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Condensation and symmetry-breaking in the zero-range process with weak site disorder

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Abstract

Condensation phenomena in particle systems typically occur as one of two distinct types: either as a *spontaneous* symmetry breaking in a homogeneous system, in which particle interactions enforce condensation in a randomly located site, or as an *explicit* symmetry breaking in a system with background disorder, in which particles condensate in the site of extremal disorder. In this paper we confirm a recent conjecture by Godrèche and Luck by showing, for a zero range process with weak site disorder, that there exists a phase where condensation occurs with an intermediate type of symmetry-breaking, in which particles condensate in a site randomly chosen from a range of sites favoured by disorder. We show that this type of condensation is characterised by the occurrence of a Gamma distribution in the law of the disorder at the condensation site. We further investigate fluctuations of the condensate size and confirm a phase diagram, again conjectured by Godrèche and Luck, showing the existence of phases with normal and anomalous fluctuations.

Keywords: Condensation; Symmetry-breaking; Extended condensate; Interacting particle system; Zero-range process; Disordered system; Weak disorder; Grand canonical ensemble; Extremes; Gamma distribution; Central limit theorem

1. Motivation and background

The purpose of this paper is two-fold. The *first* purpose is to show that for certain low-dimensional particle systems far from equilibrium the simultaneous presence of inter-particle

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interactions and interactions of particles with a spatial disorder can lead to a novel form of symmetry breaking, occurring in a phase when the two competing particle forces are of comparable strength. In these systems we observe that, when the particle density exceeds a certain threshold value, the excess fraction of the particles condensates in a single site. This site is neither chosen uniformly at random (as would be the case in systems with spontaneous symmetry breaking) nor as a function of the underlying site disorder (as would be the case in systems with explicit symmetry breaking) but by a nontrivial random mechanism favouring sites with more extreme site disorder. The existence of such systems was predicted in a recent paper by Godrèche and Luck [11]. The *second* purpose of this paper is to give a further example of the ubiquity of the Gamma distribution in particle systems with condensation, which was first observed in Dereich and Mörters [5]. In our context the Gamma distribution occurs as the universal distribution of the disorder at the condensation site.

The interacting particle model under consideration here is the *zero-range process*, first introduced in the mathematical literature by Spitzer in [16]. The zero-range process has gained importance in the statistical mechanics literature, for example as a generic model for domain wall dynamics in a system far from equilibrium [14] or as a model for granular flow [7,4]. It is also a particularly simple model undergoing a condensation transition, and widely studied for this reason alone [12,8,2]. It is related to the ideal Bose gas and to spatial permutations [6]. The zero-range process has also been studied in a disordered medium, both in infinite [1] and finite [9] geometries, and the latter situation is also the context of the present paper.

Our version of the zero-range process is a continuous time Markov process, which can be described as a system of m indistinguishable particles each located in one of n different sites. Every site can hold an arbitrary number of particles. At each time instance particles move independently given the particle configuration, and the rate at which particles hop from position i to a different position j is given as $r_{ij}u_k$, where k is the number of particles at site i. Here $R = (r_{ij}\colon 1 \le i, j \le n)$ is a Q-matrix (i.e. off-diagonal entries are nonnegative and each row sums to zero) describing the unconstrained particle motion, and $(u_k\colon k \ge 0)$ is a sequence of nonnegative weights with $u_0 = 0$, that describes the particle interactions. The term zero-range process comes from the fact that, at any given time instance, the interaction is only between particles in the same site or, in other words, the jump rate above depends on the global particle configuration only through the number k of particles on the site of departure. The case $u_k = k$ corresponds to independent movement of the particles without interaction, but our interest here is mainly in sublinear sequences $(u_k\colon k \ge 0)$, in which particles move slower if they are aggregated at a site with many other particles. One such case would be that $u_k = 1$, for all $k > 0$, meaning that at every site only one particle is free to move. The phenomena of interest in this paper occur when u_k is given as a small perturbation of this case.

Assuming that the finite state Markov chain described above is irreducible, general theory insures that the state of the zero-range process converges in law, as time goes to infinity, to a unique stationary distribution, or steady state. Denoting by Q_i the number of particles located in site i this distribution is explicitly given by

$$P\big(Q_1 = q_1, \ldots, Q_n = q_n\big) = \frac{1}{Z_{m,n}} \prod_{i=1}^{n} \pi_i^{q_i} p_{q_i} \quad \text{if } q_i \ge 0 \text{ are integers with } \sum_{i=1}^{n} q_i = m,$$

where $(\pi_i\colon 1 \le i \le n)$ is a positive left eigenvector of R for the eigenvalue zero, $(p_k\colon k \ge 0)$ are derived from $(u_k\colon k \ge 0)$ by $p_0 = 1$ and $p_k = 1/u_1 \cdots u_k$, for $k \ge 1$, and $Z_{m,n}$ is the normalisation constant, or partition function. The most studied case is that of spatial homogeneity in which $(\pi_i : 1 \leq i \leq n)$ is a vector of constant (nonzero) entries. Already in this simple case the phenomenon of condensation can occur, as established in the seminal paper by Großkinsky et al. [12]. In the set-up above, the particle system above allows for general spatial inhomogeneities encoded in the Q-matrix. Following Godrèche and Luck [11] in this point, we now simplify the analysis by focusing on relatively simple spatial inhomogeneities, which are chosen to display the full richness of possible behaviour. To this end we replace the invariant measure of a single particle motion $(\pi_i : 1 \leq i \leq n)$ by a random environment given as a product of a random site disorder. More precisely, we are assuming that $\pi_i = X_i$, for $1 \leq i \leq n$, where $(X_i : i \in \mathbb{N})$ is a sequence of independent, identically distributed random variables. We think of X_i as the fitness of site i, where fitter sites are a more attractive host for particles. One of many possible dynamics that give rise to this stationary behaviour is if sites are arranged as a circle, and particles located at site i with occupancy k hop clockwise to their nearest neighbour with rate u_k/X_i. As our results can be expressed in terms of the stationary distribution without explicit reference to any particle dynamics, we do not have to make explicit reference to the particle dynamics or the Q-matrix underlying our random environment. While this approach enables a rigorous mathematical analysis of the key phenomena, its downside is that our results contain no direct information about the kinetics of the zero-range process.

