CRITICAL PARAMETER OF RANDOM LOOP MODEL ON TREES

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ABSTRACT. We give estimates of the critical parameter for random loop models that are related to quantum spin systems. We consider here regular trees of large degrees which approximate high spatial dimensions. We find a critical parameter that indeed shares similarity with that of the cubic lattice.

1. INTRODUCTION

We consider random loop models that are motivated by quantum spin systems. A special case is the random interchange model that was first introduced by Harris [12]. Tóth showed that a variant of this model, where permutations receive the weight $2^{\#\,\text{cycles}}$, is closely related to the quantum Heisenberg ferromagnet [17]. Another loop model was introduced by Aizenman and Nachtergaele to describe the quantum Heisenberg antiferromagnet [1]. These loop models were combined in order to describe a family of quantum systems that interpolate between the two Heisenberg models, and which contains the quantum XY model [18].

Let $G = (V, E)$ be an arbitrary finite graph with vertex set V and edge set E, and $\beta > 0$, $u \in [0,1]$ be two parameters. To each edge $e \in E$ is assigned a time interval $[0,\beta]$, and an independent Poisson point process with two kinds of outcomes: "crosses" occur with intensity u and "double bars" occur with intensity $1-u$. We let $\Omega(G)$ denote the set of realizations of the combined Poisson point process on $E \times [0,\beta]$.

Given a realization $\omega \in \Omega(G)$, we consider the loop passing through a point $(x,t) \in V \times [0,\beta]$ that is defined in a natural way, as follows (see Fig. 1). The loop is a closed trajectory with support on $V \times [0,\beta]_{\text{per}}$ where $[0,\beta]_{\text{per}}$ is the interval $[0,\beta]$ with periodic boundary conditions, i.e., the torus of length β. Starting at (x,t), move "up" until meeting the first cross or double bar with endpoint x; then jump onto the other endpoint, and continue in the same direction if a cross, in the opposite direction if a double bar; repeat until the trajectory returns to (x,t).

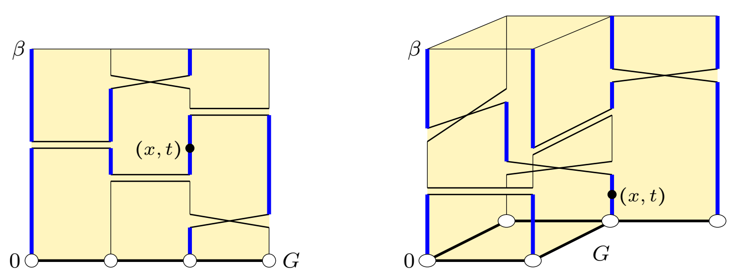

FIGURE 1. Graphs and realizations of Poisson point processes, and the loop that contains (x,t).

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In order to represent a quantum model, one should add the weight $\theta^{\#\,\text{loops}}$ with $\theta = 2, 3, 4, \dots$; quantum correlations are then given in terms of loop correlations, and magnetic long-range order is equivalent to the presence of macroscopic loops. Notice that the parameter β plays the rôle of the inverse temperature of the quantum spin system, hence the notation.

The random interchange model (i.e. the case $u = 1$ and $\theta = 1$) has been the object of several studies when the graph is a tree [3, 10, 11], the complete graph [15, 5, 6], the hypercube [13], and the Hamming graph [14]; a result for general graphs was also proposed in [2]. In the case of arbitrary θ, and on the complete graph, it has been shown that the critical parameter is identical to that of a random cluster model [7, 8]; this property is expected in graphs with diverging degrees. Another generalization of the random interchange model is Mallows permutations, studied in [16, 9].

The occurrence of macroscopic loops can be proved using the method of reflection positivity and infrared bounds in the case where $u \in [0, \frac{1}{2}]$, $\theta = 2, 3, \ldots$, and a cubic lattice of sufficiently high dimensions (depending on θ); see [18] for precise statements.

In the case where the graph is a three-dimensional cubic lattice with edges between nearest-neighbors, and with $\theta = 1$, the critical parameter $\beta_c(u)$ has been calculated numerically in [4]. The result is depicted in Fig. 2 and shows a convex curve where $\beta_c(0)$ is slightly smaller than $\beta_c(1)$ and which has a minimum at or around $u = \frac{1}{2}$. This behavior is expected to hold for all dimensions $d \geq 3$.

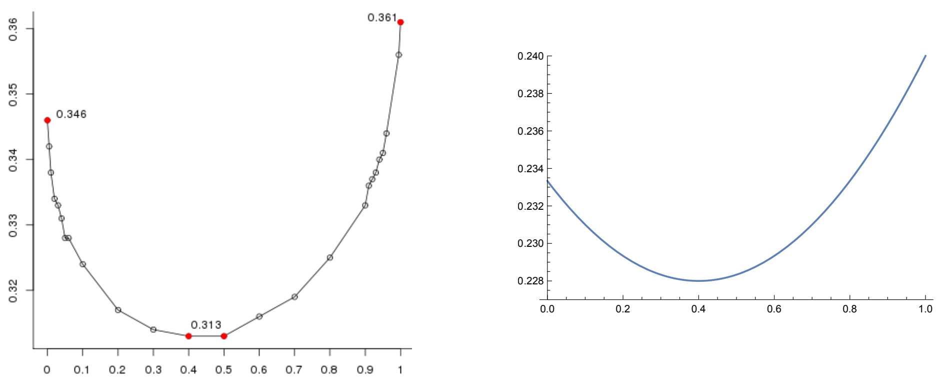

FIGURE 2. The critical parameter β_c as function of u: Left, on the three-dimensional cubic lattice (numerical results from [4]); right, Eq. (1.1) with $d = 5$.

Trees are expected to approximate high dimensions. We consider here regular trees with d offsprings. We prove that a transition takes place at the critical parameter $\beta_c = \beta_c(u, d)$ given by

$$\beta_c(u, d) = \frac{1}{d} + \frac{1 - u(1-u) - \frac{1}{6}(1-u)^2}{d^2} + o(d^{-2}). \tag{1.1}$$

The second graph of Fig. 2 shows β_c as function of u (with $d = 5$). The leading order of the critical parameter, $\frac{1}{d}$, is also the leading order for the percolation threshold in the associated percolation model where an edge is open if at least one cross or double bar is present in the corresponding interval $[0, \beta]$. The next order for β_c is a non-trivial function of u and it is smallest for $u = \frac{2}{5}$. This function can be understood by looking at edges with two links and at loop connections with two crosses, two double bars, or one each. As is explained below (see Fig. 4), loop connections are better in the latter case, with a cross and a double bar.[1]

Let E_∞ denote the event where the root of the tree (at time 0) belongs to an infinite loop.

