QUANTUM SPIN SYSTEMS & PHASE TRANSITIONS

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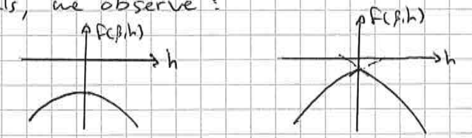
LECTURE 1 - BACKGROUND & SETTING

1. FERROMAGNETISM

The discovery of magnetic materials predates the invention of writing. On the level of thermodynamics, a ferromagnet, or magnet, is described by its free energy $f(\beta,h)$, which is a real function of the inverse temperature β and of the external magnetic field h (in some chosen direction). Such a function exists by the 2nd law of thermodynamics.
Important derived quantities are

- the magnetization $m(\beta,h) = -\frac{\partial f}{\partial h}$
- the magnetic susceptibility $\chi(\beta,h) = -\frac{\partial^2 f}{\partial h^2}$.

The function $f(\beta,h)$ is concave with respect to (β,h) and $f(\beta,-h) = f(\beta,h)$. It is piecewise analytic and phase transitions are signalled by points of non-analyticity. The ferromagnetic phase is characterized by the ability of the material to retain magnetization at $h=0$. Notice that $m(\beta,0) = 0$ whenever f is differentiable. For ferromagnetic materials, we observe:



$\beta < \beta_c$ $\qquad$ $\beta > \beta_c$

Here, $\frac{1}{\beta_c} = T_c$ is Curie temperature, and $-D_+ f(\beta,0)$, the derivative of $f(\beta,h)$ at $h=0+$, is the spontaneous magnetization. Ferromagnetic materials include Fe, Co, Ni, Gd, Dy, and composite substances.

A major challenge in theoretical and mathematical physics is to understand how magnetic properties emerge in large systems of quantum particles, that are subjected to the laws of quantum mechanics.

2. QUANTUM SPIN MODELS FOR CONDENSED MATTER SYSTEMS

The physical systems consist of atoms and electrons subjected to Coulomb interactions, whose evolution satisfies the Schrödinger equation. One expects, observes in fact, that atoms form regular structures. But this is much too complicated to describe and we start with a simplified setting. We assume that atoms remain at the sites of a regular lattice and that they host localized electrons which carry spins. The state of the system involves therefore a (quantum) spin at each site of the lattice.

Let $G = (\Lambda, \mathcal{E})$ denote a (finite) graph, with Λ the set of vertices and $\mathcal{E}$ the set of edges. For instance, $\Lambda = \{1, \dots, L\}^d$ and $\mathcal{E}$ is the set of nearest-neighbors; but more general graphs can be considered.

Let $S \in \frac{1}{2}\mathbb{N}$. The state space for the model of spin S is the Hilbert space $\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathbb{C}^{2S+1}$. On each local Hilbert space $\mathbb{C}^{2S+1}$, we consider the usual spin operators that satisfy the following properties:

- S^1, S^2, S^3 are hermitian
- $(S^1)^2 + (S^2)^2 + (S^3)^2 = S(S+1)\,\mathrm{Id}$
- $[S^1, S^2] = iS^3, \quad [S^2, S^3] = iS^1, \quad [S^3, S^1] = iS^2.$

<u>Proposition 1.1</u>

(a) For every $S \in \frac{1}{2}\mathbb{N}$, these operators exist.

(b) Each S^i has eigenvalues $\{-S, -S+1, \dots, S\}$.

The proof of (a) can be done by construction. Let $|a\rangle$, $a \in \{-S, \dots, S\}$, denote the elements of an orthonormal basis of $\mathbb{C}^{2S+1}$, and define S^3, S^+, S^- by

$$S^3 |a\rangle = a|a\rangle$$

$$S^+|a\rangle = \begin{cases} \sqrt{S(S+1) - a(a+1)}\ |a+1\rangle & \text{if } a \neq S \\ 0 & \text{if } a = S \end{cases}$$

$$S^-|a\rangle = \begin{cases} \sqrt{S(S+1) - (a-1)a}\ |a-1\rangle & \text{if } a \neq -S \\ 0 & \text{if } a = -S \end{cases}$$

Next, set $S^1 = \frac{1}{2}(S^+ + S^-)$ and $S^2 = \frac{1}{2i}(S^+ - S^-)$. When $S = \frac{1}{2}$ we get the usual Pauli matrices

$$S^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S^2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad S^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

When $S = 1$ we get

$$S^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad S^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad S^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

<u>Exercise 1.</u> Prove Proposition 1.1.

<u>Exercise 2.</u> Let $S^\pm = S^1 \pm iS^2$. Show that $\|S^\pm\| = \sqrt{2}\,S$.

[Hint: Write $\|S^+\|^2 = \frac{1}{4}(2\|S^+\|^2 + 2\|S^-\|^2)$ and use the parallelogram identity.]

Spin operators are related to rotations in $\mathbb{R}^3$. Let $\vec{S} = (S^1, S^2, S^3)$. Given $\vec{a} \in \mathbb{R}^3$, let

$$S^{\vec{a}} = \vec{a} \cdot \vec{S} = a_1 S^1 + a_2 S^2 + a_3 S^3.$$

By linearity, the spin commutation relations become $[S^{\vec{a}}, S^{\vec{b}}] = iS^{\vec{a} \times \vec{b}}$.

Let $R_{\vec{a}}\vec{b}$ denote the vector $\vec{b}$ rotated around $\vec{a}$ by the angle $\|\vec{a}\|$.

Proposition 1.2.

(a) $e^{-iS^{\vec a}} S^{\vec b} e^{iS^{\vec a}} = S^{R_{\vec a}\vec b}$.

(b) $e^{-iaS^3} S^+ e^{iaS^3} = e^{-ia} S^+$, $\quad e^{-iaS^3} S^- e^{iaS^3} = e^{ia} S^-$.

(c) If $\psi_{\vec b, c}$ is eigenvector of $S^{\vec b}$ with eigenvalue c, then $e^{-iS^{\vec a}} \psi_{\vec b, c}$ is eigenvector of $S^{R_{\vec a}\vec b}$ with eigenvalue c.

Exercise 3: Prove Proposition 1.2.

Back to $\mathcal{H}_\Lambda$. We let S_x^i denote the spin operator at site x,

$$S_x^i = S^i \otimes \mathrm{Id}_{\Lambda \setminus \{x\}}.$$

The most natural Hamiltonians are the Heisenberg ferromagnet and antiferromagnet:

$$H_{\Lambda,h}^{\text{ferro}} = -\sum_{\{x,y\}\in\mathcal{E}} \vec S_x \cdot \vec S_y - h \sum_{x\in\Lambda} S_x^3$$

$$H_{\Lambda,h}^{\text{anti}} = +\sum_{\{x,y\}\in\mathcal{E}} \vec S_x \cdot \vec S_y - h \sum_{x\in\Lambda} S_x^3.$$

Here, the notation $\vec S_x \cdot \vec S_y$ means the operator in $\mathcal{H}_x \otimes \mathcal{H}_y$ given by $S_x^1 S_y^1 + S_x^2 S_y^2 + S_x^3 S_y^3$. It is hermitian. Its spectrum can be obtained using results about addition of spins.

Proposition 1.3

(a) The eigenvalues of $(\vec S_x + \vec S_y)^2$ are $J(J+1)$, where $J = 0, 1, \dots, 2S$. The multiplicity of J is $2J+1$.

(b) $[S_x^i + S_y^i, (\vec S_x + \vec S_y)^2] = 0$, $i = 1,2,3$, and the eigenvalues of $S_x^i + S_y^i$ in the sector J are $-J, -J+1, \dots, J$.

Exercise 4. Prove Proposition 1.3.

When $S = \frac12$, we can check that the eigenvalues of $\vec S_x \cdot \vec S_y$ are $-\frac34$ (multiplicity 1) and $\frac14$ (multiplicity 3).

Let us discuss the motivation for the Heisenberg Hamiltonians with spin $S = \frac12$. In $\mathcal{H}_x \otimes \mathcal{H}_y$, consider the symmetry group of rotations, represented by operators $U^{\vec a} = e^{iS_x^{\vec a} + iS_y^{\vec a}}$, $\vec a \in \mathbb{R}^3$. One can check that the group yields the irreducible decomposition $\mathcal{H}_x \otimes \mathcal{H}_y = \text{singlet} \oplus \text{triplet}$, where singlet is the subspace spanned by $\frac{1}{\sqrt2}(|\frac12, -\frac12\rangle - |-\frac12, \frac12\rangle)$, and triplet is the subspace spanned by $|\frac12, \frac12\rangle$, $|-\frac12, -\frac12\rangle$, and $\frac{1}{\sqrt2}(|\frac12, -\frac12\rangle + |-\frac12, \frac12\rangle)$. (These correspond to $J=0$ and $J=1$ in Proposition 1.3.)

The interaction operator should be rotation invariant, so it must be of the form $c_1 P_{\text{singlet}} + c_2 P_{\text{triplet}}$. Up to constants and a shift by the identity operator, we get $\pm \vec S_x \cdot \vec S_y$. This suggests $\pm \sum_{\{x,y\}\in\mathcal{E}} \vec S_x \cdot \vec S_y$.

In these notes, we consider the following Hamiltonian:

$$\boxed{H^{(u)}_{\Lambda,h} = -2 \sum_{\{x,y\}\in\mathcal{E}} \left(S^1_x S^1_y + u S^2_x S^2_y + S^3_x S^3_y \right) - h \sum_{x\in\Lambda} S^3_x}$$

The most interesting cases are

- $u = +1$: Heisenberg ferromagnet
- $u = 0$: XY model (quantum rotators)
- $u = -1$: $(h=0)$ If Λ is bipartite (that is, Λ is the disjoint union of Λ_A and Λ_B, and every edge of $\mathcal{E}$ includes one site of Λ_A and one site of Λ_B), then

$$U^{-1} H^{(u)}_{\Lambda,0}\, U = 2 \sum (S^1_x S^1_y - u S^2_x S^2_y + S^3_x S^3_y) = H^{\text{anti}}_{\Lambda}$$

with $U = \prod_{x\in\Lambda_B} e^{i\pi S^2_x}$.

3. THERMODYNAMIC LIMIT & PHASE TRANSITIONS

The passage from the microscopic description of the system (quantum mechanics) to the macroscopic description (thermodynamics) can be done using the following formula for the free energy:

$$f_\Lambda(\beta,h) = -\frac{1}{\beta|\Lambda|} \log \operatorname{Tr} e^{-\beta H^{(u)}_{\Lambda,h}}$$

where the trace is taken in $\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathbb{C}^{2S+1}$. The justification

for this formula lies at the heart of statistical physics and it is not entirely settled. It can be shown that this free energy is a Legendre transform of the Boltzmann entropy, which is easier to comprehend heuristically.

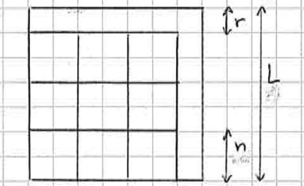
Phase transitions are related to non-analyticity of thermodynamic potentials (such as the free energy), but the function f_Λ is clearly real analytic in β, h. We are interested in large systems, and the limiting function $f = \lim_{|\Lambda|\to\infty} f_\Lambda$ may no longer be analytic. But are we allowed, physically, to take this limit? This was hotly contested in van der Waals' centenary conference, held in Amsterdam in 1937. A vote was eventually taken, resulting in a narrow victory for the thermodynamic limit. So yes, we can take it.

Can we take it mathematically? Yes, this was proved in the early days of mathematical statistical mechanics by Ruelle and Fisher. The next theorem is only a special case, but it is enough for our purpose. Existence of the thermodynamic limit is a fundamental result of this field.

Theorem 1.4

Let $\Lambda_L = \{1,\dots,L\}^d$ and $\mathcal{E}_L$ be the set of nearest-neighbors in Λ_L. Then $f_{\Lambda_L}(\beta,h)$ converges as $L\to\infty$ to a function $f(\beta,h)$. Convergence is uniform on compact sets.

Proof: Subadditive argument.



$L = kn + r, \quad 0 \leq r \leq n$

The box Λ_L is partitioned as shown above. Let

$$h_{xy} = -2\left(S_x^1 S_y^1 + u S_x^2 S_y^2 + S_x^3 S_y^3\right) + C$$

where C is a positive constant so that h_{xy} is positive-definite. Recall that $\mathrm{Tr}\, e^{A+B} \geq \mathrm{Tr}\, e^{A}$ when A, B are hermitian matrices with $B \geq 0$ (this is a consequence of the minimax principle). Then

$$\mathrm{Tr}\, e^{-\beta H^{(u)}_{\Lambda_L,h}} = e^{-\beta C|\mathcal{E}_L|}\, \mathrm{Tr}\, e^{\beta \sum_{\{x,y\}\in\mathcal{E}_L} h_{xy} + \beta h \sum_{x\in\Lambda_L} S_x^3}$$

$$\geq e^{-\beta C|\mathcal{E}_L|} \Big[\mathrm{Tr}_{\mathcal{H}_n}\, e^{\beta \sum_{\{x,y\}\in\mathcal{E}_n} h_{xy} + \beta h \sum_{x\in\Lambda_n} S_x^3} \Big]^{k^d}$$

$$= \Big[\mathrm{Tr}\, e^{-\beta H^{(u)}_{\Lambda_n,h}} \Big]^{k^d} e^{-\beta C|\mathcal{E}_L|}\, e^{k^d \beta C |\mathcal{E}_n|}.$$

(We used $\mathrm{Tr}\, e^{\beta h S_x^3} \geq 1$ for the sites in the slabs of width r.)