Our results on this model take the form of limit results where n, the number of sites, and m, the number of particles, go to infinity so that the ratio m/n converges to a fixed density $\rho > 0$. We assume that the random variable X determining the site fitness is bounded from above, without loss of generality by the value 1, and that its distribution function is regularly varying at 1 with index γ, for some $\gamma > 0$. The sequence $(p_k : k \geq 0)$ is assumed to be regularly varying with index $-\beta$, for some $\beta > 1$. The phase diagrams we identify in our main results will be given in terms of the parameters β and γ.

We first show in Theorem 2.1 that if $\beta + \gamma > 2$, there exists a positive and finite critical density $\rho^\star$ such that if $\rho > \rho^\star$, with probability going to one, there exists a unique site carrying a positive fraction of the particles. This fraction converges to $\rho - \rho^\star > 0$. This is the phenomenon of *condensation*.

If condensation occurs, we ask

(1) At which site does the condensation occur?
(2) What is the fitness of the site at which condensation occurs?
(3) How does the condensate fraction fluctuate around the limit $\rho - \rho^\star$?

Our main results answer these three questions. In Theorem 2.2 we address the first question. We show that in the case $\gamma > 1$, condensation occurs at the site with highest fitness value, revealing a case of *explicit* symmetry breaking. If $\gamma \leq 1$ and $\beta + \gamma > 2$ however, with high probability, condensation occurs at a site chosen from a range of sites with high fitness. We describe the non-degenerate limiting distribution for the rank order of the condensation site. This result establishes the novel phenomenon of *intermediate* symmetry breaking conjectured by Godrèche and Luck [11]. The second question is addressed in Theorem 2.3, where we show that in the phase of intermediate symmetry breaking the fitness of the condensation site satisfies a universal limit theorem. In fact, regardless of the underlying fitness distribution, the disorder of the condensation site converges, appropriately scaled, to a Gamma distribution. Recall that the Gamma distribution is not a classical extreme value distribution, so that its occurrence in this context may be considered surprising. In Theorem 2.4 we address the third question by studying the quenched fluctuations in the size of the condensate in the case $\gamma < 1$ of weak disorder. We show that, if $\beta + \gamma \geq 3$, the fluctuations around a disorder dependent finite size approximation of the limiting value $\rho - \rho^\star$ are normal. In contrast to this, if $2 < \beta + \gamma < 3$, the fluctuations are stable with index $\beta + \gamma - 1$. In the (easier) annealed setup such a behaviour was also conjectured by Godrèche and Luck [11].

Our proofs are mainly based on a careful analysis of a *grand-canonical ensemble*, a sequence of independent but not identically distributed random variables $Q_1, Q_2, \dots$ with the law of Q_i given explicitly in terms of the fitness X_i. Conditioning on the event $Q_1 + \cdots + Q_n = m$ we obtain the distribution of site occupancies in the stationary zero range model with m particles and n sites, often referred to as the *canonical ensemble*. Although the behaviour of the ensembles is radically different in the case of condensation, the key idea is still to derive properties of the canonical ensemble from much more accessible properties of the grand-canonical ensemble. For example, we show that the number of particles outside the condensation site in the canonical ensemble is well-approximated by the sum $Q_1 + \cdots + Q_n$ of independent random variables in the grand-canonical ensemble. The latter quantity can then be studied by classical means. This technique is inspired by ideas of Janson [13] for a model without disorder. Adaptation of these ideas to the study of disordered systems is the main technical innovation of this paper.

Notation: The symbol $\mathtt{cst}$ stands for a positive constant which may change its value at every appearance. Given two sequences $(u_n)_{n\geq 1}$ and $(v_n)_{n\geq 1}$, we write $u_n \sim v_n$ if ${u_n}/{v_n} \to 1$. We write $u_n = o(v_n)$, or $u_n \ll v_n$, if ${u_n}/{v_n} \to 0$. We use the symbol $u_n = O(v_n)$ if there exists $c > 0$ such that $|u_n| \leq c\,|v_n|$ for all sufficiently large n, and indicate by $O_{\mathbb{P}}$ if the implied constant c is allowed to be a random variable under $\mathbb{P}$. We write $u_n = \Theta(v_n)$ if both $u_n = O(v_n)$ and $v_n = O(u_n)$ hold. Finally, given a sequence $\delta_n \to 0$ and a function f, we write $u_n \approx f(v_n \pm \delta_n)$ if $f(v_n - \delta_n) \leq u_n \leq f(v_n + \delta_n)$ for all sufficiently large n.

2. Statement of the main results

Let μ be a probability distribution on $[0, 1]$ satisfying, for some $\gamma > 0$,

$$\mu([1-x, 1]) \sim \alpha_1\, x^\gamma, \quad \text{when } x \downarrow 0, \tag{RVμ}$$

and $(p_k)_{k\geq 0}$ a probability distribution on $\mathbb{N}_0 := \{0, 1, 2, \dots\}$ such that, for some $\beta > 1$,

$$p_k \sim \alpha_2\, k^{-\beta}, \quad \text{as } k \uparrow \infty. \tag{RVp}$$

We believe that all our results, except the fluctuation result at the end of this section, hold *mutatis mutandis* if the positive constants α_1, α_2 were replaced by slowly varying functions. This would require a greater technical effort, which would not help the understanding of the phenomena we are interested in, and would be detrimental to the readability of the proofs.