Theorem 1.1. *Let $[a,b]$ be an arbitrary interval and $\beta = \frac{1}{d} + \frac{\alpha}{d^2}$ with $\alpha \in [a,b]$. There exists d_0 (that depends on a, b but not on α) such that for all $d \geq d_0$, there exists $\alpha_{\rm c}(u,d)$ such that*

$$\mathbb{P}_{\beta,d,u}(E_\infty) \begin{cases} = 0 & \text{if } \alpha < \alpha_{\rm c}(u,d), \\ > 0 & \text{if } \alpha > \alpha_{\rm c}(u,d). \end{cases}$$

Further, we have $\alpha_{\rm c}(u,d) = 1 - u(1-u) - \frac{1}{6}(1-u)^2 + o(1)$.

This theorem follows from Propositions 2.1 and 3.1. Proposition 2.1 establishes the existence of $d_0(\alpha)$ such that loops occur for $\alpha > \alpha_{\rm c}$ but not for $\alpha < \alpha_{\rm c}$, if $d > d_0(\alpha)$. The case $u = 1$, that is, the interchange model on trees, was treated up to first order in d^{-1} by Hammond [10], following the work of Angel [3]. Proposition 3.1 implies that $d_0(\alpha)$ is uniform on bounded intervals. The corresponding result for the interchange model was proposed by Hammond [11]. It turns out that his method can be adapted to $u \neq 1$ with minor modifications, as explained in Section 3.

Our theorem does not exclude that other transition parameters exist away from $1/d$; this scenario is hardly believable, but a proper proof is still needed. The case $\theta \neq 1$ could probably be treated in a similar way, although a full study is needed in order to rule out extra obstacles. Another open problem that seems interesting is to establish that, in the case where the graph is a box in $\mathbb{Z}^{d'}$ with nearest-neighbor edges, the critical parameter satisfies Eq. (1.1) with $d = 2d' - 1$.

2. The critical parameter

As mentioned above, we consider an infinite rooted regular tree with d offsprings. To each edge is associated the interval $[0,1]$, and an independent Poisson point process where "crosses" occur with intensity $u\beta \in [0,\beta]$, and "double bars" occur with intensity $(1-u)\beta$. (This is a variation of the model discussed above, with β affecting the intensities rather than the time interval, which is obviously equivalent.)

Let us define $\bar\alpha(u) = 1 - u(1-u) - \frac{1}{6}(1-u)^2$. In what follows we always take α to belong to some arbitrary but fixed interval $[a,b]$, and error terms may depend on a and b.

Proposition 2.1. *Let $\beta = \frac{1}{d} + \frac{\alpha}{d^2}$ and $\delta > 0$. There exists $d_0(\delta)$ such that the following hold for all $d > d_0$.*

(a) For every $\alpha \leq \bar\alpha - \delta$, we have

$$\mathbb{P}_{\beta,d,u}\big((\rho,0) \leftrightarrow \infty\big) = 0.$$

(b) For every $\alpha \geq \bar\alpha + \delta$, we have

$$\mathbb{P}_{\beta,d,u}\big((\rho,0) \leftrightarrow \infty\big) > 0.$$

Note that we prove exponential decay for (a), that is, the loop containing $(\rho,0)$ has diameter m with probability less than $C\,\mathrm{e}^{-\eta m}$. These claims can be compared to the numerical results for three-dimensional lattices. Also, the special case $u = 1$ of our result gives a solution to Problem 10 of [3] (for large enough d).

[1] Alan Hammond pointed out to us this important observation.

2.1. **Preliminaries.** We let T denote an infinite tree where each vertex has $d \geq 2$ offspring, and write ρ for its root. For $m \geq 0$ let $T^{(m)}$ denote the subtree of T consisting of the first m generations.

We write σ_m for the probability that $(\rho, 0)$ belongs to a loop which reaches generation 0 in $T^{(m)}$, and $\zeta_m = 1 - \sigma_m$. Note that $\sigma_m \leq \sigma_{m-1}$ and that $\sigma_m \to \mathbb{P}((\rho,0) \leftrightarrow \infty)$ as $m \to \infty$. We write $B^m_{(\rho,0)}$ for the event that $(\rho, 0)$ does not belong to a loop which reaches generation 0 in $T^{(m)}$, thus $\mathbb{P}(B^m_{(\rho,0)}) = \zeta_m$.

Crosses and double-bars will be referred to collectively as *links*. If $(xy, t) \in \omega$ is a link, then in general we have that the points $(x, t+)$ and $(x, t-)$ may belong to different loops (the same is true for $(y, t+)$ and $(y, t-)$). We say that a link is a *monolink* if $(x, t+)$ and $(x, t-)$ belong to the *same* loop. The following simple observation will be useful.

Proposition 2.2. *Suppose that y is a child of x in $T^{(m)}$. If there is only one link between x and y then it is a monolink.*

Proof. Denote the link (xy, t). In the configuration obtained by removing this link, the points (x, t) and (y, t) belong to two different loops, since we are on a tree. When the link is added back in, the loops are merged to a single loop, which proves the claim. □

Write A_1 for the event that, for each child x of ρ, there is at most one link between ρ and x. Write A_2 for the event that: (i) there is a unique child x of ρ with exactly 2 links between ρ and x, (ii) for all siblings x' of x there is at most one link between ρ and x', and (iii) for all children y of x there is at most one link between x and y. See Fig. 3.

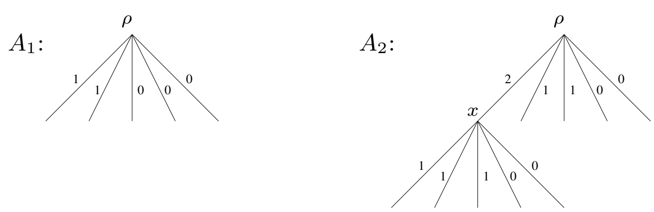

FIGURE 3. Summing over two events A_1 and A_2.

Clearly we have that

$$\zeta_m = \mathbb{P}(B^m_{(\rho,0)}) = \mathbb{P}(B^m_{(\rho,0)} \cap A_1) + \mathbb{P}(B^m_{(\rho,0)} \cap A_2) + \mathbb{P}(B^m_{(\rho,0)} \smallsetminus (A_1 \cup A_2)). \tag{2.1}$$

In the rest of this section we work with β of the form

$$\beta = \tfrac{1}{d} + \tfrac{\alpha}{d^2} \tag{2.2}$$

for $\alpha \in \mathbb{R}$.

2.2. **Occurrence of long loops.** We now prove part (b) of Proposition 2.1. For given $m \geq 1$ and $\varepsilon > 0$ we define

$$\tilde{\sigma}_m = \sigma_m \wedge \sigma_{m-1} \wedge \left(\tfrac{\varepsilon}{d}\right) = \sigma_m \wedge \left(\tfrac{\varepsilon}{d}\right). \tag{2.3}$$

In this section we show the following:

Proposition 2.3. *For all $m \geq 1$ we have*

$$\sigma_m \geq \tilde{\sigma}_{m-1} + \tfrac{\tilde{\sigma}_{m-1}}{d}(\alpha - \bar{\alpha}(u)) - \tfrac{1}{2}\tilde{\sigma}^2_{m-1} + O(d^{-3}),$$

where the $O(d^{-3})$ is uniform in m.