Next, we use

$$|\mathcal{E}_L| \leq k^d |\mathcal{E}_n| + k^d d\, n^{d-1} + d^2 r L^{d-1}.$$

Then we essentially obtain a subadditive inequality:

$$f_{\Lambda_L}(\beta,h) \leq \frac{(kn)^d}{L^d} f_{\Lambda_n}(\beta,h) + \frac{k^d d\, n^{d-1}}{L^d} C + \frac{d^2 r}{L} C.$$

We take $L \to \infty$; since $\frac{kn}{L} \to 1$, we get

$$\limsup_{L\to\infty} f_{\Lambda_L}(\beta,h) \leq f_{\Lambda_n}(\beta,h) + \frac{d}{n} C.$$

We now take $n \to \infty$ and we get

$$\limsup_{L\to\infty} f_{\Lambda_L}(\beta,h) \leq \liminf_{n\to\infty} f_{\Lambda_n}(\beta,h).$$

The limit exists.

It follows from Exercise 5 below that the family (f_{Λ_L}) is equicontinuous, so that uniform convergence on compact sets is a consequence of the Arzelá–Ascoli theorem. $\square$

Exercise 5. Assume that $\frac{|\mathcal{E}|}{|\Lambda|} \leq C$, and show that

$$|\beta f_\Lambda(\beta,h) - \beta' f_\Lambda(\beta',h')| \leq \mathrm{const}\,(|\beta-\beta'| + |h-h'|),$$

where the constant depends on C, d, S only. In particular, the constant does not depend on the size of the graph.

Lemma 1.5.
The function $\beta f_\Lambda(\beta,h)$ is concave in $(\beta, \beta h)$.

Exercise 6. Prove Lemma 1.5, e.g. by showing that the matrix $\begin{pmatrix} -\frac{\partial^2 \beta f}{\partial \beta^2} & -\frac{\partial^2 \beta f}{\partial\beta\, \partial(\beta h)} \\ -\frac{\partial^2 \beta f}{\partial\beta\, \partial(\beta h)} & -\frac{\partial^2 \beta f}{\partial(\beta h)^2} \end{pmatrix}$ is positive definite.

A state in quantum statistical mechanics is a nonnegative normalized linear functional on the space of operators in $\mathcal{H}_\Lambda$. The equilibrium states are the Gibbs states

$$\langle A \rangle_{\Lambda,\beta,h} = \frac{1}{Z(\Lambda,\beta,h)} \operatorname{Tr} A \, e^{-\beta H^{(u)}_{\Lambda,h}}$$

where A is an operator in $\mathcal{H}_\Lambda$, and the normalization is the partition function

$$Z(\Lambda,\beta,h) = \operatorname{Tr} e^{-\beta H^{(u)}_{\Lambda,h}} .$$

Next, we discuss ferromagnetic phase transitions. There are several natural definitions and it may help to consider them at the same time and to clarify their relations. Let us introduce the magnetization operator in the 3rd direction of spin,

$$M_\Lambda = \sum_{x \in \Lambda} S^3_x .$$

The three natural order parameters for ferromagnetism are

- Thermodynamic magnetization, equal to the (negative) of the right-derivative of the free energy at $h=0$:

$$m^*_{th}(\beta) = - \lim_{h \to 0+} \frac{f(\beta,h) - f(\beta,0)}{h} .$$

- Residual magnetization: Place a magnet inside a magnetic field, which is gradually removed. Is the material still magnetized? The corresponding order parameter is

$$m^*_{res}(\beta) = \lim_{h \to 0+} \liminf_{L \to \infty} \frac{1}{L^d} \langle M_{\Lambda_L} \rangle_{\Lambda_L,\beta,h} .$$

- Spontaneous magnetization. Since $\langle M_\Lambda \rangle = 0$ when $h=0$, we rather consider

$$m^*_{sp}(\beta) = \liminf_{L \to \infty} \frac{1}{L^d} \langle |M_{\Lambda_L}| \rangle_{\Lambda_L,\beta,0} .$$

Proposition 1.6

$$m^*_{th}(\beta) = m^*_{res}(\beta) \geq m^*_{sp}(\beta) .$$

Proof of $m^*_{th} = m^*_{res}$: Suppose that (f_n) is a sequence of concave, differentiable functions that converges pointwise to the function f. We show that

$$D_+ f(0) = \lim_{h \to 0+} \limsup_n f_n'(h) = \lim_{h \to 0+} \liminf_n f_n'(h) .$$

The result follows, since $D_+ f(0) = -m^*_{th}$ and $\lim_{h \to 0+} \liminf_n f_n'(h) = -m^*_{res}$.

Recall that $\limsup_m (\inf_n a_{mn}) \leq \inf_n (\limsup_m a_{mn})$,

$$\liminf_m (\sup_n a_{mn}) \geq \sup_n (\liminf_m a_{mn}) .$$

Also, $D_- f(h) = \inf_{s>0} \frac{f(h) - f(h-s)}{s}$ and $D_+ f(h) = \sup_{s>0} \frac{f(h+s) - f(h)}{s}$ when ~~since~~ f is concave.

For $h > 0$,

$$D_+ f(0) \geq D_- f(h) = \inf_{s>0} \limsup_n \frac{f_n(h) - f_n(h-s)}{s}$$

$$\geq \limsup_n f_n'(h) \geq \liminf_n f_n'(h)$$

$$\geq \sup_{s>0} \liminf_n \frac{f_n(h+s) - f_n(h)}{s} = D_+ f(h).$$

Right-derivatives of concave functions are right-continuous, so $D_+ f(h) \to D_+ f(0)$ as $h \to 0+$. $\square$

<u>Proof of $m^*_{res} \geq m^*_{sp}$</u> : Let $\{\varphi_j\}$ be an orthonormal basis of eigenvectors of $H^{(a)}_{\Lambda,0}$ and M_Λ with eigenvalues e_j and m_j, respectively. Since $H^{(a)}_{\Lambda,h} = H^{(a)}_{\Lambda,0} - hM_\Lambda$ and $[H^{(a)}_{\Lambda,0}, M_\Lambda] = 0$, the eigenvalues of $H^{(a)}_{\Lambda,h}$ are $e_j - hm_j$. Because of the spin-flip symmetry, the set $\{m_j\}$ is symmetric around 0 (with multiplicity). Let $h > 0$. We have

$$0 \leq \langle |M_{\Lambda_L}| \rangle_{\Lambda_L,\beta,h} - \langle M_{\Lambda_L} \rangle_{\Lambda_L,\beta,h} = \frac{2 \sum_{j:m_j>0} m_j \, e^{-\beta e_j - \beta h m_j}}{\sum_{j:m_j>0} e^{-\beta e_j}(e^{\beta h m_j} + e^{-\beta h m_j}) + \sum_{j:m_j=0} e^{-\beta e_j}}$$

$$\leq 2 \frac{\sum_{j:m_j>0} m_j \, e^{-\beta e_j - \beta h m_j}}{\sum_{j:m_j>0} e^{-\beta e_j + \beta h m_j}}.$$

After division by L^d, there remain only those j with $m_j \sim L^d$, which give vanishing contribution because of $e^{-\beta h m_j}$. This shows that m^*_{res} is also equal to

$$m^*_{res}(\beta) = \lim_{h\to 0+} \liminf_{L\to\infty} \frac{1}{L^d} \langle |M_{\Lambda_L}| \rangle_{\Lambda_L,\beta,h}.$$

The result follows from

$$\langle |M_\Lambda| \rangle_{\Lambda,\beta,h} = \frac{\partial}{\partial h} \underbrace{\frac{1}{\beta} \log \operatorname{Tr} e^{-\beta H^{(a)}_{\Lambda,0} + \beta h |M_\Lambda|}}_{\text{convex with respect to } h} \geq \langle |M_\Lambda| \rangle_{\Lambda,\beta,0}$$

$\square$

<u>Exercise 7.</u> (a) Show that $[H^{(a)}_{\Lambda,0}, M_\Lambda] = 0$.

(b) Show that the eigenvalues of M_Λ are symmetric around 0.

Another order parameter is the expectation of M_Λ^2, which is convenient because it can be expressed in term of the two-point correlation function (see below). It is equivalent to m^*_{sp} in the sense that both are zero or both differ from zero.

<u>Lemma 1.7.</u>

$$\left\langle \frac{|M_\Lambda|}{|\Lambda|} \right\rangle^2_{\Lambda,\beta,0} \leq \left\langle \left(\frac{M_\Lambda}{|\Lambda|}\right)^2 \right\rangle_{\Lambda,\beta,0} \leq S \left\langle \frac{|M_\Lambda|}{|\Lambda|} \right\rangle_{\Lambda,\beta,0}$$

<u>Proof</u>: For the first inequality, use $|M_\Lambda| = |M_\Lambda| \, \mathrm{Id}$ and then the following Cauchy-Schwarz inequality:

$$|\langle A^* B \rangle|^2 \leq \langle A^* A \rangle \langle B^* B \rangle$$

where $\langle \cdot \rangle$ denote any Gibbs state $\operatorname{Tr} \cdot e^{-H}$, and A, B are any operators that commute with H. For the second inequality, observe that $|M_\Lambda| \leq S|\Lambda| \, \mathrm{Id}$ implies $M_\Lambda^2 \leq S|\Lambda| \, |M_\Lambda|$, and use the fact that the Gibbs state is a positive linear functional. $\square$

We conclude this chapter by discussing correlation functions. The most common are the spin-spin correlation functions, of the form $\langle S_x^i S_y^i \rangle_{\Lambda,\beta,h}$. Notice that

$$\Big\langle \Big(\frac{M_\Lambda}{|\Lambda|} \Big)^2 \Big\rangle_{\Lambda,\beta,0} = \frac{1}{|\Lambda|^2} \sum_{x,y\in\Lambda} \langle S_x^3 S_y^3 \rangle_{\Lambda,\beta,0} .$$

For the order parameter to differ from zero, one needs that $\langle S_x^3 S_y^3 \rangle_{\Lambda,\beta,0}$ does not decay to 0 as x and y become very distant.

One expects that correlations increase when β or the interaction parameters increase. Nobody has managed to prove such correlation inequalities for quantum systems, though. But one can show that correlations are stronger in directions where interactions are stronger.

Proposition 1.8

Let $H = -\sum_{x,y\in\Lambda} \big(J_{xy}^1 S_x^1 S_y^1 + J_{xy}^2 S_x^2 S_y^2 + J_{xy}^3 S_x^3 S_y^3 \big)$ where J_{xy}^i are real numbers. Assume that for all $x,y\in\Lambda$, we have

$$|J_{xy}^2| \leq J_{xy}^1 .$$

Then for all $x,y\in\Lambda$, we have

$$|\langle S_x^2 S_y^2 \rangle| \leq \langle S_x^1 S_y^1 \rangle .$$

Further inequalities can be generated using symmetries.