We always assume, without loss of generality, that $p_0 > 0$. Denote by $\Phi\colon [0, 1] \to [0, 1]$ the generating function of the distribution $(p_k)_{k\geq 0}$, given by

$$\Phi(z) = \sum_{k=0}^{\infty} p_k z^k,$$

and define the critical density

$$\rho^\star := \int \frac{x\, \Phi'(x)}{\Phi(x)}\, \mu(dx).$$

The random disorder in our model is given by an i.i.d. sequence $\boldsymbol{X} = (X_i : i \in \mathbb{N})$ of random variables with distribution μ. Given the disorder, the stationary distribution $P_X^{m,n} = P_X$ of the disordered zero-range process is given by

$$P_X(Q_1 = q_1, \dots, Q_n = q_n) = \frac{1}{Z_{m,n}} \prod_{i=1}^{n} X_i^{q_i} p_{q_i} \mathbb{1}\{q_1 + \cdots + q_n = m\} \quad \text{for all } q_1, \dots, q_n \in \mathbb{N}_0, \tag{1}$$

where $Z_{m,n}$ is the normalisation constant. We write P_X for the 'quenched' law of $(Q_1, \dots, Q_n)$ given X and P, E for the law and expectation of the disorder X. By $P_{m,n}$ we denote the joint law of $(X_1, \dots, X_n)$ and $(Q_1, \dots, Q_n)$. We assume throughout the article that $\rho_n := {}^{m}/_{n} \to \rho > 0$ when n tends to infinity.

Let $(Q_n^{(1)}, \dots, Q_n^{(n)})$ be the order statistics of $(Q_1, \dots, Q_n)$. Our first result shows that in the condensation regime $\beta + \gamma > 2$, if the particle density ρ exceeds the critical value $\rho^\star$, the excess particles form a condensate of macroscopic occupancy in exactly one site.

Theorem 2.1 (*Condensation*)**.** *Suppose $\beta + \gamma > 2$. Then $\rho^\star < \infty$ and if $\rho > \rho^\star$ then, with high $P_{m,n}$-probability,*

$$Q_n^{(1)} = (\rho - \rho^\star)n + o(n) \quad \text{and} \quad Q_n^{(2)} = o(n).$$

The following two theorems show that in the case $\gamma < 1$ the condensate does not normally sit in the site with the largest fitness. This is called the 'extended condensate case' by Godrèche and Luck, but we prefer the term intermediate symmetry-breaking to emphasise that the condensate is still located at a single site and not extended over several sites. We say that a sequence of random variables $(Z_n)_{n\in\mathbb{N}}$ **converges in quenched distribution** to the random variable Z if, for all $\varepsilon > 0$ and all $u \in \mathbb{R}$,

$$\mathrm{P}\left(\left|P_X^{m,n}(Z_n \le u) - P_X^{m,n}(Z \le u)\right| > \varepsilon\right) \to 0, \quad \text{when } n \uparrow \infty. \tag{2}$$

We denote by I_n the index of the site of maximal occupancy, so that $Q_{I_n} = Q_n^{(1)}$. By Theorem 2.1 this eventually defines I_n uniquely in the condensation regime. We further let K_n be the rank order of the fitness of the condensation site, i.e. $K_n = k$ if and only if

$$\left|\left\{i \in \{1, \dots, n\} \colon X_i > X_{I_n}\right\}\right| = k - 1.$$

Recall that the density of a Gamma distributed random variable with parameters (γ, λ) is given by

$$p(x) = \frac{\lambda^\gamma}{\Gamma(\gamma)} x^{\gamma-1} \mathrm{e}^{-\lambda x} \quad \text{for } x \ge 0.$$

Theorem 2.2 (*Fitness Rank of the Condensate*)**.**

(i) *If $\gamma > 1$ and $\rho > \rho^\star$, then with high $P_{m,n}$-probability we have $K_n = 1$.*
(ii) *If $\gamma < 1$, $\beta + \gamma > 2$ and $\rho > \rho^\star$, then*

$$\left(n^{\gamma-1} K_n\right)^{1/\gamma} \to K$$

in quenched distribution, where K is a Gamma distributed random variable of parameters $(\gamma, \frac{\rho-\rho^\star}{\alpha_1^{1/\gamma}})$.

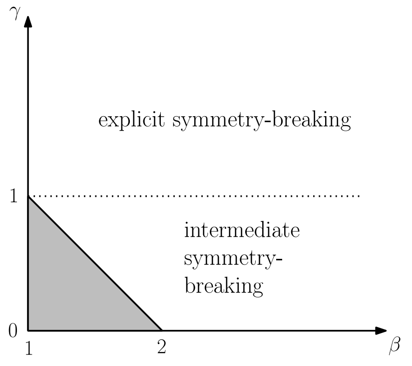

Fig. 1. This phase diagram shows the behaviour of the disordered zero-range process according to its two parameters $\beta > 1$ and $\gamma > 0$. The grey part is a zone where there is no condensation, there is condensation in the white part as soon as $\rho > \rho^\star$. The difference between explicit and intermediate symmetry-breaking is explained in Theorem 2.2 and the remark after it.

Note that the two phases described in Theorem 2.2 are both condensation phases, in case (i) explicit symmetry breaking occurs, while in case (ii) there is intermediate symmetry breaking. Fig. 1 illustrates the phase diagram established in Theorem 2.2. The next theorem gives the universal law of the fitness of the condensate.

Theorem 2.3 (*Fitness of the Condensate*)**.** *If $\gamma < 1$, $\beta+\gamma > 2$ and $\rho > \rho^\star$, denote by $F_n = X_{I_n}$ the fitness at the condensation site. Then*

$$n(1 - F_n) \to F$$

in quenched distribution, where F is a Gamma distributed random variable with parameters $(\gamma, \rho - \rho^\star)$.

Finally, we have very precise results about the asymptotic behaviour of the size of the condensate in the case of intermediate symmetry-breaking. We define random variables

$$\nu_n := \frac{1}{n} \sum_{i=1}^{n} \frac{X_i \Phi'(X_i)}{\Phi(X_i)},$$

and note that $\mathtt{E}\nu_n = \rho^\star$. The first order estimate of $Q_n^{(1)}$ given the disorder is $m - \nu_n n$, which divided by n converges in $\mathtt{P}$-probability to $\rho - \rho^\star$. The following theorem describes the fluctuations of $Q_n^{(1)}$ around the value $m - \nu_n n$ (see Fig. 2).