Given the proposition, we can establish the occurrence of infinite loops:

Proof of Proposition 2.1, part (b). We claim that if $\varepsilon < 2(\alpha - \bar{\alpha}(u))$ is small enough then, for d large enough, we have $\sigma_m \geq \tilde{\sigma}_m \geq \frac{\varepsilon}{d}$ for all m. Since $\sigma_0 = 1$ and $\sigma_1 \geq 1 - (e^{-\beta})^d$, the claim holds for $m = 1$, and Prop.2.3 gives the claim by induction. Hence $\sigma_m \geq \frac{\varepsilon}{d}$ for all m, which gives the result. □

Proof of Prop. 2.3. The starting point is the inequality

$$\zeta_m \leq \mathbb{P}(B^m_{(\rho,0)} \cap A_1) + \mathbb{P}(B^m_{(\rho,0)} \cap A_2) + 1 - \mathbb{P}(A_1) - \mathbb{P}(A_2), \tag{2.4}$$

which follows directly from (2.1). First note that

$$\mathbb{P}(A_1) = (e^{-\beta}(1+\beta))^d, \qquad \mathbb{P}(A_2) = \tfrac{1}{2} d\beta^2 e^{-\beta} (e^{-\beta}(1+\beta))^{2d-1}. \tag{2.5}$$

Next note that

$$\begin{aligned} \mathbb{P}(B^m_{(\rho,0)} \cap A_1) &= \sum_{k=0}^{d} \binom{d}{k} (\beta e^{-\beta})^k (e^{-\beta})^{d-k} (\zeta_{m-1})^k \\ &= \left(e^{-\beta}(1 + \beta\zeta_{m-1})\right)^d. \end{aligned} \tag{2.6}$$

This relies on Prop. 2.2. Indeed, if there are k children $x_1, \dots, x_k$ of ρ that are linked to ρ, with one link each, at times $t_1, \dots, t_k$ say, then $(\rho, 0)$ lies in the same loop as all of $(x_1, t_1), \dots, (x_k, t_k)$. The probability of not being connected to generation 0 is the same if one has one incoming link from a parent as if one has none, and is thus ζ_{m-1} for each of $(x_1, t_1), \dots, (x_k, t_k)$.

In obtaining a similar expression for the case A_2, it is useful to refer to Fig. 4. Let Λ_ρ and Λ_x denote the restrictions of the subset highlighted in blue to $\{\rho\} \times [0,1]$ and $\{x\} \times [0,1]$, respectively. Thus Λ_ρ and Λ_x have respective lengths X and $1 - X$ in the case of two crosses; X and X in the case of two double-bars; and 1 in the case of a mixture. It may look obvious that X is uniformly distributed in $[0,1]$; this is however incorrect, since it can be written as

$$X = \min\{U_1, U_2\} + 1 - \max\{U_1, U_2\},$$

where U_1, U_2 are independent uniform random variables on $[0,1]$; in particular $\mathbb{E}[X] = \frac{2}{3}$.

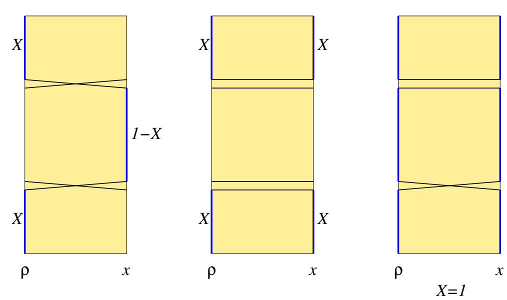

FIGURE 4. Two crosses, or two double bars, may prevent connection; one cross and one double bar is better.

As before, any link from ρ to a sibling x' of x, or from x to a child y, is a monolink. Links that fall in $\Lambda_\rho \cup \Lambda_x$ have a chance of connecting $(\rho, 0)$ to generation 0, the others do not. One thus obtains

$$\mathbb{P}(B^m_{(\rho,0)} \cap A_2) = \tfrac{1}{2} d\beta^2 e^{-\beta} \mathbb{E}\Big[\Big(e^{-\beta}(1 + \beta\zeta_{m-1}|\Lambda_\rho| + \beta(1 - |\Lambda_\rho|))\Big)^{d-1} \Big(e^{-\beta}(1 + \beta\zeta_{m-2}|\Lambda_x| + \beta(1 - |\Lambda_x|))\Big)^d\Big], \tag{2.7}$$

where the expectation is over the lengths $|\Lambda_\rho|$ and $|\Lambda_x|$. As noted above, we have

$$\begin{array}{ll} |\Lambda_\rho| = 1 - |\Lambda_x| = X, & \text{with probability } u^2, \\ |\Lambda_\rho| = |\Lambda_x| = X, & \text{with probability } (1-u)^2, \\ |\Lambda_\rho| = |\Lambda_x| = 1, & \text{with probability } 2u(1-u). \end{array} \tag{2.8}$$

We now use the inequalities

$$\zeta_{m-1} \le 1 - \tilde\sigma_{m-1}, \qquad \zeta_{m-2} \le 1 - \tilde\sigma_{m-1} \tag{2.9}$$

to obtain from (2.6) that

$$\mathbb{P}(B^m_{(\rho,0)} \cap A_1) \le \big(e^{-\beta}(1+\beta-\tilde\sigma_{m-1}\beta)\big)^d \tag{2.10}$$

and from (2.7) that

$$\mathbb{P}(B^m_{(\rho,0)} \cap A_2) \le \tfrac12 d\beta^2 e^{-\beta}\mathbb{E}\Big[\big(e^{-\beta}(1+\beta-\tilde\sigma_{m-1}\beta|\Lambda_\rho|)\big)^{d-1}\big(e^{-\beta}(1+\beta-\tilde\sigma_{m-1}\beta|\Lambda_x|)\big)^{d}\Big]. \tag{2.11}$$

In light of (2.10), (2.11) and (2.5), we will proceed by providing estimates for terms of the form

$$\big(e^{-\beta}(1+\beta-\sigma x\beta)\big)^d, \tag{2.12}$$

for $\sigma = O(d^{-1})$ and constant $x \in [0,1]$. Since $\beta = \frac1d + \frac{\alpha}{d^2}$, the following are easy to verify:

$$\begin{aligned} e^{-\beta} &= 1 - \tfrac1d + \tfrac{1}{d^2}(1/2-\alpha) + \tfrac{1}{d^3}(\alpha - 1/6) + O(d^{-4}), \\ 1+\beta-\sigma x\beta &= 1 + \tfrac1d + \tfrac{1}{d^2}(\alpha - x\sigma d) - \tfrac{1}{d^3}(\alpha x\sigma d). \end{aligned} \tag{2.13}$$

Hence

$$e^{-\beta}(1+\beta-\sigma x\beta) = 1 + \tfrac{1}{d^2}(-1/2 - x\sigma d) + \tfrac{1}{d^3}(1/3 - \alpha + x\sigma d - \alpha x\sigma d) + O(d^{-4}). \tag{2.14}$$