Proof: Consider a basis of $\mathbb{C}^{2S+1}$ where S^3 is diagonal and where the matrix elements of S^1, S^+, S^- are all nonnegative. Also, the matrix elements of S^2 are less or equal to those of S^1 in absolute value. Such a basis exists. Using the Trotter formula and multiple resolutions of the identity, we have

$$\begin{aligned} |\operatorname{Tr} S_x^2 S_y^2 e^{-\beta H}| \leq \lim_{n\to\infty} \sum_{\sigma_0,\dots,\sigma_n \in \{-S,\dots,S\}^\Lambda} \Big| & \langle \sigma_0 | S_x^2 S_y^2 | \sigma_1 \rangle \\ & \langle \sigma_1 | e^{\frac{\beta}{n} \sum J_{uv}^3 S_u^3 S_v^3} | \sigma_1 \rangle \langle \sigma_1 | \Big(1 + \frac{\beta}{n} \sum_{u,v\in\Lambda} (J_{uv}^1 S_u^1 S_v^1 + J_{uv}^2 S_u^2 S_v^2) \Big) | \sigma_2 \rangle \\ & \dots \langle \sigma_n | e^{\frac{\beta}{n} \sum J_{uv}^3 S_u^3 S_v^3} | \sigma_n \rangle \langle \sigma_n | \Big(1 + \frac{\beta}{n} \sum_{u,v\in\Lambda} (J_{uv}^1 S_u^1 S_v^1 + J_{uv}^2 S_u^2 S_v^2) \Big) | \sigma_0 \rangle \Big| . \end{aligned}$$

Observe that the matrix elements of all operators are nonnegative, except for $S_x^2 S_y^2$. Indeed, this follows from

$$\begin{aligned} J_{uv}^1 S_u^1 S_v^1 + J_{uv}^2 S_u^2 S_v^2 = & \tfrac14 (J_{uv}^1 - J_{uv}^2) (S_u^+ S_v^+ + S_u^- S_v^-) \\ & + \tfrac14 (J_{uv}^1 + J_{uv}^2) (S_u^+ S_v^- + S_u^- S_v^+) . \end{aligned}$$

We get an upper bound in the right side of the big equation above by replacing $|\langle \sigma_0 | S_x^2 S_y^2 | \sigma_1 \rangle|$ with $\langle \sigma_0 | S_x^1 S_y^1 | \sigma_1 \rangle$. We have obtained

$$|\operatorname{Tr} S_x^2 S_y^2 e^{-\beta H}| \leq \operatorname{Tr} S_x^1 S_y^1 e^{-\beta H} .$$

□

LECTURE 2 - DECAY OF CORRELATIONS IN 2D MODELS & LONG-RANGE ORDER IN 3D CLASSICAL HEISENBERG MODEL

This chapter has two distinct parts. The first part deals with the decay of two-point correlation functions in systems with continuous symmetry and $d=2$. This is usually referred to as "the" Mermin-Wagner theorem. In the second part we review the proof of Fröhlich, Simon, Spencer that the classical Heisenberg model has ferromagnetic long-range order at sufficiently low temperature, in dimensions $d \geq 3$. It introduces the method of reflection-positivity and infrared bounds. The extension to the quantum model is discussed in the next chapter.

1. DECAY OF CORRELATIONS IN 2D MODELS

The theorem of Mermin and Wagner dealt with the absence of spontaneous magnetization and it was proposed in 1966. Decay of correlations go back to Fisher and Jasnow (1971); the decay was quite weak, $(\log d(0,x))^{-\frac{1}{2}}$, but algebraic was proved in the Hubbard model by Koma and Tasaki (1992). It extends the method of "complex rotations" of McBryan and Spencer (1977) to quantum systems, and it applies to Heisenberg models as well. The present proof is simpler and it was communicated to me by Jürg Fröhlich.

The decay will be of the form $e^{-\xi_\beta(x)}$ with the function $\xi_\beta(x) : \Lambda \to \mathbb{R}$ defined by

$$\xi_\beta(x) = \sup_{\substack{(\phi_y) \in \mathbb{R}^\Lambda \\ \phi_x = 0}} \Big[\phi_0 - 2\beta S^2 \sum_{\{y,z\} \in \mathcal{E}} (\cosh(\phi_y - \phi_z) - 1) \Big]$$

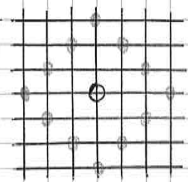
Let $d(x,y)$ denote the graph distance on $(\Lambda, \mathcal{E})$, i.e. the length of the smallest connected path between x and y. One can check that $\xi_\beta(x) \sim \frac{1}{\beta} \log(d(0,x)+1)$ in 2D-like graphs, see Exercise 8, so that $e^{-\xi_\beta(x)} \sim (d(0,x)+1)^{-\frac{\text{const}}{\beta}}$. (Notice that we often assume that Λ contains the origin 0.)

<u>Exercise 8</u>: Assume that $(\Lambda, \mathcal{E})$ is "2D-like", i.e. there exists a constant K such that $\forall \ell \geq 1$,

$$|\{x \in \Lambda : d(0,x) = \ell\}| \leq K\ell.$$

Show that there exists $C = C(\beta, S, K)$, that does not depend on x, such that

$$\xi_\beta(x) \geq \frac{1}{16\beta S^2 K} \log\big(d(0,x)+1\big) - C.$$



There are 4ℓ sites at distance ℓ of origin

Hint: Choose $\phi_y = \begin{cases} c \log \frac{d(0,x)+1}{d(0,y)+1} & \text{if } d(0,y) \leq d(0,x) \\ 0 & \text{otherwise} \end{cases}$ with suitable c.

Here is our version of the Mermin-Wagner theorem.

Theorem 2.1

For $i = 1,2,3$, we have the bound

$$|\langle S_0^i S_x^i \rangle_{\Lambda,\beta,0}| \leq 2S^2 e^{-\xi_\beta(x)}.$$

<u>Proof:</u> We use the method of complex rotations. It is convenient to exchange spin indices and to consider the Hamiltonian with interactions $-2(S_x^1 S_y^1 + S_x^2 S_y^2 + u S_x^3 S_y^3)$. Recall that $S_y^\pm = S_y^1 \pm i S_y^2$. One can check that for any $a \in \mathbb{C}$, we have

$$e^{a S_y^3} S_y^\pm e^{-a S_y^3} = e^{\pm a} S_y^\pm.$$

The Hamiltonian can be rewritten as

$$H_\Lambda = - \sum_{\{y,z\} \in \mathcal{E}} \left(S_y^+ S_z^- + S_y^- S_z^+ + 2u S_y^3 S_z^3 \right).$$

For given numbers $\{\phi_y\}$, let

$$A = \prod_{y \in \Lambda} e^{\phi_y S_y^3}.$$

Then

$$\operatorname{Tr} S_0^+ S_x^- e^{-\beta H_\Lambda} = \operatorname{Tr} A S_0^+ S_x^- A^{-1} e^{-\beta A H_\Lambda A^{-1}}.$$

Under the effects of "complex rotations", the Hamiltonian becomes

$$\begin{aligned} A H_\Lambda A^{-1} &= - \sum_{\{y,z\} \in \mathcal{E}} \left(e^{\phi_y - \phi_z} S_y^+ S_z^- + e^{-\phi_y + \phi_z} S_y^- S_z^+ + 2u S_y^3 S_z^3 \right) \\ &= H_\Lambda - \sum_{\{y,z\} \in \mathcal{E}} \Big([\cosh(\phi_y - \phi_z) - 1] [S_y^+ S_z^- + S_y^- S_z^+] \\ &\qquad - \sinh(\phi_y - \phi_z) [S_y^+ S_z^- - S_y^- S_z^+] \Big) \\ &\eqqcolon H_\Lambda + B + C. \end{aligned}$$

We used $e^x = \cosh x + \sinh x$. Notice that $B^* = B$ and $C^* = -C$. We obtain

$$\operatorname{Tr} S_0^+ S_x^- e^{-\beta H_\Lambda} = e^{\phi_0 - \phi_x} \operatorname{Tr} S_0^+ S_x^- e^{-\beta H_\Lambda - \beta B - \beta C}.$$

We now estimate the latter trace, using Trotter product formula and Hölder's inequality for traces: If $p \in [1, \infty]$, the norm of the matrix B is $\|B\|_p = (\operatorname{Tr} |B|^p)^{1/p}$ (and $\|B\|_\infty = \|B\|$, the usual operator norm. Hölder's inequality states that, for

$1 \le p,q,r \le \infty$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$, we have

$$\|AB\|_r \le \|A\|_p \|B\|_q,$$

where A,B are any matrices. (The proof is not easy, more complicated than in the case of functions, and is left as an exercise.)

We get

$$\mathrm{Tr}\, S_0^+ S_x^- e^{-\beta H_1 - \beta B - \beta C} = \lim_{N\to\infty} \mathrm{Tr}\, S_0^+ S_x^- \left(e^{-\frac{\beta}{N}H_1} e^{-\frac{\beta}{N}B} e^{-\frac{\beta}{N}C} \right)^N$$

$$\le \lim_{N\to\infty} \underbrace{\|S_0^+ S_x^-\|_\infty}_{2S^2} \underbrace{\|e^{-\frac{\beta}{N}H_1}\|_N^N}_{\mathrm{Tr}\, e^{-\beta H_1}} \underbrace{\|e^{-\frac{\beta}{N}B}\|_\infty^N}_{\le e^{\beta\|B\|}} \underbrace{\|e^{-\frac{\beta}{N}C}\|_\infty^N}_{1}$$

The theorem follows from

$$\|B\| \le 4S^2 \sum_{\{y,z\}\in\mathcal{E}} [\cosh(\phi_y-\phi_z)-1]. \qquad \square$$

The Mermin-Wagner theorem states that 2D systems with continuous symmetry do not display long-range order, at any positive temperature. The system may possess long-range in the ground state, though. And it may undergo a Kosterlitz-Thouless phase transition, where correlation functions go from exponential decay at high temperatures, to algebraic decay at low temperatures. Such a behavior is expected for the model $H^{(u)}_{1,0}$ when $-1<u<1$, but not when $|u|\ge 1$. The one result about a Kosterlitz-Thouless transition is due to Fröhlich & Spencer (1981) and concerns the classical XY model.

Exercise 9: Prove Trotter's formula,

$$e^{A+B} = \lim_{n\to\infty} \left(e^{\frac{1}{n}A} e^{\frac{1}{n}B} \right)^n = \lim_{n\to\infty} \left((1+\tfrac{1}{n}A)\, e^{\frac{1}{n}B} \right)^n,$$

when A,B are arbitrary matrices. If you are more ambitious, prove that $e^{-A-B} = \lim (e^{-\frac{1}{n}A} e^{-\frac{1}{n}B})^n$ when A,B are operators in an infinite-dimensional Hilbert space, that are bounded below (but not necessarily above).

The next exercises are about proving Hölder's inequality for matrices. The proof is due to Fröhlich (1978).

Exercise 10: (Chessboard estimates). Let $\mathcal{A}$ a $*$-algebra, and $\omega: \mathcal{A}^{2n} \to \mathbb{C}$ a functional that satisfies

- Multilinearity: $\omega(A_1,\dots,A_{2n})$ is linear with respect to each A_i, and $\overline{\omega(A_1,\dots,A_{2n})} = \omega(A_{2n}^*,\dots,A_1^*)$.
- Cyclicity: $\omega(A_1,\dots,A_{2n}) = \omega(A_{2n},A_1,\dots,A_{2n-1})$.
- Reflection positivity:
$$|\omega(A_1,\dots,A_{2n})|^2 \le |\omega(A_1,\dots,A_n,A_n^*,\dots,A_1^*)|\,|\omega(A_{2n}^*,\dots,A_{n+1}^*,A_{n+1},\dots,A_{2n})|$$

Show that

$$|\omega(A_1,\dots,A_{2n})| \le \prod_{i=1}^{2n} \omega(A_i,A_i^*,\dots,A_i,A_i^*)^{1/2n}.$$

Exercise 11: Prove Hölder's inequality for matrices.

2. LONG-RANGE ORDER IN 3D CLASSICAL HEISENBERG MODEL

The method of reflection positivity and infrared bounds is not easy, and it helps to learn it first with the classical model. Let us recall the setting of the classical Heisenberg ferromagnet.

The graph is $\Lambda = \{-\frac{L}{2}+1, \dots, \frac{L}{2}\}^d$, with $L \in 2\mathbb{N}$, and $\mathcal{E}$ the set of nearest-neighbors with periodic boundary conditions. At each site x there is a "classical spin" $\sigma_x \in \mathbb{S}^2 \subset \mathbb{R}^3$. One should not confuse the dimension of the space, d, with the dimension of the spin space, 3. The state space is

$$\Omega_\Lambda = \bigtimes_{x\in\Lambda} \mathbb{S}^2 = (\mathbb{S}^2)^\Lambda.$$

The Hamiltonian is the function $\Omega_\Lambda \to \mathbb{R}$ given by

$$H(\underline{\sigma}) = -2 \sum_{\{x,y\}\in\mathcal{E}} \sigma_x \cdot \sigma_y - h \sum_{x\in\Lambda} \sigma_x^3 .$$

The partition function is $Z(\Lambda,\beta,h) = \int_{\Omega_\Lambda} d\underline{\sigma}\, e^{-\beta H(\underline{\sigma})}$, where $d\underline{\sigma}$ denotes the Lebesgue measure on Ω_Λ, normalized so that $\int d\underline{\sigma} = 1$. An order parameter for ferromagnetism is

$$\Big\langle \Big(\frac{1}{|\Lambda|} \sum_{x\in\Lambda} \sigma_x \Big)^2 \Big\rangle = \frac{1}{|\Lambda|^2} \sum_{x,y\in\Lambda} \sum_{i=1}^3 \langle \sigma_x^i \sigma_y^i \rangle ,$$

where $\langle \cdot \rangle = \frac{1}{Z(\Lambda,\beta,h)} \int_{\Omega_\Lambda} d\underline{\sigma} \cdot e^{-\beta H(\underline{\sigma})}$.

Let $\mathcal{X}(x) = \langle \sigma_0^3 \sigma_x^3 \rangle = \langle \sigma_y^3 \sigma_{y+x}^3 \rangle$ denote the two-point correlation function. We shall prove that it does not decay to 0.