Theorem 2.4 (*Quenched Fluctuations of the Condensate*)**.** *Assume that $\gamma < 1$.*

(i) *If $2 < \beta + \gamma < 3$ and $\rho > \rho^\star$, let $\kappa = \frac{1}{\beta+\gamma-1}$. Then,*

$$\frac{Q_n^{(1)} - m + \nu_n n}{n^\kappa} \to W_\kappa$$

in quenched distribution, where W_κ is a $1/\kappa$-stable random variable.

(ii) *If $\beta + \gamma \geq 3$ and $\rho > \rho^\star$, then*

$$\frac{Q_n^{(1)} - m + \nu_n n}{\sqrt{n}} \to W$$

in quenched distribution, where W is a normal random variable.

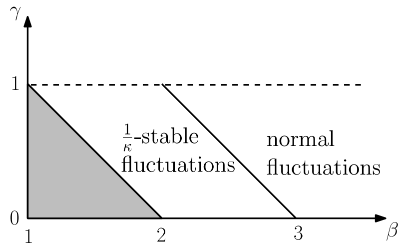

Fig. 2. This phase diagram shows the fluctuations of the size of the condensate according to the values of the two parameters β and γ.

Remark. Note that the *quenched* fluctuation result gives information on the size of the condensate for fixed instances of the disorder and is much more subtle than the *averaged* fluctuation results that would allow averaging over the disorder. Averaged fluctuations are centred around $(\rho - \rho^\star)n$ and hold without the restriction $\gamma < 1$, the distinction of the normal and anomalous regime persists in this situation, as predicted by Godrèche and Luck.

Remark. We discuss the fluctuations in the strong disorder case $\gamma > 1$ in Section 7.

The following four sections are devoted to the proofs of our main theorems. Section 3 presents the grand canonical framework used in our proofs. It contains a fairly standard technical proof of a central limit theorem for independent random variables that may be skipped on first reading. Section 4 contains the proof of condensation, i.e. of Theorem 2.1. Section 5 is devoted to intermediate symmetry-breaking and contains the proofs of Theorems 2.2 and 2.3. Section 6 deals with fluctuations, this is where Theorem 2.4 is proved. We list some interesting open problems in Section 7, and in the Appendix we collect general results on the limit behaviour of the fitnesses, which are used throughout the paper. As results on i.i.d. random variables regularly varying near their essential supremum are difficult to find in the literature, this may be of independent interest.

3. The grand canonical ensemble

Given the sequence $X_1, X_2, \ldots$ of random variables with distribution μ we now define another model, the *grand canonical ensemble*, as the sequence $Q_1, Q_2, \ldots$ of conditionally independent random variables with the law of Q_i given by

$$\mathbb{P}_X(Q_i = k) = \frac{p_k X_i^k}{\Phi(X_i)}. \tag{3}$$

Given positive integers n, m we can recover $P_{m,n}$ as the law of $(Q_1, \ldots, Q_n, X_1, \ldots, X_n)$ conditioned on the event $\{Q_1 + \cdots + Q_n = m\}$. In this framework the random variables ν_n can be described as

$$\nu_n = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_X Q_i.$$

We now show that the sequence $(\nu_n)_{n\in\mathbb{N}}$ satisfies a law of large numbers.

Lemma 3.1 (*Natural Density*). *If* $\beta + \gamma > 2$, *then* P-*almost surely* $\nu_n \to \rho^\star < \infty$.

Proof. Denote $G(x) := \frac{x\Phi'(x)}{\Phi(x)}$, for all $x \in [0, 1]$. We show that $G(X)$ is integrable, so that the result follows from an application of Kolmogorov's law of large numbers. In the case $\beta > 2$, we have that G is bounded and hence integrable. In the case $2 - \gamma < \beta < 2$, we can use integral bounds to get $\Phi'(x) = \Theta\left((1-x)^{\beta-2}\right)$, so that $G(x) = \Theta((1-x)^{\beta-2})$, as $x \uparrow 1$. Letting $G^{-1}(u) = \inf\{x : G(x) > u\}$, we have $\mathtt{P}(G(X) > u) \le \mathtt{P}(X \ge G^{-1}(u))$. Observe that $G^{-1}(u) \uparrow 1$, as $u \uparrow \infty$, which tells us in view of (RVμ) that $\mathtt{P}(X \ge G^{-1}(u)) \sim \alpha_1(1-G^{-1}(u))^\gamma$, as $u \uparrow \infty$. As $1 - G^{-1}(u) = \Theta(u^{\frac{1}{\beta-2}})$, we obtain $\mathtt{P}(G(X) > u) = O(u^{-\frac{\gamma}{2-\beta}})$. Integrability follows since $\frac{\gamma}{2-\beta} > 1$. In the case $\beta = 2$, we have $\Phi'(x) \sim -\log(1-x)$ and integrability follows using a similar argument as above. □

Limit theorems for the independent (but not identically distributed) random variables $(Q_i)_{i\ge1}$ under $\mathbb{P}_X$ are nontrivial, but can be obtained by classical methods. We abbreviate the partial sums as

$$S_n := \sum_{i=1}^{n} Q_i.$$

Lemma 3.2 (*Grand Canonical Law of Large Numbers*)**.** *If $\beta + \gamma > 2$, then $\frac{1}{n} S_n - \nu_n \to 0$ in $\mathbb{P}_X$-probability.*

Combining Lemmas 3.1 and 3.2 we see that, if $\rho > \rho^\star$, the probability $\mathbb{P}_X(S_n = m)$ is going to zero as $n \to \infty$. We shall see later[1] that, with high $\mathtt{P}$-probability, this decay is polynomial if $\gamma \le 1$, but stretched exponential if $\gamma > 1$.

The law of large numbers, Lemma 3.2, follows from the central limit theorem for the grand canonical ensemble, which we now state. The central limit theorem for the grand canonical ensemble prepares the proof of Theorem 2.4 for the canonical ensemble. The proof is a direct application of classical techniques for independent (but not identically distributed) random variables, and may be omitted on first reading.

