{1}{2} \langle [A^2, A^2] \rangle \langle B, [B, H], [B, H] \rangle$$

One of the inequalities above gives an upper bound for the Duhamel function, namely $(A, A) \leq \frac{1}{2} \langle [A^2, A] \rangle$. But we need a lower bound in order to extend the IRE bound to the usual correlation function. This is provided by Falk-Brock inequality (see also [305]).

Let $\Phi(s) = \int_0^s \cosh \frac{\alpha}{\beta} d\alpha$.

Exercise: check that

$\sqrt{s} \leq \Phi(s) \leq \sqrt{s} + s$.
Since $\mathbf{\Phi}$ is monotone increasing, the following is a lower bound for $(A,A)$:

**Lemma (Folk-Bruch inequality)**

$$2 \langle \mathbf{A}^{\mathbf{A}} + \mathbf{A}^{\mathbf{A}} \rangle \leq \mathbf{\Phi} \left( \frac{b_4 (A, A)}{2 (A, A) + \mathbf{A}^{\mathbf{A}} + \mathbf{A}^{\mathbf{A}}} \right)$$

**Proof:** Recall the function $F_\mathbf{A}$. The inequality can be written as

$$2 \frac{F_\mathbf{A}(\mathbf{A}) + F_\mathbf{A}(\mathbf{A})}{F_\mathbf{A}(\mathbf{A}) + F_\mathbf{A}(\mathbf{A})} \leq \mathbf{\Phi} \left( \frac{b_4 (A, A)}{2 (A, A) + \mathbf{A}^{\mathbf{A}} + \mathbf{A}^{\mathbf{A}}} \right)$$

If $\mathbf{V}$ is an orthonormal pair of eigenvalues of $\mathbf{H}$ with eigenvalues $\mathbf{V}$, we can write

$$F_\mathbf{A}(\mathbf{A}) = \sum_{\mathbf{V}} \frac{1}{2} (\mathbf{A}, \mathbf{A}_{\mathbf{V}}) e^{\mathbf{A}_{\mathbf{V}}} + (\mathbf{A}, 2 \mathbf{A}_{\mathbf{V}})$$

$$= \int_0^\infty e^{\mathbf{A}_t} d\mu(t),$$

where $\mu$ is a positive measure. We have

$$\mathbf{F}_\mathbf{A}(\mathbf{A}) = \sum_{\mathbf{V}} f_\mathbf{A}(\mathbf{V}) d\mu(\mathbf{V})$$

$$= \int_0^\infty f_\mathbf{A}(\mathbf{A}_t) d\mu(\mathbf{A}_t)$$

$$= \int_0^\infty f_\mathbf{A}(\mathbf{A}_t) d\mu(\mathbf{A}_t)$$

Define the measure $d\mu(t) = \left( \int f_\mathbf{A}(\mathbf{A}_t) d\mu(\mathbf{A}_t) \right)^{-1} f_\mathbf{A}(\mathbf{A}_t) d\mu(\mathbf{A}_t)$.

We have

$$\frac{\mathbf{F}_\mathbf{A}(\mathbf{A})}{\mathbf{F}_\mathbf{A}(\mathbf{A}) + \mathbf{F}_\mathbf{A}(\mathbf{A})} = \int_0^\infty \frac{1}{2} d\mu(t)$$

$$\mathbf{F}_\mathbf{A}(\mathbf{A}) = \int_0^\infty \frac{1}{2} c \mu(t) \frac{1}{2} d\mu(t)$$

The last step is to use Jensen's inequality (recall that $\mathbf{\Phi}$ is convex):

$$\mathbf{\Phi} \left( \frac{b_4 (A, A)}{F_\mathbf{A}(\mathbf{A}) + F_\mathbf{A}(\mathbf{A})} \right) = \mathbf{\Phi} \left( \int \frac{1}{2} d\mu(t) \right) \geq \int \mathbf{\Phi} \left( \frac{1}{2} \right) d\mu(t)$$

$$= \int \frac{1}{2} c \mu(t) \frac{1}{2} d\mu(t) = \frac{1}{2} \mathbf{F}_\mathbf{A}(\mathbf{A}) + \mathbf{F}_\mathbf{A}(\mathbf{A}).$$

In order to see that the Fourier transform of $(S^2, S^2)$ can be written in the form $(A, A)$, let us define the Fourier transform of operators: Let

$$\mathbf{S}^2_\mathbf{A} = \sum_{\mathbf{V}} \mathbf{V} \mathbf{V}^\dagger + \mathbf{V} \mathbf{V}^\dagger \mathbf{V} \mathbf{V}^\dagger, \quad \mathbf{V} \in \mathbf{A}^\dagger$$

$$\mathbf{S}^2_\mathbf{A} = \frac{1}{2} \sum_{\mathbf{V} \in \mathbf{A}^\dagger} \mathbf{V} \mathbf{V}^\dagger \mathbf{V} \mathbf{V}^\dagger, \quad \mathbf{V} \in \mathbf{A}^\dagger.$$