Let us introduce the dual lattice for Fourier theory, Λ^*, and the lattice <u>dispersion relation</u> $\varepsilon(k)$, $k \in \Lambda^*$:

$$\Lambda^* = \frac{2\pi}{L} \{-\tfrac{L}{2}+1, \dots, \tfrac{L}{2}\}^d$$

$$\varepsilon(k) = 2 \sum_{i=1}^d (1-\cos k_i)$$

Notice that $\varepsilon(k) \sim k^2$ around $k=0$.

Theorem 2.2 (Long-range order, Fröhlich-Simon-Spencer 1976)
Assume $h=0$ and L even. Then we have the lower bound

$$\frac{1}{|\Lambda|} \sum_{x\in\Lambda} \mathcal{X}(x) \geq \frac{1}{3} - \frac{1}{\beta|\Lambda|} \sum_{k\in\Lambda^*\setminus\{0\}} \frac{1}{\varepsilon(k)} .$$

As a consequence, we obtain that in the infinite-volume limit

$$\liminf_{L\to\infty} \Big\langle \Big(\frac{1}{|\Lambda|} \sum_{x\in\Lambda} \sigma_x \Big)^2 \Big\rangle \geq 1 - \underbrace{\frac{3}{(2\pi)^d\beta}}_{\text{small when } \beta \text{ large}} \underbrace{\int_{[-\pi,\pi]^d} \frac{dk}{\varepsilon(k)}}_{<\infty \text{ if } d\geq 3} .$$

The right side is positive when β is large enough, if $d \geq 3$, and the classical Heisenberg model exhibits spontaneous magnetization!

Structure of proof:

$$\begin{array}{ll} \text{LRO} & (\text{Theorem 2.2}) \\ \Uparrow \text{Fourier} & \\ \text{IRB} & \hat{\chi}(k) \le \frac{1}{\beta \mathcal{E}(k)} \\ \Uparrow v_x = \eta \cos kx & \\ \text{GD} & Z(\underline{v}) \le Z(0) \\ \Uparrow \text{cunning!} & \\ \text{RP} & Z(v_1,v_2)^2 \le Z(v_1, Rv_1)\, Z(Rv_2, v_2) \end{array}$$

These steps are explained in paragraphs I – IV below.

I. Infrared bound (IRB) implies LRO

It is necessary to define precisely the Fourier transform. If $f \in \ell^2(\Lambda)$,

$$\hat{f}(k) = \sum_{x \in \Lambda} e^{-ikx} f(x), \qquad k \in \Lambda^*$$

$$f(x) = \frac{1}{|\Lambda|} \sum_{k \in \Lambda^*} e^{ikx} \hat{f}(k), \qquad x \in \Lambda$$

Lemma 2.3 (IRB $\Rightarrow$ LRO)

Assume that $\hat{\chi}(k) \le \frac{1}{\beta \mathcal{E}(k)}$. Then we get the lower bound of Theorem 2.2.

Proof: Since $\int_{S^2} d\sigma_0\, (\sigma_0^3)^2 = \frac{1}{3} \int_{S^2} d\sigma_0\, \|\sigma_0\|^2 = \frac{1}{3}$, we have

$$\frac{1}{3} = \chi(0) \underset{\text{inv. Fourier}}{=} \frac{1}{|\Lambda|} \hat{\chi}(0) + \frac{1}{|\Lambda|} \sum_{k \in \Lambda^* \setminus \{0\}} \hat{\chi}(k).$$

Then $\frac{1}{|\Lambda|} \sum_{x \in \Lambda} \chi(x) = \frac{1}{|\Lambda|} \hat{\chi}(0) \ge \frac{1}{3} - \frac{1}{\beta |\Lambda|} \sum_{k \in \Lambda^* \setminus \{0\}} \frac{1}{\mathcal{E}(k)}$. □

Remarks: $\hat{\chi}(k)$ is real because of lattice symmetries.

II. Gaussian domination (GD) implies IRB

We introduce a partition function $Z(\underline{v})$ that depends on an external real field $\underline{v} = (v_x)_{x \in \Lambda}$. We need the discrete Laplacian Δ:

$$(\Delta v)_x = \sum_{y : \{x,y\} \in \mathcal{E}} (v_y - v_x)$$

As a matrix in $\ell^2(\Lambda)$: $\Delta_{xy} = \begin{cases} -2d & \text{if } x = y \\ 1 & \text{if } \{x,y\} \in \mathcal{E} \\ 0 & \text{oth.} \end{cases}$

Exercise 12: Check the useful identity

$$(f, -\Delta g) = \sum_{\{x,y\} \in \mathcal{E}} (f_x - f_y)(g_x - g_y)$$

where $(f,g) = \sum_{x \in \Lambda} \bar{f}_x g_x$ denotes the inner product in $\ell^2(\Lambda)$. This identity is the discrete analogue of $-\int f \frac{d^2}{dx^2} g = \int \frac{df}{dx} \frac{dg}{dx}$.

At $h = 0$, the Heisenberg Hamiltonian is

$$H(\underline{\sigma}) = -2 \sum_{\{x,y\} \in \mathcal{E}} \sigma_x \sigma_y = \sum_{\{x,y\} \in \mathcal{E}} (\sigma_x - \sigma_y)^2 - 2d|\Lambda|$$

Definition: $Z(\underline{v}) = \int_{\Omega_\Lambda} d\underline{\sigma}\, \exp\Big\{-\beta \sum_{\{x,y\} \in \mathcal{E}} (\sigma_x + \underbrace{v_x}_{\equiv v_x \vec{e}_3} - \sigma_y - v_y)^2\Big\}$

Notice that $Z(0) = Z(\Lambda, \beta, 0)\, e^{2d\beta|\Lambda|}$.

Lemma 2.4 (GD $\Rightarrow$ IRB)

Assume that $Z(\underline{v}) \le Z(0)\ \forall \underline{v}$. Then $\hat{\chi}(k) \le \frac{1}{\beta\,\varepsilon(k)}$.

Proof: $$Z(\underline{v}) = \int_{\Omega_\Lambda} d\sigma \, \exp\Big\{ -\beta \sum (\sigma_x-\sigma_y)^2 + \underbrace{2\beta \sum (\sigma_x^3-\sigma_y^3)(v_x-v_y)}_{-(\underline{\sigma},\Delta\underline{v})} - \underbrace{\beta \sum (v_x-v_y)^2}_{(\underline{v},\Delta\underline{v})} \Big\}$$

Let $\underline{h} = -\Delta\underline{v}$. Then

$$Z(\underline{v}) \le Z(0) \iff \langle e^{-2\beta(\underline{\sigma},\underline{h})} \rangle \le e^{-\beta(\underline{v},\Delta\underline{v})}$$

We use the inequality with $v_x = \cos kx$, $k \in \Lambda^*$ fixed.

$$(-\Delta v)_x = -\mathrm{Re}\,(\Delta e^{ikx}) = -\mathrm{Re}\Big(\sum_{j=1}^d e^{ikx+ik_j} + e^{ikx-ik_j} - 2e^{ikx} \Big)$$

$$= -2\,\mathrm{Re}\, e^{ikx} \Big(\sum_{j=1}^d \cos k_j - 1 \Big) = \varepsilon(k)\, v_x .$$

We see that $\underline{h} = \varepsilon(k)\,\underline{v}$. Consider the field $\eta\underline{v}$, η small.

$$\langle e^{-2\beta\eta\,\varepsilon(k)\,(\underline{\sigma},\underline{v})} \rangle \le e^{\beta\eta^2 \varepsilon(k) \|\underline{v}\|^2}$$

$$\parallel$$

$$1 - 2\beta\eta\,\varepsilon(k) \underbrace{\langle(\underline{\sigma},\underline{v})\rangle}_{=0} + \beta^2\eta^2 \varepsilon(k)^2 \underbrace{\sum_{x,y} \cos kx \cos ky \,\langle \sigma_x^3 \sigma_y^3 \rangle}_{= \sum_{x,z} \cos kx \cos k(x-z)\, \chi(z)} + O(\eta^4)$$

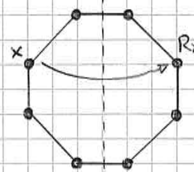
$$\sum_z \cos k(x-z)\,\chi(z) = \mathrm{Re} \sum_z e^{ikx} e^{-ikz} \chi(z) = \mathrm{Re}\, e^{ikx} \underbrace{\hat{\chi}(k)}_{\in\mathbb{R}} = \cos kx\, \hat{\chi}(k)$$

We get $1 + \beta^2\eta^2 \varepsilon(k)^2 \hat{\chi}(k) \|\underline{v}\|^2 + O(\eta^4) \le 1 + \beta\eta^2 \varepsilon(k) \|\underline{v}\|^2 + O(\eta^4)$,

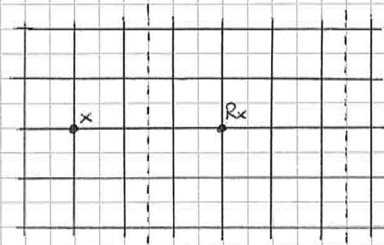
so that $\beta\,\varepsilon(k)\,\hat{\chi}(k) \le 1$.

III. Reflection positivity (RP) implies GD.

Let $R: \Lambda \to \Lambda$ denote the following reflection map:



$d=1$



$d=2$

Let us write $\Lambda = \Lambda_1 \cup \Lambda_2$ with $\Lambda_1 \cap \Lambda_2 = \emptyset$, Λ_i connected, $\Lambda_2 = R\Lambda_1$.

Denote $\underline{v} = (v_1, v_2)$, where v_1 is a field on Λ_1 and v_2 is a field on Λ_2.

Lemma 2.5 (RP $\Rightarrow$ GD)

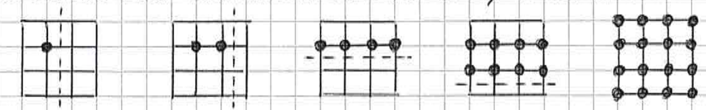
Assume that $Z(v_1,v_2)^2 \le Z(v_1,Rv_1)\, Z(Rv_2,v_2)$ for any reflection R across edges, in all lattice directions. Then $Z(\underline{v}) \le Z(0)$ $\forall$ real field $\underline{v}$.

Proof: Since $Z(\underline{v}+\text{const}) = Z(\underline{v})$, we can fix $v_0 = 0$.

The existence of maximizers follows from $\lim_{\|\underline{v}\|_\infty \to \infty} Z(\underline{v}) = 0$.

Let v^* a maximizer. Then (v_1^*, Rv_1^*) is also a maximizer, $\forall R$.

It is then possible to construct a sequence of maximizers and reflections so that maximizers are eventually constant:



$\square$

IV. Proof of reflection positivity.

Lemma 2.6
For every reflection R, we have $Z(v_1,v_2)^2 \le Z(v_1,Rv_1)\, Z(Rv_2,v_2)$.

Proof: The partition function $Z(v_1,v_2)$ has the form

$$Z(v_1,v_2) = \int_{\Omega_{\Lambda_1}} d\sigma_1 \int_{\Omega_{\Lambda_2}} d\sigma_2\, e^{-F(\sigma_1,v_1)-F(\sigma_2,v_2)} \exp\Big\{-\beta \sum_{\substack{x\in\Lambda_1\\ y\in\Lambda_2\\ \{x,y\}\in\mathcal{E}}} \sum_{i=1}^{3} \big(\sigma^i_{1,x}+v^i_{1,x}-\sigma^i_{2,y}-v^i_{2,y}\big)^2\Big\}$$

The last exponential couples the two half domains Λ_1 and Λ_2. A key idea is to introduce fields to decouple them. Recall that

$$e^{-\frac12 a^2} = \int_{-\infty}^{\infty} \frac{d\xi}{\sqrt{2\pi}}\, e^{-\frac12 \xi^2}\, e^{i\xi a}.$$

Introducing a field for each term of the exponential, we get

$$Z(v_1,v_2) = \Big(\prod_{\substack{x\in\Lambda_1\\ y\in\Lambda_2\\ \{x,y\}\in\mathcal{E}}} \prod_{i=1}^{3} \int_{-\infty}^{\infty} \frac{d\xi^i_{xy}}{\sqrt{2\pi}}\, e^{-\frac12 {\xi^i_{xy}}^2}\Big)$$

$$\cdot \int_{\Omega_{\Lambda_1}} d\sigma_1\, e^{-F(\sigma_1,v_1)} \exp\Big\{ i\sqrt{2\beta} \sum_{\substack{x\in\Lambda_1\\ y\in\Lambda_2\\ \{x,y\}\in\mathcal{E}}} \sum_{i=1}^{3} \xi^i_{xy}\, (\sigma^i_{1,x}+v^i_{1,x})\Big\} \int_{\Omega_{\Lambda_2}} d\sigma_2 \dots$$

Using Cauchy-Schwarz inequality,

$$Z(v_1,v_2)^2 \le \Big(\prod\prod \int \frac{d\xi^i_{xy}}{\sqrt{2\pi}}\, e^{-\frac12 {\xi^i_{xy}}^2}\Big) \int_{\Omega_{\Lambda_1}} d\sigma_1\, e^{-F(\sigma_1,v_1)}\, e^{i\sqrt{2\beta}\sum\sum \xi^i_{xy}(\sigma^i_{1x}+v^i_{1x})}$$

$$\cdot \int_{\Omega_{\Lambda_1}} d\sigma_1'\, e^{-F(\sigma_1',v_1)}\, e^{-i\sqrt{2\beta}\sum\sum \xi^i_{xy}(\sigma'^i_{1x}+v^i_{1x})} \cdot \text{same in } \Lambda_2$$

$$= Z(v_1,Rv_1)\ Z(Rv_2,v_2). \qquad \square$$

LECTURE 3 – LONG-RANGE ORDER IN 3D QUANTUM HEISENBERG MODEL

We now describe the extension of the method of reflection positivity and infrared bounds to quantum systems. We mainly follow the original article of Dyson, Lieb, Simon (1978). As we shall see, the method is restricted to parameters $u\in[-1,0]$, thus covering models that interpolate between the Heisenberg antiferromagnet and the XY model. To this day, there exists no proofs of phase transition at positive temperatures for the model $H^{(u)}_{\Lambda,h}$ on $\mathbb{Z}^d$, when $u\in(0,1]$. (For $u>1$, ferromagnetism was proved by Kennedy, 1985.)