Proposition 3.3 (*Grand Canonical Central Limit Theorem*)**.**

(i) *If $2 < \beta + \gamma < 3$, let $\kappa = \frac{1}{\beta+\gamma-1}$. Then, in quenched distribution,*[2]

$$\frac{\sum_{i=1}^{n} Q_i - \nu_n n}{n^\kappa} \to W_\kappa,$$

where W_κ is a $1/\kappa$-stable random variable.

(ii) *If $\beta + \gamma \ge 3$, then, in quenched distribution,*

$$\frac{\sum_{i=1}^{n} Q_i - \nu_n n}{\sqrt{n}} \to W,$$

where W is a Gaussian random variable.

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[1] See, in particular, Lemma 4.3.

[2] To define convergence in quenched distribution in the grand-canonical framework, one has to replace P_X by $\mathbb{P}_X$ in (2).

Proof of Proposition 3.3. (ii) This is a direct application of the central limit theorem for sums of independent but non identical random variables based on Lindeberg's condition; that is, it is sufficient to show that for all $\varepsilon > 0$, we have

$$\lim_{n\uparrow\infty} \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\left[(Q_i - \mathbb{E}_X Q_i)^2 \mathbb{1}\{|Q_i - \mathbb{E}_X Q_i| > \varepsilon\sqrt{n}\}\right] = 0.$$

Recall that $\mathbb{E}_X Q_i = \frac{X_i \Phi'(X_i)}{\Phi(X_i)} = G(X_i)$ and that, if $\beta > 2$, the function $G(x) = \frac{x\Phi'(x)}{\Phi(x)}$ is bounded on [0, 1], behaves as $O((1-x)^{\beta-2})$ if $\beta < 2$ and as $O(-\log(1-x))$ if $\beta = 2$. Therefore, in view of Lemma A.1 and using that $\beta + \gamma \geq 3$ and $\beta > 1$, we have $\max_{i=1}^{n} G(X_i) = o(\sqrt{n})$ in $\mathtt{P}$-probability. Therefore, for all large enough n,

$$\begin{aligned}\sum_{i=1}^{n} \mathbb{E}\left[(Q_i - \mathbb{E}_X Q_i)^2 \mathbb{1}\{|Q_i - \mathbb{E}_X Q_i| > \varepsilon\sqrt{n}\}\right] &= \sum_{i=1}^{n} \sum_{k=\mathbb{E}_X Q_i + \varepsilon\sqrt{n}}^{\infty} \frac{p_k X_i^k}{\Phi(X_i)}(k - \mathbb{E}_X Q_i)^2 \\ &\leq \mathtt{cst.} \sum_{k=\varepsilon\sqrt{n}}^{\infty} k^{2-\beta} \sum_{i=1}^{n} X_i^k.\end{aligned}$$

First note that assuming $\beta > 3$ leads to

$$\frac{1}{n}\sum_{k=\varepsilon\sqrt{n}}^{\infty} k^{2-\beta}\sum_{i=1}^{n} X_i^k \leq \sum_{k=\varepsilon\sqrt{n}}^{\infty} k^{2-\beta} \to 0,$$

and hence Lindeberg's condition is verified. We may assume now that $\beta \leq 3$ and write

$$\begin{aligned}\frac{1}{n}\sum_{k=\varepsilon\sqrt{n}}^{\infty} k^{2-\beta}\sum_{i=1}^{n} X_i^k &= \frac{1}{n}\sum_{k=\varepsilon\sqrt{n}}^{\frac{n^{1/\gamma}}{\log n}} k^{2-\beta}\sum_{i=1}^{n} X_i^k + \frac{1}{n}\sum_{k=\frac{n^{1/\gamma}}{\log n}}^{n^{1/\gamma}\log^2 n} k^{2-\beta}\sum_{i=1}^{n} X_i^k \\ &\quad + \frac{1}{n}\sum_{k=n^{1/\gamma}\log^2 n}^{\infty} k^{2-\beta}\sum_{i=1}^{n} X_i^k,\end{aligned}$$

where the first and second term on the right are void if $\gamma > 2$. Applying Lemma A.3(ii) allows to bound the inner sum of the first term by a constant multiple of $nk^{-\gamma}$, showing that the term tends to zero because $\beta + \gamma > 3$. The second term is bounded from above by (we assume here that $\beta < 3$, the case $\beta = 3$ can be treated similarly)

$$\frac{1}{n}\sum_{i=1}^{n} X_i^{\frac{n^{1/\gamma}}{\log n}} \sum_{k=1}^{n^{1/\gamma}\log^2 n} k^{2-\beta} \sim \frac{\log^\gamma n}{n}\, n^{\frac{3-\beta}{\gamma}} \log^{2(3-\beta)} n,$$

using Lemma A.3(ii) applied to $s_n = \frac{n^{1/\gamma}}{\log n}$. Hence the second term also tends to zero as $n \uparrow \infty$. Finally, the third term is, by Lemmas A.3(i), and A.1, asymptotically bounded by

$$\begin{aligned}\frac{1}{n}\sum_{k=n^{1/\gamma}\log^2 n}^{\infty} k^{2-\beta}\big(X_n^{(1)}\big)^k V_k^{(n)} &\leq \mathtt{cst.}\frac{1}{n}\int_{\mathtt{cst.}n^{1/\gamma}\log^2 n}^{\infty} x^{2-\beta} \mathrm{e}^{-xn^{-1/\gamma}}\,dx \\ &\leq \mathtt{cst.}n^{\frac{3-\beta}{\gamma}-1}\int_{\log^2 n}^{\infty} u^{2-\beta}\mathrm{e}^{-u}\,du,\end{aligned}$$

which also goes to zero, because $\beta + \gamma \geq 3$. Therefore, Lindeberg's condition is verified concluding the proof of (ii). Note that the variance of the limit normal distribution is given by $\mathtt{E}\,\mathtt{Var}_X Q_i$.