We can then check that

$$\mathbf{\Delta} \mathbf{S}^2_\mathbf{A} \mathbf{S}^2_\mathbf{A} \mathbf{S}^2_\mathbf{A} = \sum_{\mathbf{V} \in \mathbf{A}^\dagger} \mathbf{V} \mathbf{V}^\dagger \mathbf{V} \mathbf{V}^\dagger \mathbf{V} \mathbf{V}^\dagger = \frac{1}{2} \sum_{\mathbf{V} \in \mathbf{A}^\dagger} \mathbf{V} \mathbf{V}^\dagger \mathbf{V} \mathbf{V}^\dagger \mathbf{V} \mathbf{V}^\dagger = \frac{1}{2} \mathbf{S}^2_\mathbf{A} \mathbf{S}^2_\mathbf{A} \mathbf{S}^2_\mathbf{A}.$$

The same relation applies to other correlation functions, including Duhemal. Notice that $(S^2)^2 = S^4$, and that $(S^4, S^4)^\dagger = (S^4, S^4)^\dagger$. (Here, no $-k$ because of the definition of the Duhemal inner product).

We can now apply the Folk-Bruch inequality, and using $\mathbf{\Phi}(\mathbf{S}^2) \leq \mathbf{S}^2 + 5$, we get
Lemma (IRB: For usual correlation function)

\[
\langle \hat{S}_z \hat{S}_z \rangle (k) \leq \sqrt{E(k)} + \frac{1}{2} \beta \chi(k)
\]

where \( \chi(k) = \sqrt{E(k)} (1 - \text{sign}(k)) \langle \hat{S}_x \hat{S}_x + \text{sign}(k) \langle \hat{S}_z \hat{S}_z \rangle \rangle \).

As will be apparent in the proof of this lemma, \( \chi(k) \) can be written as \( \langle \hat{S}_x \hat{S}_x \rangle + \langle \hat{S}_z \hat{S}_z \rangle \).

When \( L = L_1 = \ldots = L_d \), we can check that \( \chi(k) = \langle \hat{S}_z \hat{S}_z \rangle \) and \( E(k) = - \langle \hat{S}_z \hat{S}_z \rangle E(k) \).

To prove this, use \( \hat{H}^{(0)} = \hat{H}^{(\text{spin})} \), and spin rotation invariance.

Proof: Let \( A = \hat{S}_z \) and \( H = \beta \hat{H}^{(\text{spin})} \). Let us calculate the double commutator in the Frohlich inequality,

\[
[A, [A, A]] = \sum_{k \in \Lambda} e^{ik \cdot \mathbf{k}} [A, A]
\]

\[
= -2 \beta \sum_{k \in \Lambda} e^{ik \cdot \mathbf{k}} \left[ \hat{S}_x \hat{S}_x + u \hat{S}_y \hat{S}_y + u \hat{S}_x \hat{S}_z, \hat{S}_z \right]
\]

\[
= -2 \beta \sum_{k \in \Lambda} e^{ik \cdot \mathbf{k}} \left[ -i \hat{S}_z \hat{S}_y + u \hat{S}_x \hat{S}_z \right]
\]

\[
[A, [A, A]] = 2 \beta \sum_{k \in \Lambda} e^{ik \cdot \mathbf{k}} \left[ -i \hat{S}_z \hat{S}_y + u \hat{S}_x \hat{S}_z \right]
\]

\[
= 4 \beta \sum_{k \in \Lambda} \left( (1 - \text{sign}(k)) \hat{S}_x \hat{S}_z + (u - \text{sign}(k)) \hat{S}_x \hat{S}_z \right)
\]

\[
= -37
\]

We have found \( \langle [A, [A, A]] \rangle = 4 \beta \ln E(k) \). The lemma now follows. From

\[
4 \beta \ln E(k) \leq \varphi(\frac{\ln (\hat{S}_z \hat{S}_z)}{4 \beta \ln E(k)}) \Rightarrow \frac{\ln (\hat{S}_z \hat{S}_z)}{4 \beta \ln E(k)} \leq \varphi^{-1}(4 \beta \ln E(k))
\]

and from \( \varphi(x) \leq \sqrt{x + 1} \).

We finally obtain the lower bound of the theorem of p. 26.

\[
\frac{1}{2} \hat{S}(\mathbf{k})^2 \leq \langle \hat{S}_z \hat{S}_z \rangle = \frac{1}{4 \pi} \sum_{\mathbf{k} \in \Lambda} \frac{\langle \hat{S}_z \hat{S}_z \rangle (\mathbf{k})}{E(k)}
\]

Then \( \frac{1}{4 \pi} \sum_{\mathbf{k} \in \Lambda} \frac{\langle \hat{S}_z \hat{S}_z \rangle (\mathbf{k})}{E(k)} \) (0) = \( \frac{1}{2} \hat{S}(\mathbf{k})^2 \) - \( \frac{1}{4 \pi} \sum_{\mathbf{k} \in \Lambda} \langle \hat{S}_z \hat{S}_z \rangle (\mathbf{k}) \)

Using the IRB of the previous lemma, the theorem follows.