The main theorem about existence of long-range order is presented in Section 3.1, and it is proved in the rest of the chapter. The proof is similar to that of the classical model, but there is one main extra difficulty: the "natural" infrared bound holds only for the Duhamel two-point function, and some efforts are needed in order to obtain another, weaker, infrared bound for the usual two-point function. The key is Falk-Bruch inequality (1969), that was independly proposed in DLS '78.

This time, we follow the logical order, and the structure of proof is the following:

$$\begin{array}{ll} \text{RP} & \text{(requires } u \leq 0\text{)} \\ \Downarrow & \\ \text{GD} & \\ \Downarrow & \\ \text{IRB} & \text{for Duhamel fct} \\ \Downarrow & \\ \text{IRB} & \text{for usual correl. fct} \\ \Downarrow & \\ \text{LRO} & \end{array}$$

1. MAIN RESULT: LONG-RANGE ORDER

We choose the following order parameter for spontaneous magnetization in the quantum model $H^{(u)}_{\Lambda,0}$:

$$\Big\langle \Big(\frac{1}{|\Lambda|} \sum_{x\in\Lambda} \vec{S}_x \Big)^2 \Big\rangle_{\Lambda,\beta,0} = \frac{1}{|\Lambda|^2} \sum_{x,y\in\Lambda} \sum_{i=1}^{3} \langle S^i_x S^i_y \rangle_{\Lambda,\beta,0}$$

As before, we take $\Lambda = \{-\frac{L}{2}+1, \dots, \frac{L}{2}\}^d$ with periodic boundary conditions and L even. Using translation invariance, we can fix $y=0$ above.

Here is the beautiful result of Dyson, Lieb, and Simon.

Theorem 3.1 (Long-range order in quantum systems)
Assume that $h=0$, L is even, and $u\in[-1,0]$.
Then we have the lower bound

$$\frac{1}{|\Lambda|} \sum_{x\in\Lambda} \langle S^3_0 S^3_x \rangle_{\Lambda,\beta,0} \geq \frac{1}{3} S(S+1) - \frac{1}{\sqrt{2}\,|\Lambda|} \sum_{k\in\Lambda^*\setminus\{0\}} \sqrt{\frac{\varepsilon_u(k)}{\varepsilon(k)}} - \underbrace{\frac{1}{2\beta|\Lambda|} \sum_{k\in\Lambda^*\setminus\{0\}} \frac{1}{\varepsilon(k)}}_{\substack{\text{small for } \beta \text{ large,} \\ \text{if } \frac{1}{\varepsilon(k)} \text{ is integrable}}}$$

Here, $\varepsilon_u(k) = \sum_{i=1}^{d} \Big((1-u\cos k_i) \langle S^1_0 S^1_{e_i} \rangle + (u-\cos k_i) \langle S^2_0 S^2_{e_i} \rangle \Big)$, with e_i the unit vector in $\mathbb{Z}^d$ in the direction i.

It is easy to check that $\varepsilon_u(k) \leq S(S+1)$, so that the lower bound is positive for $d\geq 3$ and S large enough, uniformly in Λ. Kennedy, Lieb, Shastry (1988) have improved the bound so that it is positive when $d\geq 3$ and all $S\in\frac{1}{2}\mathbb{N}$.

When $d=2$, long-range order cannot hold because of Theorem 2.1 (Mermin-Wagner). But it can hold in the ground state, that is, if the limit $\beta\to\infty$ is taken before the limit $L\to\infty$. Motivated by Neves and Perez (1986), Kennedy, Lieb, Shastry (1988) have proved a positive lower bound in $d=2$, $S\geq 1$, $u\in[-1,0]$. The bound is also positive when $S=\frac{1}{2}$ and $u \lesssim 0$. This leaves the important case of the ground state of the 2D Heisenberg antiferromagnet, where antiferromagnetism is expected but not proved.

It is perhaps worth emphasizing that Theorem 3.1 establish the presence of a ferromagnetic phase in the model $H_{1,0}^{(u)}$. Recall that the Heisenberg antiferromagnet is related to $H_{1,0}^{(-1)}$ by a unitary transformation (rotation of the spins on a sublattice) and the corresponding result is that

$$\frac{1}{|\Lambda|} \sum_{x\in\Lambda} (-1)^{\|x\|_1} \langle S_0^3 S_x^3 \rangle_{\text{Heis.AF}} \geqslant \text{lower bound of Thm 3.1}$$

This is antiferromagnetic ordering, also called Néel order.

As already mentioned, the proof of Theorem 3.1 mirrors the one of FSS for classical systems, but one big difficulty is that the infrared bound does not hold for the usual correlation function. The reason is that (with $S=\frac{1}{2}$)

$$\underbrace{\langle \vec{S}_0 \cdot \vec{S}_0 \rangle}_{=3/4} - \underbrace{\langle \vec{S}_0 \cdot \vec{S}_x \rangle}_{\leq \frac{1}{4} \text{ by Prop. 1.3}} \geqslant \frac{1}{2}.$$

On the other hand, assuming the IRB,

$$\left| \langle \vec{S}_0 \cdot \vec{S}_0 \rangle - \langle \vec{S}_0 \cdot \vec{S}_x \rangle \right| = \frac{1}{|\Lambda|} \left| \sum_{k\in\Lambda^*} (1-e^{ikx}) \widehat{\langle \vec{S}_0 \cdot \vec{S}_x \rangle}(k) \right|$$

$$\leq \frac{1}{|\Lambda|} \frac{\text{const}}{\beta} \sum_{k\in\Lambda^*\setminus\{0\}} \frac{|1-e^{ikx}|}{\varepsilon(k)} \longrightarrow 0 \text{ as } \beta\to\infty,$$

contradiction!

2. REFLECTION POSITIVITY FOR QUANTUM SYSTEMS

It is enough for our purpose to consider the claim of Dyson, Lieb, Simon (1978). A more general approach can be found in Fröhlich, Israel, Lieb, Simon (1978-80).

Lemma 3.2 (Reflection positivity)
Hilbert space $\mathcal{H} = h \otimes h$, $\dim h < \infty$. Matrices A, B, C_i, D_i, $i=1,\dots,k$, in the small space h. Then

$$\left| \operatorname{Tr}_{\mathcal{H}} e^{A\otimes \mathrm{Id} + \mathrm{Id}\otimes B - \sum_{i=1}^k (C_i\otimes \mathrm{Id} - \mathrm{Id}\otimes D_i)^2} \right|^2$$

$$\leq \operatorname{Tr}_{\mathcal{H}} e^{A\otimes \mathrm{Id} + \mathrm{Id}\otimes \bar{A} - \sum^k (C_i\otimes \mathrm{Id} - \mathrm{Id}\otimes \bar{C}_i)^2}$$

$$\cdot \operatorname{Tr}_{\mathcal{H}} e^{\bar{B}\otimes \mathrm{Id} + \mathrm{Id}\otimes B - \sum^k (\bar{D}_i\otimes \mathrm{Id} - \mathrm{Id}\otimes D_i)^2}$$

where $\bar{A}$ denotes the complex conjugate of A (not the adjoint) with respect to a fixed basis (the inequality holds $\forall$ basis).

Proof: As before, we introduce fields to decouple the two Hilbert spaces. We also use Trotter's formula. Here, we assume $k=1$ (otherwise, use more fields).

$$\left| \operatorname{Tr} e^{A\otimes Id + Id\otimes B - (C\otimes Id - Id\otimes D)^2} \right| =$$

$$= \lim_{n\to\infty} \left| \operatorname{Tr}\left(e^{\frac{1}{n}A\otimes Id}\, e^{\frac{1}{n}Id\otimes B}\, e^{-\frac{1}{n}(C\otimes Id - Id\otimes D)^2} \right)^n \right|^2$$

$$= \lim_{n\to\infty} \left| \int d\nu(s_1)\dots d\nu(s_n)\, \operatorname{Tr} \prod_{j=1}^{n} e^{\frac{1}{n}A\otimes Id}\, e^{\frac{1}{n}Id\otimes B}\, e^{\frac{is_j}{\sqrt{n}}(C\otimes Id - Id\otimes D)} \right|^2$$

$$\boxed{e^{-M^2} = \int_{-\infty}^{\infty} \frac{ds}{2\sqrt{\pi}}\, e^{-\frac{s^2}{4}}\, e^{isM} \doteq \int d\nu(s)\, e^{isM}}$$

$$= \lim_{n\to\infty} \left| \int d\nu(s_1)\dots d\nu(s_n)\, \operatorname{Tr}_{h} \prod_{j=1}^{n} e^{\frac{1}{n}A}\, e^{\frac{is_j}{\sqrt{n}}C}\; \operatorname{Tr}_{h} \prod_{j=1}^{n} e^{\frac{1}{n}B}\, e^{-\frac{is_j}{\sqrt{n}}D} \right|^2$$

$$\boxed{\operatorname{Tr}_{\mathcal{H}} M\otimes N = \operatorname{Tr}_{h} M \, \operatorname{Tr}_{h} N}$$

$$\leq \lim_{n\to\infty} \left(\int d\nu(s_1)\dots d\nu(s_n)\, \operatorname{Tr}_{h} \prod_{j=1}^{n} e^{\frac{1}{n}A}\, e^{\frac{is_j}{\sqrt{n}}C}\; \operatorname{Tr}_{h} \prod_{j=1}^{n} e^{\frac{1}{n}\bar{A}}\, e^{-\frac{is_j}{\sqrt{n}}\bar{C}} \right)$$

$$\left(\quad \dots \quad e^{\frac{1}{n}\bar{B}}\, e^{\frac{is_j}{\sqrt{n}}\bar{D}} \qquad e^{\frac{1}{n}B}\, e^{-\frac{is_j}{\sqrt{n}}D} \right)$$

$\boxed{\text{Cauchy-Schwarz}}$

We used $\overline{MN} = \bar{M}\,\bar{N}$. It is necessary to preserve the order of the matrices, hence the complex conjugates. Retracing the steps backwards, we get the lemma. $\square$

3. PARTITION FUNCTION WITH FIELDS & GAUSSIAN DOMINATION

Given a field $\underline{v} = (v_x)_{x\in\Lambda}$ with $v_x \in \mathbb{R}$, let $\underline{h} = \Delta \underline{v}$ and

$$H(\underline{v}) = H^{(u)}_{\Lambda,0} - \sum_{x\in\Lambda} h_x S^3_x .$$

The partition function with field is

$$Z(\underline{v}) = \operatorname{Tr}_{\mathcal{H}_\Lambda} e^{-\beta H(\underline{v})}$$

Let us also define

$$Z'(\underline{v}) = Z(\underline{v})\, e^{\frac{1}{4}\beta(\underline{v},\Delta\underline{v})} .$$

The property of "Gaussian domination" is

$$Z(\underline{v}) \leq Z(0)\, e^{-\frac{1}{4}\beta(\underline{v},\Delta\underline{v})} \iff Z'(\underline{v}) \leq Z'(0).$$

As in the classical case, it is a consequence of the following lemma:

<u>Lemma 3.3</u> (RP of quantum Heisenberg model)
If $u \leq 0$, then for any reflection R across edges,

$$Z'(v_1, v_2)^2 \leq Z'(v_1, Rv_1)\, Z'(Rv_2, v_2).$$

Proof: We have

$$H(\underline{v}) = \sum_{\{x,y\}\in\mathcal{E}} \left((S^1_x - S^1_y)^2 + (\sqrt{u}\, S^2_x - \sqrt{u}\, S^2_y)^2 + \left(S^3_x + \frac{v_x}{2} - S^3_y - \frac{v_y}{2}\right)^2 \right) + E_\Lambda - \frac{1}{4} \sum_{\{x,y\}\in\mathcal{E}} (v_x - v_y)^2$$

with $E_\Lambda = -2 \sum_{x\in\Lambda} \left((S^1_x)^2 + u (S^2_x)^2 + (S^3_x)^2 \right)$ a sum of local terms.