(i) We apply the very general [10, Section 25, Theorem 2]. Using this it is enough to show that, asymptotically as $n \uparrow \infty$, there are constants $C_1, C_2 \geq 0$ such that

$$\sum_{i=1}^{n} \mathbb{P}_X\left(Q_i - \mathbb{E}_X Q_i \geq x n^{\kappa}\right) \to \frac{C_1}{x^{1/\kappa}}, \quad \text{for all } x > 0, \tag{4}$$

$$\sum_{i=1}^{n} \mathbb{P}_X\left(Q_i - \mathbb{E}_X Q_i \leq x n^{\kappa}\right) \to \frac{C_2}{|x|^{1/\kappa}}, \quad \text{for all } x < 0, \tag{5}$$

$$\lim_{\varepsilon\downarrow 0} \limsup_{n\uparrow\infty} \frac{1}{n^{2\kappa}} \sum_{i=1}^{n} \mathtt{Var}_X\big((Q_i - \mathbb{E}_X Q_i)\mathbb{1}\{|Q_i - \mathbb{E}_X Q_i| < \varepsilon n^{\kappa}\}\big) = 0. \tag{6}$$

First remark that, as above, we have $\sup_{i=1}^n \mathbb{E}_X Q_i = o(n^{\kappa})$. Hence, (5) is (trivially) verified with $C_2 = 0$. Now recall that $p_k \sim \alpha_2 k^{-\beta}$ when k tends to infinity. Thus, for all $\varepsilon > 0$ there exists an integer $k(\varepsilon)$ such that, for all $k \geq k(\varepsilon)$, we have $p_k \approx (1 \pm \varepsilon)\alpha_2 k^{-\beta}$. Choose n such that $xn^{\kappa} > k(\varepsilon)$ and such that $\sup_{i=1}^n \mathbb{E}_X Q_i \leq x n^{\kappa}$. Then

$$\begin{aligned}\sum_{i=1}^{n} \mathbb{P}_X\left(Q_i - \mathbb{E}_X Q_i \geq x n^{\kappa}\right) &\approx \alpha_2(1\pm\varepsilon)\sum_{i=1}^{n}\sum_{k\geq xn^{\kappa}+\mathbb{E}_X Q_i} k^{-\beta}\frac{X_i^k}{\Phi(X_i)}\\ &\approx \alpha_2(1\pm\varepsilon)\sum_{i=1}^{n}\sum_{k\geq xn^{\kappa}} (k+\mathbb{E}_X Q_i)^{-\beta}\frac{X_i^{k+\mathbb{E}_X Q_i}}{\Phi(X_i)}.\end{aligned} \tag{7}$$

To show that $(k + \mathbb{E}_X Q_i)^{-\beta} \approx (1 \pm \varepsilon)k^{-\beta}$ for all $k \geq xn^{\kappa}$, for all $i \in \{1, \ldots, n\}$, and large enough n, note that

$$k^{-\beta}\left(1 + \frac{\sup_{i=1..n} \mathbb{E}_X Q_i}{xn^{\kappa}}\right)^{-\beta} \leq (k + \mathbb{E}_X Q_i)^{-\beta} \leq k^{-\beta},$$

and use that $\sup_{i=1}^n \mathbb{E}_X Q_i = o(n^{\kappa})$. For all $i \in \{1, \ldots, n\}$, we bound $X_i^{\mathbb{E}_X Q_i}$ from above and below by

$$X_i^{\sup_{i=1..n} \mathbb{E}_X Q_i} \leq X_i^{\mathbb{E}_X Q_i} \leq 1.$$

Plugging these bounds into (7) we get the following lower and upper bound for $\sum_{i=1}^n \mathbb{P}_X(Q_i - \mathbb{E}_X Q_i \geq xn^{\kappa})$ with $\sigma_n := \sup_{i=1..n} \mathbb{E}_X Q_i$ in the lower bound and $\sigma_n := 0$ in the upper bound,

$$\begin{aligned}&\alpha_2(1\pm\varepsilon)^2\sum_{i=1}^{n}\sum_{k\geq xn^{\kappa}} k^{-\beta}\frac{X_i^{k+\sigma_n}}{\Phi(X_i)}\\ &\quad\approx \alpha_2(1\pm\varepsilon)^2\sum_{k\geq xn^{\kappa}} k^{-\beta}\sum_{i=1}^{n}\frac{X_i^{k+\sigma_n}}{\Phi(X_i)}\end{aligned}$$

$$\approx \alpha_2(1\pm\varepsilon)^2\Bigg(\sum_{k=xn^\kappa}^{\frac{n^{1/\gamma}}{\log n}} nk^{-\beta-\gamma}U^{(n)}_{k+\sigma_n} + \sum_{k=\frac{n^{1/\gamma}}{\log n}}^{n^{1/\gamma}\log n} k^{-\beta}\sum_{i=1}^n \frac{X_i^{k+\sigma_n}}{\Phi(X_i)}$$
$$+ \sum_{k=n^{1/\gamma}\log n}^{\infty} k^{-\beta}\left(X_n^{(1)}\right)^{k+\sigma_n} V^{(n)}_{k+\sigma_n}\Bigg),$$

using Lemma A.3 notations. Using that $\sigma_n = o(n^{1/\gamma}/\log n)$ it can be checked easily that the second and third terms are $o(1)$–terms, independent of x. Thus only the first term of the above sum needs to be considered. Note that there exists two integers $m_n, M_n \in [xn^\kappa, n^{1/\gamma}/\log n]$ such that $U^{(n)}_{m_n} \le U^{(n)}_k \le U^{(n)}_{M_n}$, for all $n \ge 1$ and $k \in [xn^\kappa, n^{1/\gamma}/\log n]$. In view of Lemma A.3(ii), we have $U^{(n)}_{M_n} \sim U^{(n)}_{m_n} \sim \alpha_1\Gamma(1+\gamma)$ as $n\to\infty$. Thus,