Combining this with

$$\big(1 + \tfrac{a}{n^2} + \tfrac{b}{n^3} + O(n^{-4})\big)^n = 1 + \tfrac{a}{n} + \tfrac{1}{n^2}(b + a^2/2) + O(n^{-3}) \tag{2.15}$$

we see that

$$\big(e^{-\beta}(1+\beta-\sigma x\beta)\big)^d = 1 - \tfrac1d(1/2 + x\sigma d) + \tfrac{1}{d^2}\big(1/3 - \alpha + x\sigma d - \alpha x\sigma d + \tfrac12(1/2 + x\sigma d)^2\big) + O(d^{-3}). \tag{2.16}$$

Applying this to (2.10) and (2.5) we obtain

$$\mathbb{P}(A_1) - \mathbb{P}(B^m_{(\rho,0)} \cap A_1) \ge \tilde\sigma_{m-1} - \tfrac{\tilde\sigma_{m-1}}{d}(3/2 - \alpha) - \tfrac12\tilde\sigma^2_{m-1} + O(d^{-3}). \tag{2.17}$$

Now consider the case of $\mathbb{P}(A_2)$ and (2.11). Since

$$\tfrac12 d\beta^2 e^{-\beta} = \tfrac{1}{2d} + O(d^{-2}) \tag{2.18}$$

it suffices in this case to use (2.16) to order $\frac1d$. We may also replace the $d-1$ in the exponent by d. We obtain that

$$\begin{aligned} &\mathbb{P}(A_2) - \mathbb{P}(B^m_{(\rho,0)} \cap A_2) \\ &\ge \big(\tfrac{1}{2d} + O(d^{-2})\big)\mathbb{E}\Big[\big(1-\tfrac1d\big) - \big(1 - \tfrac1d(1/2 + |\Lambda_\rho|\tilde\sigma_{m-1}d)\big)\big(1 - \tfrac1d(1/2 + |\Lambda_x|\tilde\sigma_{m-1}d)\big)\Big] + O(d^{-3}) \\ &= \tfrac{\tilde\sigma_{m-1}}{2d}\mathbb{E}\big[|\Lambda_\rho| + |\Lambda_x|\big] + O(d^{-3}) \\ &= \tfrac{\tilde\sigma_{m-1}}{2d}\big(u^2 + \tfrac43(1-u)^2 + 4u(1-u)\big) + O(d^{-3}) \\ &= \tfrac{\tilde\sigma_{m-1}}{d}\big(\tfrac12 + u(1-u) + \tfrac16(1-u)^2\big) + O(d^{-3}). \end{aligned} \tag{2.19}$$

Here we used (2.8) and $\mathbb{E}(X) = \frac23$. Adding this to (2.17) and substituting in (2.4), we obtain the claim. □

2.3. **Absence of long loops.** Interestingly, the absence of large loops for $\alpha < \bar{\alpha}(u)$ seems harder to establish than their occurrence for $\alpha > \bar{\alpha}(u)$. Intuitively, this is because for part of the range of α that we consider (namely, for $\alpha > 1/2$) the percolation-tree is infinite with positive probability, yet we still need to show that the loops are always blocked.

We will use the notations $p_0, p_1, p_2, \dots$ and $p_{\geq 2}, p_{\geq 3}, \dots$ for the probabilities for a Poisson(β) random variable. We also use the shorthand

$$q = p_{\geq 3} = \sum_{j=3}^{\infty} e^{-\beta} \frac{\beta^j}{j!} = \frac{1}{6d^3} + O(d^{-4}), \tag{2.20}$$

and define for $m \geq 3$

$$\check{\sigma}_{m-1} = \sum_{\ell=3}^{m} (dq)^{\ell-3} \sigma_{m-\ell}. \tag{2.21}$$

In this section we prove:

Proposition 2.4. *For all $m \geq 3$ we have*

$$\sigma_m \leq \check{\sigma}_{m-1}\big(1 + \tfrac{\alpha - \bar{\alpha}(u)}{d} + O(d^{-2})\big),$$

where the $O(d^{-2})$ is uniform in m.

The proposition implies the remaining part of Proposition 2.1:

Proof of Proposition 2.1, part (a). Suppose $\alpha - \bar{\alpha}(u) \leq -2\varepsilon < 0$. For d large enough we have $dq \leq 1/d^2$ and, by Prop. 2.4, that

$$\sigma_m \leq (1 - \tfrac{\varepsilon}{d})\check{\sigma}_{m-1} \leq (1 - \tfrac{\varepsilon}{d}) \sum_{\ell=3}^{m} \sigma_{m-\ell} \big(\tfrac{1}{d^2}\big)^{\ell-3}, \tag{2.22}$$

for all $m \geq 3$. We show, by induction over m, that if d is large enough, then there are constants $C = C(d) > 0$ and $\sigma = \sigma(d) \in (0,1)$ such that

$$\sigma_k \leq C\sigma^k \text{ for all } k \geq 0. \tag{2.23}$$

This clearly implies the result. We choose $\sigma = 1 - \frac{\varepsilon}{6d}$, and by choosing C appropriately we can assume that (2.23) holds for $k = 0, 1, 2$. Suppose that it holds for $k \leq m-1$ for some $m \geq 3$. Then by (2.22)

$$\sigma_m \leq C\sigma^m \,(1 - \tfrac{\varepsilon}{d})\big(\tfrac{1}{\sigma}\big)^3 \sum_{l=3}^{m} \big(\tfrac{1}{\sigma d^2}\big)^{\ell-3} \leq C\sigma^m \,(1 - \tfrac{\varepsilon}{d})\big(\tfrac{1}{\sigma}\big)^3 \tfrac{1}{1 - 1/\sigma d^2}. \tag{2.24}$$

But here the factor

$$(1 - \tfrac{\varepsilon}{d})\big(\tfrac{1}{\sigma}\big)^3 \tfrac{1}{1-1/\sigma d^2} = (1 - \tfrac{\varepsilon}{d})(1 + 3\tfrac{\varepsilon}{6d} + O(d^{-2}))(1 + O(d^{-2})) = 1 - \tfrac{\varepsilon}{2d} + O(d^{-2}) \leq 1, \tag{2.25}$$

provided d is large enough. Hence (2.23) follows for $k = m$, as required. □

Lemma 2.5. *Assume that*

$$\mathbb{P}(B^m_{(\rho,0)} \cap (A_1 \cup A_2)^c) \geq \mathbb{P}((A_1 \cup A_2)^c)[1 - c\check{\sigma}_{m-1}]$$

for some constant $c > 0$. Then the bound of Proposition 2.4 holds true.