Let $H'(\underline{v}) = H(\underline{v}) + \frac{1}{4} \sum (v_x - v_y)^2$. Then $Z'(\underline{v}) = \mathrm{Tr}\, e^{-\beta H'(\underline{v})}$ and it can be cast in a reflection positive form, by defining

$$A = -\beta \underbrace{\sum_{\{x,y\}\in\mathcal{E}_1}}_{\text{edges in } \Lambda_1} \left((S^1_x - S^1_y)^2 + (\sqrt{u}\, S^2_x - \sqrt{u}\, S^2_y)^2 + \left(S^3_x + \frac{v_x}{2} - S^3_y - \frac{v_y}{2}\right)^2 \right) - \beta E_{\Lambda_1}$$

B = same in Λ_2

$C^1_i = \sqrt{\beta}\, S^1_{x_i}$, $D^1_i = \sqrt{\beta}\, S^1_{y_i}$, $C^2_i = \sqrt{\beta u}\, S^2_{x_i}$, $D^2_i = \sqrt{\beta u}\, S^2_{y_i}$,

$C^3_i = \sqrt{\beta}\,(S^3_{x_i} + \frac{1}{2} v_{x_i})$, $D^3_i = \sqrt{\beta}\,(S^3_{y_i} + \frac{1}{2} v_{y_i})$

where $\{x_i, y_i\}$ denote the edges crossing the reflection plane, with $x_i \in \Lambda_1$ and $y_i \in \Lambda_2$.

We have $\overline{S^1_x} = S^1_x$, $\overline{S^3_x} = S^3_x$, and $\overline{\sqrt{u}\, S^2_x} = \sqrt{u}\, S^2_x$ in the usual basis, if $u \leq 0$.

Lemma 3.3 now follows from Lemma 3.2. □

4. GD IMPLIES IRB FOR DUHAMEL CORRELATION FUNCTION

It is time to introduce Duhamel correlation function:

$$(S^3_0, S^3_x) = \frac{1}{Z(\Lambda,\beta,0)} \frac{1}{\beta} \int_0^\beta ds\, \mathrm{Tr}\, S^3_0\, e^{-s H^{(u)}_{\Lambda,0}}\, S^3_x\, e^{-(\beta-s) H^{(u)}_{\Lambda,0}}$$

Lemma 3.4 (GD $\Rightarrow$ IRB for Duhamel)

$$\widehat{(S^3_0, S^3_x)}(k) \leq \frac{1}{2\beta\,\varepsilon(k)}, \qquad k \in \Lambda^* \setminus \{0\}$$

Proof: Choose $v_x = \eta \cos kx$ in GD inequality. We have

$$Z(\underline{v}) = Z(0) + \frac{1}{2} \left(\underline{h}, \frac{\partial^2 Z(\underline{v})}{\partial h_x \partial h_y}\Big|_{\underline{h}=0} \underline{h} \right) + O(\eta^4)$$

(Recall that $\underline{h} = \Delta \underline{v} = -\varepsilon(k)\underline{v}$.) Also, $Z(\underline{v}) = Z(\Lambda,\beta,\underline{h}) = \mathrm{Tr}\, e^{-\beta H^{(u)}_{\Lambda,\underline{h}}}$.

Using Duhamel's formula, we can check that

$$\frac{1}{Z(\Lambda,\beta,\underline{h})} \frac{\partial^2}{\partial h_x \partial h_y} Z(\Lambda,\beta,\underline{h})\Big|_{\underline{h}=0} = \beta^2 (S^3_x, S^3_y)$$

Then

$$Z(\underline{v}) = Z(0) + Z(0)\frac{1}{2}\eta^2 \varepsilon(k)^2 \sum_{x,y\in\Lambda} \beta^2 \cos kx \cos ky\, (S^3_x, S^3_y) + O(\eta^4)$$

$$= Z(0) + Z(0) \frac{1}{2} \beta^2 \eta^2 \varepsilon(k)^2\, \widehat{(S^3_0, S^3_x)}(k) \sum_x \cos^2 kx + O(\eta^4).$$

Further, $e^{-\frac{1}{4}\beta(\underline{v}, \Delta\underline{v})} = e^{\frac{1}{4}\beta\,\varepsilon(k)\,\eta^2 \sum_x \cos^2 kx}$.

Looking at order η^2, we get the lemma. □

5. DUHAMEL FUNCTION & FALK-BRUCH INEQUALITY

We need to transfer the IRB for the Duhamel correlation function to the usual correlation function. This is done using Falk-Bruch's inequality.

We consider a more general setting, this has the advantage to alleviate the notation.

Let $\mathcal{H}$ a finite-dimensional Hilbert space, and H a Hermitian operator. The set of linear maps $\mathcal{H} \to \mathcal{H}$ is also a Hilbert space. On this space, we consider the <u>Duhamel inner product</u>

$$(A, B) = \frac{1}{Z} \int_0^1 ds \; \mathrm{Tr}_{\mathcal{H}} \, e^{-(1-s)H} A^* e^{-sH} B$$

where $Z = \mathrm{Tr}\, e^{-H}$. (Exercise: check it is inner product!)

We have

$$\frac{d}{ds} \mathrm{Tr}\, e^{-(1-s)H} A^* e^{-sH} B = \mathrm{Tr}\, e^{-(1-s)H} [H, A^*] e^{-sH} B = \mathrm{Tr}\, e^{-(1-s)H} A^* e^{-sH} [B, H],$$

and we obtain a useful identity:

$$([A,H], B) = (A, [B,H]).$$

Further,

$$(A, [B,H]) = \frac{1}{Z} \int_0^1 ds \frac{d}{ds} \mathrm{Tr}\, e^{-(1-s)H} A^* e^{-sH} B = \langle [B, A^*] \rangle,$$

where $\langle \cdot \rangle = \frac{1}{Z} \mathrm{Tr} \cdot e^{-H}$.

Let $F(s) = \mathrm{Tr}\, e^{-(1-s)H} A^* e^{-sH} A$.

$$\frac{d^2}{ds^2} F(s) = \mathrm{Tr}\, e^{-(1-s)H} [A,H]^* e^{-sH} [A,H] \geq 0, \quad \text{(why?)}$$

so $F(s)$ is convex (it is actually log-convex, $F''F - (F')^2 \geq 0$). Then

$$\frac{1}{2} \langle A^*A + AA^* \rangle = \frac{1}{2Z} \big(F(0) + F(1) \big) \geq \frac{1}{Z} \int_0^1 F(s)\, ds = (A, A)$$

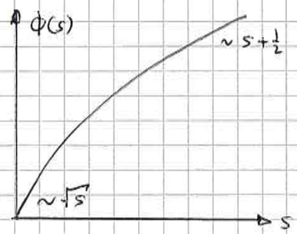
with identity iff $[A,H] = 0$. It is worth remarking that we get <u>БОГОЛЮБОВ inequality</u> from Cauchy-Schwarz:

$$\underbrace{|(A, [B,H])|^2}_{= |\langle [B, A^*] \rangle|^2} \leq \underbrace{(A, A)}_{\leq \frac{1}{2}\langle A^*A + AA^* \rangle} \; \underbrace{([B,H], [B,H])}_{= (B, [[B,H], H]) = \langle [[B,H], B^*] \rangle}$$

We have already obtained above an upper bound for the Duhamel function, namely $(A,A) \leq \frac{1}{2} \langle A^*A + AA^* \rangle$. But we need a lower bound in order to extend the IRB to the usual correlation function.

Let $\phi(s) = \sqrt{s} \coth \frac{1}{\sqrt{s}}$.

Exercise: check that $\sqrt{s} \leq \phi(s) \leq \sqrt{s} + s$.



Since ϕ is monotone increasing, the following is a lower bound for (A,A):

Lemma 3.5 (Falk-Bruch inequality)

$$\frac{2\langle A^*A + AA^*\rangle}{\langle [A^*,[H,A]]\rangle} \leq \phi\left(\frac{4(A,A)}{\langle [A^*,[H,A]]\rangle}\right)$$

Proof: Recall the function $F(s)$. The inequality can be written as

$$2\,\frac{F(0)+F(1)}{F'(1)-F'(0)} \leq \phi\left(\frac{4\int_0^1 F(s)\,ds}{F'(1)-F'(0)}\right)$$

If $\{e_i\}$ is an orthonormal set of eigenvectors of H with eigenvalues λ_j, we can write

$$\begin{aligned} F(s) &= \sum_{i,j} |(e_i, A e_j)|^2 e^{-\lambda_j} e^{(\lambda_j-\lambda_i)s} \\ &= \int_{-\infty}^{\infty} e^{st}\,d\mu(t), \end{aligned}$$

where μ is a positive measure. We have

$$\int_0^1 F(s)\,ds = \int \frac{e^t-1}{t}\,d\mu(t)$$

$$F(0)+F(1) = \int (e^t+1)\,d\mu(t)$$

$$F'(1)-F'(0) = \int t(e^t-1)\,d\mu(t)$$

Define the probability measure $d\nu(t) = \left(\int t(e^t-1)\,d\mu(t)\right)^{-1} t(e^t-1)\,d\mu(t)$.

We have $\dfrac{\int F(s)\,ds}{F'(1)-F'(0)} = \int \frac{1}{t^2}\,d\nu(t)$, $\quad \dfrac{F(0)+F(1)}{F'(1)-F'(0)} = \int \frac{1}{t}\coth\frac{t}{2}\,d\nu(t)$.

The last step is to use Jensen's inequality (ϕ is concave).

$$\begin{aligned} \phi\left(\frac{4\int_0^1 F(s)\,ds}{F'(1)-F'(0)}\right) &= \phi\left(\int \frac{4}{t^2}\,d\nu(t)\right) \geq \int \phi\left(\frac{4}{t^2}\right) d\nu(t) \\ &= \int \frac{2}{t}\coth\frac{t}{2}\,d\nu(t) = 2\,\frac{F(0)+F(1)}{F'(1)-F'(0)}. \quad \square \end{aligned}$$

The Falk-Bruch inequality is saturated when the measure $d\mu$ is a Dirac on a single value. This is the case if H is the Hamiltonian of the harmonic oscillator, and A the creation or annihilation operator.

The following inequality is immediate, and enough for our purpose. It follows from $\phi(s) \leq \sqrt{s} + s$.

Corollary 3.6

$$\tfrac{1}{2}\langle A^*A + AA^*\rangle \leq \tfrac{1}{2}\sqrt{(A,A)\,\langle [A^*,[H,A]]\rangle} + (A,A).$$

6. INFRARED BOUND FOR THE USUAL CORRELATION FCT

In order to see that the Fourier transform of (S_0^3, S_x^3) can be written in the form (A,A), let us define Fourier transforms of operators. Let

$$\hat{S}_k^3 = \sum_{x\in\Lambda} e^{-ikx} S_x^3 , \quad k \in \Lambda^*$$

$$S_x^3 = \frac{1}{|\Lambda|} \sum_{k\in\Lambda^*} e^{ikx} \hat{S}_k^3 , \quad x \in \Lambda .$$

Then we check that

$$\widehat{\langle S_0^3 S_x^3\rangle}(k) = \sum_{x\in\Lambda} e^{-ikx} \langle S_0^3 S_x^3\rangle = \frac{1}{|\Lambda|}\sum_{x,y\in\Lambda} e^{-ik(x-y)} \langle S_x^3 S_y^3\rangle = \frac{1}{|\Lambda|}\langle \hat{S}_{-k}^3 \hat{S}_k^3\rangle .$$

The same relation applies to other correlation functions, including Duhamel. Notice that $(\hat{S}_k^3)^* = \hat{S}_{-k}^3$, and that

$$\widehat{(S_0^3, S_x^3)}(k) = \frac{1}{|\Lambda|} (\hat{S}_k^3 , \hat{S}_k^3)$$

(there is no $-k$ because of the $*$ in the definition of the Duhamel inner product).

Lemma 3.7. (IRB for usual correlation fct)

$$\widehat{\langle S_0^3 S_x^3\rangle}(k) \leq \sqrt{\frac{\varepsilon_u(k)}{2\,\mathcal{E}(k)}} + \frac{1}{2\beta\,\mathcal{E}(k)} .$$

As can be seen in the proof, we have $\varepsilon_u(k) = ([H_{\Lambda,0}^{(u)}, \hat{S}_k^3], [H_{\Lambda,0}^{(u)}, \hat{S}_k^3])$ up to positive factors, so $\varepsilon_u(k) \geq 0$ for all $k \in \Lambda^*$.

Using spin rotation invariance, we can also check that $\varepsilon_1(k) = \langle S_0^1 S_{e_1}^1\rangle\, \mathcal{E}(k)$ and $\varepsilon_{-1}(k) = \langle S_0^1 S_{e_1}^1\rangle\, \mathcal{E}(k+\pi)$.