$$\sum_{k=xn^\kappa}^{\frac{n^{1/\gamma}}{\log n}} nk^{-\beta-\gamma}U^{(n)}_{m_n} \le \sum_{k=xn^\kappa}^{\frac{n^{1/\gamma}}{\log n}} nk^{-\beta-\gamma}U^{(n)}_{k} \le \sum_{k=xn^\kappa}^{\frac{n^{1/\gamma}}{\log n}} nk^{-\beta-\gamma}U^{(n)}_{M_n},$$

both bounds being then equivalent to $\alpha_1\Gamma(\gamma+1)n(xn^\kappa)^{1-\beta-\gamma} \sim \alpha_1\Gamma(\gamma+1)x^{-1/\kappa}$ when n tends to infinity. We eventually get that, for all n large enough,

$$\sum_{i=1}^n \mathbb{P}_X\left(Q_i - \mathbb{E}_X Q_i \ge xn^\kappa\right) \approx \frac{\alpha_1\alpha_2\Gamma(\gamma+1)(1\pm\varepsilon)^2}{x^{1/\kappa}},$$

which implies (4) with $C_1 := \alpha_1\alpha_2\Gamma(\gamma+1)$. Finally, for all large enough n,

$$\frac{1}{n^{2\kappa}}\sum_{i=1}^n \mathtt{Var}_X\big((Q_i-\mathbb{E}_XQ_i)\mathbb{1}\{|Q_i-\mathbb{E}_XQ_i|<\varepsilon n^\kappa\}\big)$$
$$\le \mathtt{cst}.n^{-2\kappa}\sum_{i=1}^n\sum_{k\le 2\varepsilon n^\kappa}(k-\mathbb{E}_XQ_i)^2k^{-\beta}X_i^k$$
$$\le \mathtt{cst}.n^{-2\kappa}\sum_{i=1}^n\sum_{k\le \mathbb{E}_XQ_i}(\mathbb{E}_XQ_i)^2k^{-\beta}X_i^k + \mathtt{cst}.n^{-2\kappa}\sum_{k=0}^{2\varepsilon n^\kappa}k^{2-\beta}\sum_{i=1}^n X_i^k$$
$$\le \mathtt{cst}.n^{-2\kappa}\sum_{i=1}^n G(X_i)^2 + \mathtt{cst}.n^{1-2\kappa}\sum_{k=0}^{2\varepsilon n^\kappa}k^{2-\beta-\gamma},$$

in view of Lemma A.3(ii) and (iii). Recall that G is bounded if $\beta > 2$, has exponential tails if $\beta = 2$, and has tails of polynomial order $-\frac{\gamma}{2-\beta}$ if $\beta < 2$. Hence $\sum_{i=1}^n G(X_i)$ is $O(n)$ if $\gamma > 2(2-\beta)$, and $O_{\mathrm{P}}(n^{2(2-\beta)/\gamma})$ otherwise. From this we derive that the first term above goes to zero as n goes to infinity. Moreover, the second term is a constant multiple of $\varepsilon^{3-\beta-\gamma}$, which verifies (6) and completes the proof of (i). □

4. The condensation effect

In this section we not only prove Theorem 2.1 but also provide crucial information about the position of the condensate, which will enter into the proofs of our main theorems.

We choose $\delta_n \downarrow 0$ such that $\mathbb{P}_X(|S_n - n\nu_n| \leq \frac{1}{2}n\delta_n) \to 1$, in P-probability. With $\kappa = \max\{\frac{1}{2}, \frac{1}{\beta+\gamma-1}\}$ we can achieve this for a sequence satisfying $n^\kappa \ll n\delta_n$. If $1 < \gamma < 2$ we make the stronger assumption that $n^{1/\gamma} \ll n\delta_n$. We assume $\beta + \gamma > 2$, $\rho > \rho^\star$ and fix $\varepsilon > 0$ such that $\varepsilon < \frac{\beta+\gamma-2}{\beta+\gamma}(\rho - \rho^\star)$ if $\gamma \leq 1$, and $\varepsilon < \frac{\beta-1}{\beta\gamma}(\rho - \rho^\star)$ if $\gamma > 1$.

We partition the event $\{S_n = m\}$ into four disjoint events,

$$\begin{aligned}
\mathcal{E}_1 &= \{S_n = m, \exists i \in \{1, \dots, n\} \text{ such that } |Q_i - (m - \nu_n n)| \leq \delta_n n, \text{ and } \forall j \neq i,\ Q_j \leq \varepsilon n\},\\
\mathcal{E}_2 &= \{S_n = m, \exists i \neq j \in \{1, \dots, n\} \text{ such that } |Q_i - (m - \nu_n n)| \leq \delta_n n \text{ and } Q_j > \varepsilon n\},\\
\mathcal{E}_3 &= \{S_n = m, \forall i \in \{1, \dots, n\}, |Q_i - (m - \nu_n n)| > \delta_n n \text{ and } \exists j \in \{1, \dots, n\}\\
&\qquad \text{such that } Q_j > \varepsilon n\},\\
\mathcal{E}_4 &= \{S_n = m \text{ and, for all } i \in \{1, \dots, n\},\ Q_i \leq \varepsilon n\}.
\end{aligned}$$

The idea is to prove that, asymptotically as n tends to infinity, $\mathcal{E}_1$ is the dominating event. We further define the following events, for all $i, j \in \{1, \dots, n\}$,

$$\begin{aligned}
\mathcal{E}_{1,i} &= \{S_n = m,\ |Q_i - (m - \nu_n n)| \leq \delta_n n \text{ and } Q_j \leq \varepsilon n \text{ for all } j \neq i\},\\
\mathcal{E}^*_{1,i} &= \{S_n = m \text{ and } |Q_i - (m - \nu_n n)| \leq \delta_n n\},\\
\mathcal{E}^*_{3,i} &= \{S_n = m,\ |Q_i - (m - \nu_n n)| > \delta_n n \text{ and } Q_i > \varepsilon n\},\\
\mathcal{D}_{i,j} &= \{S_n = m,\ |Q_i - (m - \nu_n n)| \leq \delta_n n \text{ and } Q_j > \varepsilon n\}.
\end{aligned}$$

Recall that $u_n \approx f(\nu_n, \mp\delta_n)$ means that $f(\nu_n, \delta_n) \leq u_n \leq f(\nu_n, -\delta_n)$ for all sufficiently large n.