Proof. We note that, by (2.5), (2.6) and (2.7), we have that

$$\begin{aligned} \mathbb{P}(B^m_{(\rho,0)} \cap A_1) &\geq \mathbb{P}(A_1)\big(1 - \sigma_{m-1}\big(1 + \tfrac{\alpha-1}{d} + O(d^{-2})\big)\big), \\ \mathbb{P}(B^m_{(\rho,0)} \cap A_2) &\geq \mathbb{P}(A_2)\big(1 - \sigma_{m-2}\big(1 + 2u(1-u) + \tfrac{1}{3}(1-u)^2 + O(d^{-1})\big)\big). \end{aligned} \tag{2.26}$$

This uses the inequalities $\sigma_{m-1} \leq \sigma_{m-2}$ and $(1-x)^n \geq 1 - nx$ for $x \in [0,1]$ and $n \geq 1$, as well as the asymptotics

$$\frac{\beta}{1+\beta} = \frac{1}{d} + \frac{\alpha - 1}{d^2} + O(d^{-3}). \tag{2.27}$$

We have $\check{\sigma}_{m-1} \geq \sigma_{m-2} \geq \sigma_{m-1}$ and $\mathbb{P}(A_1) = 1 - \frac{1}{2d} + O(d^{-2})$ and $\mathbb{P}(A_2) = \frac{1}{2d} + O(d^{-2})$. Together with the assumption of the lemma, we have, using (2.1),

$$\begin{aligned}\sigma_m &\leq 1 - \mathbb{P}(A_1)\big[1 - \sigma_{m-1}(1 + \tfrac{\alpha-1}{d} + O(d^{-2}))\big] \\ &\quad - \mathbb{P}(A_2)\big[1 - \sigma_{m-2}(1 + 2u(1-u) + \tfrac{1}{3}(1-u)^2 + O(d^{-1}))\big] - \mathbb{P}((A_1 \cup A_2)^c)\big[1 - c\check{\sigma}_{m-1}\big] \\ &= \sigma_{m-1}\,\mathbb{P}(A_1)\big[1 + \tfrac{\alpha-1}{d} + O(d^{-2})\big] + \sigma_{m-2}\,\mathbb{P}(A_2)\big[1 + 2u(1-u) + \tfrac{1}{3}(1-u)^2 + O(d^{-1})\big] \\ &\quad + c\check{\sigma}_{m-1}\mathbb{P}((A_1 \cup A_2)^c) \\ &\leq \check{\sigma}_{m-1}\Big[1 + \tfrac{\alpha-1}{d} - \tfrac{1}{2d} + O(d^{-2}) + \tfrac{1}{2d} + \tfrac{u(1-u) + \frac{1}{6}(1-u)^2}{d} + O(d^{-2})\Big].\end{aligned} \tag{2.28}$$

This is indeed the upper bound of Proposition 2.4. □

The rest of this section will be devoted to the proof of the assumption of Lemma 2.5.

We write $(A_1 \cup A_2)^c$ as a union

$$(A_1 \cup A_2)^c = \bigcup_{k=1}^{d} (A'_k \cup A''_k), \tag{2.29}$$

of the disjoint events

- A'_1: that ρ has exactly one child with ≥ 3 links and all other children of ρ have 0 or 1 links;
- A'_k for $k \geq 2$: that ρ has exactly k children with ≥ 2 links;
- A''_k for $k \geq 1$: that ρ has exactly one child x with exactly 2 links, all other children of ρ have 0 or 1 links, and x has exactly k children with ≥ 2 links.

The following bounds will be useful later:

Lemma 2.6. *For d large enough we have*

$$\sum_{k=1}^{d} k\mathbb{P}(A'_k) \leq 2\sum_{k=1}^{d}\mathbb{P}(A'_k), \qquad \text{and} \qquad \sum_{k=1}^{d} k\mathbb{P}(A''_k) \leq 2\sum_{k=1}^{d}\mathbb{P}(A''_k).$$

Proof. We start with the A''_k:s, which is actually the simpler case. For convenience, we write A''_0 for the event that ρ has exactly one child x with exactly 2 links, and that the other children of ρ have 0 or 1 links. Then

$$\sum_{k=1}^{d} k\mathbb{P}(A''_k) = \mathbb{P}(A''_0)\sum_{k=1}^{d} k\mathbb{P}(A''_k \mid A''_0) = \mathbb{P}(A''_0)dp_{\geq 2}, \tag{2.30}$$

since the last sum is the expected number of children of x with two links or more. Similarly

$$\sum_{k=1}^{d} \mathbb{P}(A''_k) = \mathbb{P}(A''_0)\sum_{k=1}^{d} \mathbb{P}(A''_k \mid A''_0) = \mathbb{P}(A''_0)\;(1 - (1-p_{\geq 2})^d). \tag{2.31}$$

It is easy to deduce that

$$\frac{\sum_{k=1}^{d} k\mathbb{P}(A''_k)}{\sum_{k=1}^{d} \mathbb{P}(A''_k)} \to 1, \text{ as } d \to \infty, \tag{2.32}$$

which gives the claim for the A_k''. For the A_k' a straightforward but tedious calculation gives that

$$\sum_{k=1}^{d} k\mathbb{P}(A_k') = dp_{\geq 2} - dp_2(1-p_{\geq 2})^{d-1} = \frac{5/12}{d^2} + O(d^{-3}), \tag{2.33}$$

and

$$\sum_{k=1}^{d} \mathbb{P}(A_k') = 1-(1-p_{\geq 2})^d - dp_2(1-p_{\geq 2})^{d-1} = \frac{7/24}{d^2} + O(d^{-3}). \tag{2.34}$$

Hence

$$\frac{\sum_{k=1}^{d} k\mathbb{P}(A_k')}{\sum_{k=1}^{d} \mathbb{P}(A_k')} \to \frac{10}{7} < 2, \text{ as } d\to\infty, \tag{2.35}$$

which gives the claim for the A_k'. □

The main idea in establishing the assumption of Lemma 2.5 is to use a certain random subtree $\check{T}$ of $T^{(m)}$. In order to define $\check{T}$, it helps to think that it consists of "bulk sites" and "end sites". The root ρ is a bulk site by definition. Assume that the tree has been defined up to level k, and let x be a bulk site at level k. An offspring y is

(a) a bulk site if the number of links n_{xy} on the edge x is equal to 3,4,...;
(b) a bulk site if $x=\rho$, $n_{xy}=2$, and all siblings z of y satisfy $n_{xz}\in\{0,1\}$;
(c) an end site if $n_{xy}\in\{0,1,2\}$, unless there is situation (b).

Notice that the event $(A_1\cup A_2)^c$ is measurable with respect to $\check{T}$.

We write $\check{\omega}$ for the configuration of crosses and double-bars within $\check{T}$. For $j=1,2$ and $1\leq \ell\leq m-1$ we write $\mathcal{E}_\ell^{(j)}$ for the set of leaves (end sites) of $\check{T}$ at distance ℓ from ρ and with j incoming links. If $x\in\mathcal{E}_\ell^{(1)}$ we write $t(x)$ for the time-coordinate of the incoming link, and if $y\in\mathcal{E}_\ell^{(2)}$ we write $t_1(y)$ and $t_2(y)$ for the time-coordinates of the two incoming links. We also let $\mathcal{E}_\ell=\mathcal{E}_\ell^{(1)}\cup\mathcal{E}_\ell^{(2)}$ $(1\leq\ell\leq m-1)$ and we let $\mathcal{E}_m$ be the set of vertices of $\check{T}$ at distance m from ρ.