Proof: Let $A = \hat{S}_k^3$ and $H = \beta H_{\Lambda,0}^{(u)}$. We calculate the double commutator.

$$[H,A] = \sum_{x\in\Lambda} e^{-ikx} [H, S_x^3]$$

$$= -2\beta \sum_{x,y:\{x,y\}\in\mathcal{E}} e^{-ikx} [S_x^1 S_y^1 + u S_x^2 S_y^2 + S_x^3 S_y^3 , S_x^3]$$

$$= -2\beta \sum_{x,y:\{x,y\}\in\mathcal{E}} e^{-ikx} (-i S_x^2 S_y^1 + u i S_x^1 S_y^2)$$

$$[A^*, [H,A]] = 2\beta i \sum_{x,y:\{x,y\}\in\mathcal{E}} e^{-ikx} [e^{ikx} S_x^3 + e^{iky} S_y^3 , S_x^2 S_y^1 - u S_x^1 S_y^2]$$

$$= 2\beta \sum_{x,y:\{x,y\}\in\mathcal{E}} \left(S_x^1 S_y^1 - e^{-ik(x-y)} S_x^2 S_y^2 + u S_x^2 S_y^2 - u e^{-ik(x-y)} S_x^1 S_y^1 \right)$$

$$= 4\beta \sum_{\{x,y\}\in\mathcal{E}} \Big((1-u\cos k(x-y))\, S_x^1 S_y^1 + (u-\cos k(x-y))\, S_x^2 S_y^2 \Big).$$

We have found $\langle [A^*, [H,A]]\rangle = 4\beta|\Lambda|\,\varepsilon_u(k)$. The result follows from Corollary 3.6. □

Proof of Theorem 3.1:

$$\tfrac{1}{3} S(S+1) = \langle S_0^3 S_0^3\rangle = \frac{1}{|\Lambda|}\sum_{k\in\Lambda^*} \widehat{\langle S_0^3 S_x^3\rangle}(k) .$$

Then $\frac{1}{|\Lambda|}\sum_{x\in\Lambda} \langle S_0^3 S_x^3\rangle = \frac{1}{|\Lambda|}\widehat{\langle S_0^3 S_x^3\rangle}(0) = \frac{1}{3}S(S+1) - \frac{1}{|\Lambda|}\sum_{k\in\Lambda^*\setminus\{0\}} \widehat{\langle S_0^3 S_x^3\rangle}(k)$.

The theorem follows from Lemma 3.7. □

Exercise 13. Let $S=\frac{1}{2}$ and $x \neq 0$. We claimed somewhere that $\langle \vec{S}_0 \cdot \vec{S}_x \rangle \leq \frac{1}{4}$, where $\langle \cdot \rangle$ is a quantum state (i.e. a nonnegative normalized linear functional on the space of operators). Prove it! (Proposition 1.3 may help.)

Exercise 14. Check that $(MN)^* = N^* M^*$ but $\overline{MN} = \overline{M}\,\overline{N}$, for any complex matrices M, N (this exercise is trivial, actually).

Exercise 15. Check that $\operatorname{Tr} e^{-(1-s)H} A^* e^{-sH} A \geq 0$ for any operator A (H is hermitian).

Exercise 16. Check that $\sqrt{s} \leq \phi(s) \leq \sqrt{s} + s$, where $\phi(s) = \sqrt{s} \coth \frac{1}{\sqrt{s}}$ is the function in the Falk-Bruch inequality.

LECTURE 4 - RANDOM LOOP REPRESENTATION OF QUANTUM HEISENBERG MODELS

1. INTRODUCTION.

These representations go back to Feynman's approach to the interacting Bose gas (1953), using what came to be known as the Feynman-Kac formula. Similar descriptions of quantum lattice systems were introduced by Ginibre (1969) in order to prove the occurrence of phase transitions in anisotropic systems, and by Conlon and Solovej (1993) in order to get a lower bound for the free energy of the Heisenberg ferromagnet. This result was improved by Tóth, who introduced a random loop representation for the spin $\frac{1}{2}$ Heisenberg ferromagnet (1993). Another representation was proposed by Aizenman & Nachtergaele for the Heisenberg antiferromagnet (1994), in a remarkable work about spin chains. It was recently used by Bachmann & Nachtergaele in the context of gapped ground states (2013).

We describe here a synthesis of the two representations that applies to models that interpolate between the ferromagnet and the antiferromagnet. It also applies to the spin $\frac{1}{2}$

quantum XY model. It also applies to certain SU(2) invariant spin 1 models, where it allows to use the method of reflection positivity and infrared bounds in order to prove quadrupolar long-range order, that is characteristic of a spin nematic phase. See Ueltschi (2013) for more information. Crawford, Ng, and Starr, have used it to prove bounds on the probability of emptyness formation (in preparation, 2013).

Recently, mathematicians have used similar probabilistic representations in order to study the quantum Ising model. See articles of Ioffe, Crawford, Grimmett, Björnberg of the past few years.

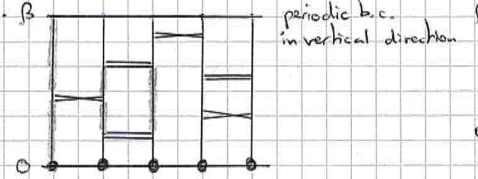
2. <u>RANDOM LOOP MODELS & RELEVANT RANDOM VARIABLES</u>

The representation holds for arbitrary finite graphs. Let $\beta > 0$ and $u \in [0,1]$. To each edge $\{x,y\} \in \mathcal{E}$ is associated the interval $[0,\beta]$ and a Poisson point process on this interval. There occur two kinds of events:

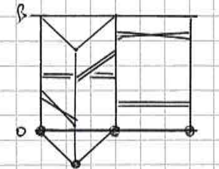
- <u>crosses</u> $\times$ occur with intensity u
- <u>double bars</u> $=$ occur with intensity $1-u$.

That is, if $[t, t+dt]$ is a tiny interval in $[0,\beta]$, there is a cross with probability $u\,dt$, a double bar with probability

$(1-u)\,dt$, and nothing with probability $1-dt$. The probability of multiple events is negligible, of order $(dt)^2$. Given a graph $(\Lambda, \mathcal{E})$, we let ρ denote the measure of independent Poisson processes at each edge of $\mathcal{E}$. Let ω be a realization of this probability measure. It contains finitely many objects with probability 1. To ω corresponds the set of loops $\mathcal{L}(\omega)$, defined in a natural way:



There are two loops here, $|\mathcal{L}(\omega)| = 2$



This graph is not linear.
How many loops?

The <u>partition function</u> $Z(\Lambda,\beta)$ of the model of random loops is defined by

$$Z(\Lambda,\beta) = \int 2^{|\mathcal{L}(\omega)|}\, d\rho(\omega).$$

The relevant measure is

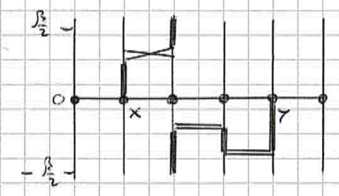
$$d\mu(\omega) = \frac{1}{Z(\Lambda,\beta)}\, 2^{|\mathcal{L}(\omega)|}\, d\rho(\omega).$$

That is, the realizations ω of the Poisson process ρ receive the weight $2^{|\mathcal{L}(\omega)|}$.

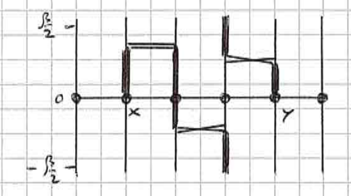
The relevant quantities, "random variables", are the following :

- $1_{x \leftrightarrow y}(\omega)$ is the indicator function for the sites x and y to belong to the same loop ;
- $1_{x \Lsh y}(\omega)$ is the indicator function for x and y to belong to the same loop, with identical vertical direction ;
- $1_{x \curvearrowright y}(\omega)$ is the indicator function for x and y to belong to the same loop, with opposite vertical directions ;
- $L_x(\omega)$ is the length of the loop that contains x (the length is defined as the sum of the lengths of vertical intervals of the loops).

Illustration for the events $x \Lsh y$ and $x \curvearrowright y$:



Event $x \Lsh y$



Event $x \curvearrowright y$

Notice the obvious relations : $1_{x \leftrightarrow y} = 1_{x \Lsh y} + 1_{x \curvearrowright y}$ and $0 \leq L_x(\omega) \leq \beta|\Lambda|$. If ω has only crosses, then $1_{x \curvearrowright y}(\omega) = 0$. If the graph is bipartite and ω has only double bars, then $1_{x \curvearrowright y}(\omega) = 0$ when x, y belong to the same sublattice, and $1_{x \Lsh y}(\omega) = 0$ when, x, y belong to different sublattices.

3. RELATIONS BETWEEN RANDOM LOOPS & QUANTUM MODELS

Let $S = \frac{1}{2}$; the Hilbert space is $\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathbb{C}^2$. We consider the Hamiltonian

$$H_\Lambda^{(u)} = -2 \sum_{\{x,y\} \in \mathcal{E}} \left(S_x^1 S_y^1 + (2u-1) S_x^2 S_y^2 + S_x^3 S_y^3 - \tfrac{1}{4} \right).$$

It differs a bit from the Hamiltonian $H_{1,h}^{(u)}$ introduced previously. An unimportant constant has been added. The parameter $u=1$ still corresponds to the Heisenberg ferromagnet, but the antiferromagnet has $u=0$ and the XY model has $u=\frac{1}{2}$.

Here are the precise relations between loops and quantum spins.

Theorem 4.1

Assume that $u \in [0,1]$.

(a) Partition functions: $Z(\Lambda, \beta) = \int 2^{|\mathcal{L}(\omega)|} d\rho(\omega) = \operatorname{Tr} e^{-\beta H_\Lambda^{(u)}}$

(b) Correlation in spin directions 1 and 3:

$$\langle S_x^1 S_y^1 \rangle = \langle S_x^3 S_y^3 \rangle = \tfrac{1}{4} \mathbb{P}(x \leftrightarrow y)$$

quantum Gibbs state — random loops

(c) Correlations in spin direction 2:

$$\langle S_x^2 S_y^2 \rangle = \tfrac{1}{4} \left[\mathbb{P}(x \Lsh y) - \mathbb{P}(x \curvearrowright y) \right].$$

We prove Theorem 4.1 by first rewriting the Hamiltonian in terms of the suitable interaction operators T_{xy} and Q_{xy}, introduced below. The exponential of a sum of matrices can be expressed with the help of a Poisson point process. The partition function has then an expansion in terms of "space-time spin configurations".

Let T_{xy} be the transposition operator on $\mathbb{C}^2 \otimes \mathbb{C}^2$:

$$T_{xy}\,|a,b\rangle = |b,a\rangle \quad , \quad a,b = \pm\tfrac{1}{2}$$

The operator Q_{xy} has the following matrix elements:

$$\langle a,b|\, Q_{xy}\, |c,d\rangle = \delta_{a,b}\,\delta_{c,d} \quad , \quad a,b,c,d = \pm\tfrac{1}{2}.$$

It is not too hard to check that

$$\vec{S}_x \cdot \vec{S}_y = \tfrac{1}{2} T_{xy} - \tfrac{1}{4}$$
$$S_x^1 S_y^1 - S_x^2 S_y^2 + S_x^3 S_y^3 = \tfrac{1}{2} Q_{xy} - \tfrac{1}{4}.$$

The Hamiltonian $H_\Lambda^{(u)}$ can then be expressed in terms of T_{xy} and Q_{xy}:

$$H_\Lambda^{(u)} = -\sum_{\{x,y\}\in\mathcal{E}} \left(u\,T_{xy} + (1-u)\,Q_{xy} - 1 \right).$$

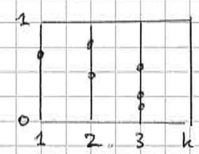
This form of the Hamiltonian is well suited for a Feynman-Kac type of expansion. The next lemma goes back at least to Aizenman - Nachtergaele (1994).

Lemma 4.2

Let $A_1, \dots, A_k$ be bounded operators, and ρ a Poisson point process on $\{1,\dots,k\} \times [0,1]$ of intensity 1. Then

$$e^{\sum_{j=1}^k (A_j - 1)} = \int d\rho(\omega) \prod^{*}_{(j,t)\in\omega} A_j .$$

Here, the product is over the events of ω in increasing times.



This realization gives the product $A_3 A_3 A_2 A_3 A_1 A_2$.

Proof: We start with the Poisson process. Discretizing the time direction, the expectation can be written as

$$\int d\rho(\omega) \prod^{*}_{(j,t)\in\omega} A_j = \lim_{n\to\infty} \sum_{m=0}^{n} \sum_{\substack{(j_1,t_1),\dots,(j_m,t_m) \\ t_1<t_2<\dots<t_m}} \left(\tfrac{1}{n}\right)^m \left(1-\tfrac{1}{n}\right)^{kn-m} \prod_{i=1}^m A_{j_i}$$

$$= \lim_{n\to\infty} \prod_{t=1}^{n} \prod_{j=1}^{k} \left(1 - \tfrac{1}{n} + \tfrac{1}{n} A_j\right)$$

$$= \lim_{n\to\infty} \left(\prod_{j=1}^{k} \left(1 + \tfrac{1}{n}(A_j - 1)\right) \right)^n$$

$$= \lim_{n\to\infty} \left(1 + \tfrac{1}{n} \sum_{j=1}^{k} (A_j - 1) \right)^n$$

$$\overset{\text{Trotter}}{=} e^{\sum (A_j - 1)}.$$

$\square$

<u>Proof of Theorem 4.1</u>: We only prove (a), the extension to (b) and (c) are left as an exercise. Given a realization ω of the Poisson point process on $\mathcal{E} \times [0,\beta]$, we let $m = m(\omega)$ denote the number of events. It follows from Lemma 4.2 that

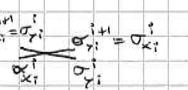
$$e^{-\beta H_\Lambda^{(u)}} = e^{\sum_{\{x,y\} \in \mathcal{E}} (u T_{xy} + (1-u) Q_{xy} - 1)}$$

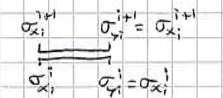
$$= \int d\rho(\omega) \prod^{*}_{\{x,y,t\} \in \omega} \left\{ \begin{matrix} T_{xy} & \text{if event is a cross} \\ Q_{xy} & \text{if event is a double bar} \end{matrix} \right\}$$

We actually used a straightforward generalization of Lemma 4.2 with two Poisson processes of intensities u and $1-u$. Using a basis of classical spin configurations, $|\sigma\rangle$ with $\sigma \in \{-\frac{1}{2}, \frac{1}{2}\}^\Lambda$, we get

$$\operatorname{Tr} e^{-\beta H_\Lambda^{(u)}} = \int d\rho(\omega) \sum_{\sigma^1, \dots, \sigma^m} \langle \sigma^1 | R_{x_1 y_1} | \sigma^2 \rangle \langle \sigma^2 | R_{x_2 y_2} | \sigma^3 \rangle \dots \langle \sigma^m | R_{x_m y_m} | \sigma^1 \rangle,$$

where $R_{x_i y_i}$ is either $T_{x_i y_i}$ or $Q_{x_i y_i}$. Observe that the product of matrix elements is zero, unless $\sigma_z^i = \sigma_z^{i+1}$ $\forall z \neq x_i, y_i$, and the constraints at x_i, y_i are as follows:

$T_{x_i y_i}$: 

$Q_{x_i y_i}$: 

Given a realization ω, the constraint is easily expressed in terms of the loops: the space-time configuration is constant on each loop. There two possibility, $\pm\frac{1}{2}$, for each loop, hence the factor $2^{|\mathcal{L}(\omega)|}$. $\square$

Another natural random variable in the loop model is the length L_x of the loop that contains the site x. This too is related to a physical quantity, the magnetic susceptibility. Here are precise equivalences.

<u>Theorem 4.3</u>

(a) $$\mathbb{E}\Big(\sum_{x \in \Lambda} L_x \Big) = \frac{4}{\beta} \frac{\partial^2}{\partial h^2} \log \operatorname{Tr} e^{-\beta H_\Lambda^{(u)} + \beta h \sum_{x \in \Lambda} S_x^3} \Big|_{h=0}$$

(b) There exists a constant K such that

$$4\beta \sum_{x,y \in \Lambda} \langle S_x^3 S_y^3 \rangle - K\beta \sqrt{(1-u)|\mathcal{E}|} \sqrt{\mathbb{E}(\textstyle\sum_x L_x)} \leq \mathbb{E}\Big(\sum_{x \in \Lambda} L_x\Big) \leq 4\beta \sum_{x,y \in \Lambda} \langle S_x^3 S_y^3 \rangle$$

The theorem holds for any finite graph, but it is perhaps better understood in the case of a regular lattice such as a cube of size L, with periodic boundary conditions. Using translation invariance, the claim (a) can be rewritten

$$\mathbb{E}(L_0) = \frac{4}{\beta|\Lambda|} \chi_\Lambda(\beta, 0) = \frac{4}{\beta} \frac{\partial^2}{\partial h^2} f_\Lambda(\beta, h) \Big|_{h=0},$$

where L_0 is the length of the loop containing 0, and χ_Λ and f_Λ are finite-volume susceptibility and free energy. We see that $f = \lim_{|\Lambda| \to \infty} f_\Lambda$ is not analytic at $h = 0$ when the lengths of the loops diverge.

The claim (b) can be written as

$$4\beta \left\langle \left(\frac{M_\Lambda}{|\Lambda|}\right)^2 \right\rangle - \frac{K\beta}{|\Lambda|} \sqrt{(1-u)d}\, \sqrt{\mathbb{E}\left(\frac{L_0}{|\Lambda|}\right)} \leq \mathbb{E}\left(\frac{L_0}{|\Lambda|}\right) \leq 4\beta \left\langle \left(\frac{M_\Lambda}{|\Lambda|}\right)^2 \right\rangle$$

There is spontaneous magnetization iff there are <u>macroscopic</u> loops: $\frac{L_0}{|\Lambda|}$ converges to a non-trivial random variable as $|\Lambda| \to \infty$.

<u>Proof of Theorem 4.3</u>: We have

$$\frac{4}{\beta} \frac{\partial^2}{\partial h^2} \log \operatorname{Tr} e^{-\beta H_\Lambda^{(u)} + \beta h \sum_x S_x^3} \Big|_{h=0} = \frac{4}{Z(\Lambda,\beta)} \sum_{x,y\in\Lambda} \int_0^\beta ds \, \operatorname{Tr} S_x^3 e^{-sH_\Lambda^{(u)}} S_y^3 e^{-(\beta-s)H_\Lambda^{(u)}}$$

One can extend Theorem 4.1 to such "Schwinger correlations", so that the right side is equal to

$$\sum_{x,y\in\Lambda} \int_0^\beta \mathbb{P}((x,0) \leftrightarrow (y,t))\, dt = \sum_{x\in\Lambda} \mathbb{E}\Big(\underbrace{\sum_{y\in\Lambda} \int_0^\beta 1_{(x,0)\leftrightarrow(y,t)}\, dt}_{= L_x} \Big).$$

The claim (b) follows from the inequalities for the Duhamel function that we saw in Chapter 3, and it is left as an exercise. $\square$

4. THE HARD-CORE LATTICE BOSE GAS

Following the first experimental realizations of Bose condensates, in 1995, there has been considerable interest in theoretical physics for the interacting Bose gas. It is natural to consider lattice models, and it is quite reasonable to assume that, in some cases of physical relevance, there is strong on-site repulsion so that at most one boson can occupy a given site. The resulting model is then mathematically equivalent to a spin system with $S=\frac{1}{2}$. In order to see that, let us define the annihilation and creation operators by

$$a_x = S_x^1 + i S_x^3$$
$$a_x^* = S_x^1 - i S_x^3$$

These operators satisfy the (anti)commutation relations

$$[a_x, a_y^{(*)}] = 0 \quad \forall x \neq y$$
$$\{a_x, a_x^*\} = \mathrm{Id} \quad \forall x$$

The operator for the number of particles at x is

$$n_x = a_x^* a_x = S_x^2 + \tfrac{1}{2}.$$

Its eigenvalues are 0 and 1, as they should be. The Hamiltonian can be rewritten using these operators as

$$H_\Lambda^{(u)} = -\sum_{\{x,y\}\in\mathcal{E}} \left(a_x^* a_y + a_y^* a_x + 2(2u-1)(n_x - \tfrac{1}{2})(n_y - \tfrac{1}{2}) - \tfrac{1}{2} \right).$$

The first terms are called "hopping operators". The last terms involve nearest-neighbor interactions. They disappear when $u=\frac{1}{2}$.

The relevant correlation functions are those representing "off-diagonal long-range order" that signal the occurrence of Bose-Einstein condensation (Penrose, Onsager, 1956):

$$\langle a_x^* a_y \rangle = 2 \langle S_x^3 S_y^3 \rangle .$$

The density-density correlation function, that signals the occurrence of a solid phase, is given by

$$\langle n_x n_y \rangle - \langle n_x \rangle \langle n_y \rangle = \langle S_x^2 S_y^2 \rangle$$

Exercise 17. Prove the relations above. Notice that some terms vanish because of symmetries.

A fascinating question is about the possible existence of a "supersolid" phase, where both order parameters above differ from 0 (as $\|x-y\| \to \infty$, after taking $|\Lambda| \to \infty$). I am grateful to A. Giuliani and R. Seiringer for discussions about this in June at IHP. The physics community is bitterly divided, and it would help if the situation was clarified in some relevant models such as the hard-core lattice Bose gas (possibly with nearest-neighbor interactions).

Consider the Hamiltonian $H_\Lambda^{(u)} = -2 \sum_{\{x,y\} \in \mathcal{E}} (S_x^1 S_y^1 + u S_x^2 S_y^2 + S_x^3 S_y^3)$, with general spin $S \in \frac{1}{2}\mathbb{N}$. It follows from RP and GD that for all $u \in [-1, 0)$, we have the IRB

$$\widehat{(-1)^x (S_0^2, S_x^2)}(k) \leq \frac{1}{2\beta |u| \, \varepsilon(k)} .$$

The double commutator in the Falk-Bruch inequality gives, with $A = \sum_x e^{-ikx} (-1)^x S_x^2$,

$$\langle [A^*, [H, A]] \rangle = 4\beta |\Lambda| \, \varepsilon(k+\pi) \, \langle S_0^3 S_{e_1}^3 \rangle .$$

The IRB for the usual correlation function is then

$$\widehat{(-1)^x \langle S_0^2 S_x^2 \rangle}(k) \leq \sqrt{\frac{\langle S_0^3 S_{e_1}^3 \rangle}{2|u|}} \, \frac{1}{|\Lambda|} \sqrt{\frac{\varepsilon(k+\pi)}{\varepsilon(k)}} + \frac{1}{2\beta |u| \, \varepsilon(k)} .$$

We then obtain the lower bound for long-range order:

$$\lim_{\beta \to \infty} \lim_{|\Lambda| \to \infty} \frac{1}{|\Lambda|} \sum_{x \in \Lambda} (-1)^x \langle S_0^2 S_x^2 \rangle \geqslant \frac{1}{3} S(S+1) - \sqrt{\frac{\langle S_0^3 S_{e_1}^3 \rangle}{2|u|}} \, J_d ,$$

where $J_d = \frac{1}{(2\pi)^d} \int \sqrt{\frac{\varepsilon(k+\pi)}{\varepsilon(k)}} \, dk$. Since $\langle S_0^3 S_{e_1}^3 \rangle \leq \|S_0^3\|^2 = S^2$, there is long-range order in the weaker direction of spins for any $u \neq 0$, if S is large enough. For $S = \frac{1}{2}$ we can use a better bound from DLS:

$$\langle S_0^3 S_{e_1}^3 \rangle = \frac{1}{4d} \langle \underbrace{S_0^1 \sum_{\|e\|=1} S_e^1 + S_0^3 \sum_{\|e\|=1} S_e^3}_{\text{max. eigenvalue} \leq \frac{1}{2}\sqrt{d(d+1)}} \rangle \leq \frac{1}{8} \left(1 + \frac{1}{d}\right)^{1/2} .$$

The lower bound for long-range order is then

$$\frac{1}{4} - \frac{1}{4\sqrt{|u|}} \left(1 + \frac{1}{d}\right)^{1/4} J_d .$$

One can check that $J_d \geq 1$, so the lower bound is always negative. But $J_d \to 1$ as $d \to \infty$ (DLS), so the bound goes to 0 as $|u| \to 1$ and $d \to \infty$! A slightly better bound would give long-range order. Is that possible? Probably not, because of the following conjecture.

Conjecture. In the random loop model, for any $S \in \frac{1}{2}\mathbb{N}$, any $u \in (0,1)$, any $\beta \in (0, \infty)$, any graph, the correlation function $\mathbb{P}(x \overset{\frown}{\smile} y) - \mathbb{P}(x \overset{\frown}{\frown} y)$ has exponential decay with respect to the graph distance.

5. CONCLUSION

The study of quantum spin systems is far from completed. Their importance cannot be exagerated because of their applications to condensed matter physics. The Mermin-Wagner theorem, and the proofs of existence of long-range order in systems with continuous symmetry by Fröhlich, Simon, Spencer, and Dyson, Lieb, Simon, form an essential part of our understanding of these systems.

Probabilistic approaches such as models of random loops could contribute much in the future. There are intriguing questions about the joint distributions of the lengths of macroscopic loops, which should be surprisingly simple and general; Poisson-Dirichlet distributions. This has been proved by Schramm in the random interchange model on the complete graph, following a conjecture of Aldous (2005). They should also occur in random loop models with genuine spatial structure, see the heuristic discussion in Goldschmidt, U, Windridge (2011). This is backed by numerical results, see Gandolfo, Ruiz, U (2007), Grosskinsky, Lovisolo, U (2012), Nahum, Chalker, et.al. (2013).

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