For $y\in T^{(m)}$ we let T_y be the subtree rooted at y, consisting of y and all its descendants in $T^{(m)}$. For a sub-tree T' of $T^{(m)}$ we write $\Omega(T')$ for the set of configurations of crosses and double-bars in T'. In particular, $\Omega(T_y)$ is the set of configurations in the subtree rooted at y. We write $B^k_{(y,t)}\subseteq\Omega(T_y)$ for the set of configurations in T_y such that the loop of (y,t) visits no vertex $z\in T_y$ at distance k from y (note that we do not consider any incoming links to y from its parent). And we write $B^m_{(\rho,0)}(y)\subseteq\Omega(T^{(m)})$ for the event that the loop of $(\rho,0)$ visits no vertex $z\in T_y$ at distance m from ρ, i.e. the loop does not reach distance m in the subtree rooted at y.

The next lemma concerns the probability of blocking a loop at a vertex y when there are two incoming links.

Lemma 2.7. *Let y be a vertex of $T^{(m)}$ at distance ℓ from ρ, let $0<t_1<t_2<1$, let $\omega'\in\Omega(T^{(m)}\setminus T_y)$ be arbitrary, and let $\omega''\in B^{m-\ell}_{(y,t_1)}\cap B^{m-\ell}_{(y,t_2)}$. Form a configuration $\omega\in\Omega(T^{(m)})$ by adjoining ω' and ω'' as well as two links to y from its parent, at times t_1 and t_2. Then $\omega\in B^m_{(\rho,0)}(y)$.*

This lemma is useful since the event $B^{m-\ell}_{(y,t_1)}\cap B^{m-\ell}_{(y,t_2)}$ is defined entirely in the subgraph T_y, which is disjoint from $T^{(m)}\setminus T_y$, and due to the bound

$$\mathbb{P}(B^{m-\ell}_{(y,t_1)}\cap B^{m-\ell}_{(y,t_2)})\geq 1-2\sigma_{m-\ell}. \tag{2.36}$$

Proof. Write x for the parent of y. In ω' the points (x,t_1) and (x,t_2) belong to some loops L_1', L_2', where possibly $L_1' = L_2'$. Similarly, in ω'' the points (y,t_1) and (y,t_2) belong to some loops L_1'', L_2'', possibly equal. Note that neither L_1'' nor L_2'' reaches distance $m-\ell$ from y in T_y.

We can form ω by starting with $\omega' \cup \omega''$, and putting in the links (xy,t_1) and (xy,t_2) one at a time. When putting in (xy,t_1) we necessarily merge L_1' and L_1'', since they were disjoint before. When we then put in (xy,t_2) we either cause another merge, involving L_2'', or we cause a loop to split. In either case, no loop of $T^{(m)} \smallsetminus T_y$ ever merges with a loop which reaches distance m from ρ in T_y. □

Note that, writing $\check{\mathbb{P}}(\cdot)$ for $\mathbb{P}(\cdot \mid \check{T}, \check{\omega})$,

$$\begin{aligned}\mathbb{P}(B^m_{(\rho,0)} \cap (A_1 \cup A_2)^c) &= \mathbb{E}[\mathbb{1}_{(A_1\cup A_2)^c} \check{\mathbb{P}}(B^m_{(\rho,0)})] \\ &= \mathbb{E}\Big[\mathbb{1}_{(A_1\cup A_2)^c} \check{\mathbb{P}}\Big(\bigcap_{\ell=1}^m \bigcap_{y\in\mathcal{E}_\ell} B^m_{(\rho,0)}(y)\Big)\Big].\end{aligned} \tag{2.37}$$

But by Lem. 2.7 and (2.36) we have

$$\begin{aligned}\check{\mathbb{P}}\Big(\bigcap_{\ell=1}^m \bigcap_{y\in\mathcal{E}_\ell} B^m_{(\rho,0)}(y)\Big) &\geq \check{\mathbb{P}}\Big(\bigcap_{\ell=1}^{m-1} \bigcap_{x\in\mathcal{E}_\ell^{(1)}} B^{m-\ell}_{(x,t(x))} \bigcap_{y\in\mathcal{E}_\ell^{(2)}} \big(B^{m-\ell}_{(y,t_1(y))} \cap B^{m-\ell}_{(y,t_2(y))}\big)\Big)\mathbb{1}\{\mathcal{E}_m = \varnothing\} \\ &= \prod_{\ell=1}^{m-1} \prod_{x\in\mathcal{E}_\ell^{(1)}} \check{\mathbb{P}}(B^{m-\ell}_{(x,t(x))}) \prod_{y\in\mathcal{E}_\ell^{(2)}} \check{\mathbb{P}}\big(B^{m-\ell}_{(y,t_1(y))} \cap B^{m-\ell}_{(y,t_2(y))}\big)\mathbb{1}\{\mathcal{E}_m = \varnothing\} \\ &\geq \prod_{\ell=1}^{m-1} (1-\sigma_{m-\ell})^{|\mathcal{E}_\ell^{(1)}|} (1-(2\sigma_{m-\ell})\wedge 1)^{|\mathcal{E}_\ell^{(2)}|} \mathbb{1}\{\mathcal{E}_m = \varnothing\} \\ &\geq 1 - 2\sum_{\ell=1}^m \sigma_{m-\ell}|\mathcal{E}_\ell|.\end{aligned} \tag{2.38}$$

(Here $\sigma_0 = 1$, and the last line is negative when $\mathcal{E}_m \neq \varnothing$.) Hence

$$\mathbb{P}(B^m_{(\rho,0)} \cap (A_1 \cup A_2)^c) \geq \mathbb{E}\Big[\mathbb{1}_{(A_1\cup A_2)^c}\Big(1 - 2\sum_{\ell=1}^m \sigma_{m-\ell}|\mathcal{E}_\ell|\Big)\Big], \tag{2.39}$$

and the assumption of Lemma 2.5 follows if we show that

$$\sum_{\ell=1}^m \sigma_{m-\ell}\mathbb{E}\big[\mathbb{1}_{(A_1\cup A_2)^c}|\mathcal{E}_\ell|\big] \leq 48\mathbb{P}((A_1\cup A_2)^c)\check{\sigma}_{m-1}. \tag{2.40}$$

The following will let us establish (2.40) (and hence Prop. 2.4):

Lemma 2.8. *For d large enough, $k \geq 1$, $m \geq 3$, and $1 \leq \ell \leq m$, we have*

$$\mathbb{E}\big[\mathbb{1}_{A_k'}|\mathcal{E}_\ell|\big] \leq 4k\mathbb{P}(A_k')a_\ell' \qquad \text{and} \qquad \mathbb{E}\big[\mathbb{1}_{A_k''}|\mathcal{E}_\ell|\big] \leq 4k\mathbb{P}(A_k'')a_\ell'',$$

where $a_1' = 1$, $a_\ell' = (dq)^{\ell-2}$ for $\ell \geq 2$, $a_1'' = a_2'' = 1$, and $a_\ell'' = (dq)^{\ell-3}$ for $\ell \geq 3$.

Proof. It suffices to bound the conditional expectations

$$\mathbb{E}\big[|\mathcal{E}_\ell| \,\big|\, A_k'\big] \quad \text{and} \quad \mathbb{E}\big[|\mathcal{E}_\ell| \,\big|\, A_k''\big] \tag{2.41}$$

by the appropriate functions. We prove the result for the A_k', the arguments for the A_k'' are similar.

There are several cases to consider, we start with $\ell = 1$. Given A'_k, the number of 1's in generation $\ell = 1$ has distribution $\mathrm{Bin}(d-k, p_1/(p_0+p_1))$, and it follows that

$$\mathbb{E}\big[|\mathcal{E}_1^{(1)}|\,\big|\,A'_k\big] = (d-k)\frac{p_1}{p_0+p_1} \le \frac{p_1 d}{p_0+p_1} \tag{2.42}$$

which is trivially bounded by $2k = 2ka'_1$. Next, we have $\mathbb{E}\big[|\mathcal{E}_1^{(2)}|\,\big|\,A'_1\big] = 0$, whereas for $k \ge 2$ the number of 2's in generation $\ell = 1$ has distribution $\mathrm{Bin}(k, p_2/p_{\ge 2})$, so that

$$\mathbb{E}\big[|\mathcal{E}_1^{(2)}|\,\big|\,A'_k\big] = k\frac{p_2}{p_{\ge 2}} \le k \le 2ka'_1. \tag{2.43}$$

For $2 \le \ell \le m-1$ we argue as follows. We consider the subtree of $\check{T}$ formed by edges with ≥ 3 links; the number of 1's (respectively, 2's) in generation ℓ of $\check{T}$ equals the size of generation $\ell-1$ of the subtree times an independent $\mathrm{Bin}(d, p_1)$ (respectively, $\mathrm{Bin}(d, p_2)$) random variable. Each edge with ≥ 3 links from ρ is the root of a Galton–Watson tree of (≥ 3):s; these Galton–Watson trees have offspring distribution $\mathrm{Bin}(d, q)$, and hence on average $(dq)^r$ descendants after r steps. For $k = 1$ we get simply

$$\mathbb{E}\big[|\mathcal{E}_\ell^{(j)}|\,\big|\,A'_1\big] = (dp_j)(dq)^{\ell-2} \le 2a'_\ell. \tag{2.44}$$

For $k \ge 2$ there are $\mathrm{Bin}(k, p_{\ge 3}/p_{\ge 2})$ Galton–Watson trees to consider, hence

$$\mathbb{E}\big[|\mathcal{E}_\ell^{(j)}|\,\big|\,A'_k\big] = (k\tfrac{p_{\ge 3}}{p_{\ge 2}})(dp_j)(dq)^{\ell-2} \le 2ka'_\ell. \tag{2.45}$$

For $\ell = m$ a similar argument gives

$$\mathbb{E}\big[|\mathcal{E}_m|\,\big|\,A'_1\big] = ((1-p_0)d)(dq)^{m-2} \text{ and } \mathbb{E}\big[|\mathcal{E}_m|\,\big|\,A'_k\big] = (k\tfrac{p_{\ge 3}}{p_{\ge 2}})((1-p_0)d)(dq)^{m-2}. \tag{2.46}$$

□

Proof of Prop. 2.4. As mentioned, it is enough to establish (2.40). Using Lemmas 2.6 and 2.8 as well as the inequalities $a'_\ell \le a''_\ell$ and $\sigma_{m-1} \le \sigma_{m-2} \le \sigma_{m-3}$, we see that

$$\begin{aligned}\sum_{\ell=1}^m \sigma_{m-\ell}\mathbb{E}\big[\mathbb{1}_{(A_1\cup A_2)^c}|\mathcal{E}_\ell|\big] &= \sum_{\ell=1}^m \sigma_{m-\ell}\sum_{k=1}^d \big(\mathbb{E}\big[\mathbb{1}_{A'_k}|\mathcal{E}_\ell|\big] + \mathbb{E}\big[\mathbb{1}_{A''_k}|\mathcal{E}_\ell|\big]\big)\\ &\le 4\sum_{\ell=1}^m \sigma_{m-\ell}(a'_\ell + a''_\ell)\sum_{k=1}^d k\big(\mathbb{P}(A'_k) + \mathbb{P}(A''_k)\big)\\ &\le 16\sum_{\ell=1}^m \sigma_{m-\ell}a''_\ell \sum_{k=1}^d \big(\mathbb{P}(A'_k) + \mathbb{P}(A''_k)\big)\\ &= 16\Big(\sigma_{m-1} + \sigma_{m-2} + \sum_{\ell=3}^m \sigma_{m-\ell}(dq)^{\ell-3}\Big)\mathbb{P}((A_1\cup A_2)^c)\\ &\le 48\mathbb{P}((A_1\cup A_2)^c)\check{\sigma}_{m-1},\end{aligned} \tag{2.47}$$

for d large enough, as required. □

3. Sharpness of the transition

The arguments of Hammond [11] can straightforwardly be adapted to our setting. We thus obtain the following 'sharpness' result, which shows that (in the interval $\beta \in [d^{-1}, d^{-1} + 2d^{-2}]$) there is a unique β_c such that $\sigma(\beta) = \mathbb{P}((\rho, 0) \leftrightarrow \infty)$ satisfies $\sigma = 0$ for $\beta < \beta_c$ and $\sigma > 0$ for $\beta > \beta_c$:

Proposition 3.1. *For d large enough, the function $\beta \mapsto \sigma(\beta)$ is non-decreasing on the interval $\beta \in [d^{-1}, d^{-1} + 2d^{-2}]$.*

Sketch proof. Hammond's arguments [11] are written for the case $u = 1$ when there are only crosses, but they apply (almost verbatim) to the general case $u \in [0,1]$. We provide here a synopsis of the proof, for the reader's benefit.

The starting point is a formula for the derivative $\frac{d\sigma_n}{d\beta}$, involving the concept of 'the added link' (called the added *bar* by Hammond). In addition to the Poisson process ω of links (i.e. crosses and double-bars), let $\mathfrak{a}$ be an independently and uniformly placed link in $T^{(n)}$, which is a cross with probability u and otherwise a double-bar. Let P^+ and P^- denote the following *pivotality events*:

$$P^+ = \{(\rho,0) \overset{\omega}{\not\leftrightarrow} n, (\rho,0) \overset{\omega\cup\{\mathfrak{a}\}}{\leftrightarrow} n\}, \quad P^- = \{(\rho,0) \overset{\omega}{\leftrightarrow} n, (\rho,0) \overset{\omega\cup\{\mathfrak{a}\}}{\not\leftrightarrow} n\}. \tag{3.1}$$

In words, P^+ is the event that $\mathfrak{a}$ creates a connection to level n that was not present in ω, and P^- is the event that $\mathfrak{a}$ breaks a connection to level n. We say that $\mathfrak{a}$ is on-pivotal if P^+ happens and off-pivotal if P^- happens. Then we have [11, Lemma 1.7]:

$$\frac{d\sigma_n}{d\beta} = |\mathcal{E}_n|\big(\mathbb{P}(P^+) - \mathbb{P}(P^-)\big). \tag{3.2}$$

Here $\mathcal{E}_n$ denotes the set of edges of $T^{(n)}$.

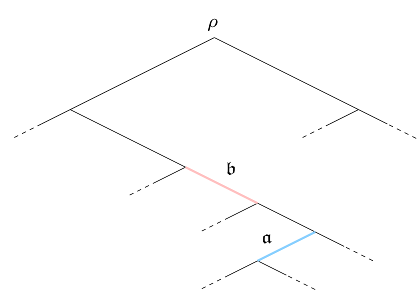

FIGURE 5. Illustration for the random link $\mathfrak{a}$ and the bottleneck-link $\mathfrak{b}$.

Hammond shows that the difference on the right-hand-side of (3.2) is positive on the interval in β considered (when d is large enough). The result then follows by letting $n \to \infty$. To show that $\mathbb{P}(P^+) - \mathbb{P}(P^-) \geq 0$, Hammond introduces the following events. Firstly, the *crossing-event* C that the loop $\mathcal{L}_{(\rho,0)}(\omega)$ of $(\rho,0)$ in ω visits an end-point of the added link $\mathfrak{a}$ before reaching level n. Note that $P^\pm \subseteq C$, since if C does not happen then the added link has no effect on whether or not $\mathcal{L}_{(\rho,0)}$ reaches level n. Secondly, the *bottleneck-event* B that some edge of $T^{(n)}$ on the (unique) path from ρ to $\mathfrak{a}$ supports only one link. On the event B, let the bottleneck-link $\mathfrak{b}$ be the furthest such link from ρ. And thirdly, the *no-escape-event* $N \subseteq B$ that the loop $\mathcal{L}_{(\rho,0)}(\omega \setminus \mathfrak{b})$ of $(\rho,0)$ in $\omega \setminus \mathfrak{b}$ does not reach level n.

Note that $P^\pm$ can be written as a disjoint union

$$P^\pm = (P^\pm \cap C \cap B^c) \cup (P^\pm \cap C \cap B \cap N). \tag{3.3}$$

Indeed, one only needs to check that $C \cap B \subseteq N$, that is, if C happens and there is a bottle-neck, then the no-escape-event happens. But if C happens and $\mathfrak{b}$ is a bottle-neck, then in $\omega \setminus \mathfrak{b}$ the loop $\mathcal{L}_{(\rho,0)}$ cannot reach level n, because if it did then it would reach level n in both ω and $\omega \cup \mathfrak{a}$ also, since $\mathfrak{b}$ is a monolink (Proposition 2.2).

Hence it suffices to provide lower bounds on the differences

$$\begin{aligned} \delta_1 &= \mathbb{P}(P^+ \cap C \cap B^c) - \mathbb{P}(P^- \cap C \cap B^c), \\ \delta_2 &= \mathbb{P}(P^+ \cap C \cap B \cap N) - \mathbb{P}(P^- \cap C \cap B \cap N). \end{aligned} \tag{3.4}$$

It is easy to give a lower bound on the first term in δ_1. Indeed, suppose the following happen: (i) in ω there is no link adjacent to ρ, (ii) $\mathfrak{a}$ is adjacent to ρ, (iii) the other endpoint of $\mathfrak{a}$ is connected by a loop to level n. Then $P^+ \cap C \cap B^c$ happens. It follows that

$$\mathbb{P}(P^+ \cap C \cap B^c) \geq (e^{-\beta})^d \frac{d}{|\mathcal{E}_n|} \sigma_{n-1}. \tag{3.5}$$

It turns out that the second term in δ_1 satisfies

$$\mathbb{P}(P^- \cap C \cap B^c) \leq c \frac{\sigma_{n-1}}{|\mathcal{E}_n|}, \tag{3.6}$$

for some constant c independent of d. The detailed argument for this is more involved, see [11, Lemma 4.5], but no changes are required compared to Hammond's original argument. Very briefly, the reason that one gets a constant factor c rather than a factor which grows with d as in (3.5) is as follows. If P^- happens, then necessarily the edge supporting $\mathfrak{a}$ also supports some link of ω: if it did not then adding $\mathfrak{a}$ would necessarily merge two loops, thereby preserving any connections to level n. If also B^c happens, i.e. there is no bottleneck, then necessarily $\mathfrak{a} \in \mathcal{M} \cup \mathcal{S}$ where $\mathcal{M}$ is the connected cluster of ρ consisting of edges which support ≥ 2 links in ω, and $\mathcal{S}$ is the set of edges that are adjacent to an edge of $\mathcal{M}$ and support exactly one link in ω. Now $\mathcal{M}$ is a very sub-critical Galton–Watson tree, and is therefore of at most constant (expected) size, and $\mathcal{S}$ is an approximately constant ($\mathrm{Bin}(d, \beta e^{-\beta})$) multiple of the number of leaves of $\mathcal{M}$, and is thus also small. Hence there is an approximately constant number of locations for $\mathfrak{a}$ which are consistent with the event $P^- \cap C \cap B^c$, giving the factor $c/|\mathcal{E}_n|$. The factor σ_{n-1} appears in (3.6) since some link of $\mathcal{S}$ is connected to level n.

Putting together (3.5) and (3.6) we obtain that, for d large enough,

$$\delta_1(n) \geq \tfrac{d}{2} (e^{-\beta})^d \frac{\sigma_{n-1}}{|\mathcal{E}_n|}. \tag{3.7}$$

Now consider the other term $\delta_2(n)$, where the bottleneck- and no-escape-events B and N happen. Since N happens, any connections to level n must occur in the subtree rooted at the bottleneck edge $\mathfrak{b}$, which is some (random) distance $n' \leq n$ from level n. Since $\mathfrak{b}$ was defined as the *furthest* bottleneck from ρ, there is no bottleneck in this subtree. We thus essentially have that $\delta_2(n) = \delta_1(n')$, so we can use the bounds on δ_1 that were already established. The only n-dependence in those bounds was in the factors $\sigma_{n-1}/|\mathcal{E}_n|$. It follows that for large enough d we certainly have $\delta_2(n) \geq 0$. Together with (3.7) and (3.2) this gives

$$\frac{d\sigma_{n-1}}{d\beta} \geq \tfrac{d}{2} (e^{-\beta})^d \sigma_{n-1} \geq 0, \tag{3.8}$$

which as explained gives the result. □

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