Lemma 4.1. *For all $i \in \{1, \dots, n\}$, with high* P*-probability,*

$$\mathbb{P}_X(\mathcal{E}^*_{1,i}) \approx \alpha_2(\rho - \rho^\star)^{-\beta} n^{-\beta} \frac{X_i^{(\rho_n - \nu_n \mp \delta_n)n}}{\Phi(X_i)}(1 + o(1)),$$

with an error $o(1)$ which is uniform in i.

Proof. For all $i \in \{1, \dots, n\}$, we denote $S^{(i)}_{n-1} = \sum_{\substack{j=1\\ j\neq i}}^n Q_j$. Hence

$$\begin{aligned}
\mathbb{P}_X(\mathcal{E}^*_{1,i}) &= \sum_{k:|k-(m-\nu_n n)|\leq\delta_n n} \mathbb{P}_X(Q_i = k \text{ and } S_n = m)\\
&= \sum_{k:|k-(m-\nu_n n)|\leq\delta_n n} \mathbb{P}_X(Q_i = k)\mathbb{P}_X\Big(\sum_{\substack{j=1\\ j\neq i}}^n Q_j = m - k\Big)\\
&= \sum_{k:|k-(m-\nu_n n)|\leq\delta_n n} \frac{p_k X_i^k}{\Phi(X_i)} \mathbb{P}_X(S^{(i)}_{n-1} = m - k).
\end{aligned}$$

For all integers k such that $|k - (m - \nu_n n)| \leq \delta_n n$, we have $p_k \sim \alpha_2(m - \nu_n n)^{-\beta}$ as $n \uparrow \infty$. Thus,

$$\begin{aligned}
\mathbb{P}_X(\mathcal{E}^*_{1,i}) &= \sum_{k:|k-(m-\nu_n)n|\leq\delta_n n} \alpha_2(m - \nu_n n)^{-\beta} \frac{X_i^k}{\Phi(X_i)} \mathbb{P}_X(S^{(i)}_{n-1} = m - k)\,(1 + o(1))\\
&\approx \alpha_2(m - \nu_n n)^{-\beta} \frac{X_i^{m-\nu_n n \mp \delta_n n}}{\Phi(X_i)} \mathbb{P}_X(|S^{(i)}_{n-1} - \nu_n n| \leq \delta_n n)\,(1 + o(1)).
\end{aligned}$$

As the tails $\mathbb{P}_{\boldsymbol{X}}(Q_i > x)$ are going to zero uniformly in $\boldsymbol{X}$ we have that $Q_i = o(n\delta_n)$ in $\mathbb{P}_{\boldsymbol{X}}$-probability. Hence $\mathbb{P}_{\boldsymbol{X}}(|S_{n-1}^{(i)} - \nu_n n| \le \delta_n n) = \mathbb{P}_{\boldsymbol{X}}(|S_n - \nu_n n - Q_i| \le \delta_n n)$ is bounded from below by $\mathbb{P}_{\boldsymbol{X}}(|S_n - \nu_n n| \le \frac{1}{2}\delta_n n) - o(1)$, where the o-term is independent of i, and this bound converges to one by choice of δ_n. This implies the statement. □

Lemma 4.2. *For all $i \neq j \in \{1, \dots, n\}$, with high $\mathtt{P}$-probability,*

$$\mathbb{P}_{\boldsymbol{X}}(\mathcal{D}_{i,j}) = O(n^{-2\beta}) X_i^{m-\nu_n n-\delta_n n} \sum_{k>\varepsilon n} X_j^k,$$

where the implied constant is independent of i and j.

Proof. For all $i \neq j \in \{1, \dots, n\}$, abbreviating again $S_{n-1}^{(i)} = \sum_{\substack{j=1\\ j\neq i}}^n Q_j$, we have

$$\mathbb{P}_{\boldsymbol{X}}(\mathcal{D}_{i,j}) = \sum_{k:|k-(m-\nu_n n)|\le\delta_n n} \mathbb{P}_{\boldsymbol{X}}(Q_i = k,\, Q_j > \varepsilon n \text{ and } S_n = m)$$
$$= \sum_{k:|k-(m-\nu_n n)|\le\delta_n n} \frac{p_k X_i^k}{\Phi(X_i)} \mathbb{P}_{\boldsymbol{X}}(Q_j > \varepsilon n \text{ and } S_{n-1}^{(i)} = m-k).$$

We now use that $0 < p_0 \le \Phi(z)$, for all $z \ge 0$, together with the asymptotic behaviour of (p_k) to bound this by a constant multiple of

$$n^{-\beta} X_i^{m-\nu_n n-\delta_n n} \mathbb{P}_{\boldsymbol{X}}(Q_j > \varepsilon n \text{ and } |S_{n-1}^{(i)} - \nu_n n| \le \delta_n n) \le \mathtt{cst}.n^{-2\beta} X_i^{m-\nu_n n-\delta_n n} \sum_{k>\varepsilon n} X_j^k,$$

as required. □

Lemma 4.3. *If $\beta + \gamma > 2$, then, with high $\mathtt{P}$-probability,*

$$\mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_1) = \left[\sum_{i=1}^n \mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_{1,i}^*)\right](1 + o(1)).$$

(i) *Moreover, if $\gamma > 1$,*

$$\mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_1) = \mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_{1,J_n}^*)(1+o(1)) \approx \alpha_2(\rho - \rho^\star)^{-\beta}\big(X_n^{(1)}\big)^{(\rho_n-\nu_n\mp\delta_n)n} n^{-\beta}(1+o(1)),$$

where $J_n \in \{1, \dots, n\}$ is the index realising the maximum fitness, i.e. $X_{J_n} = X_n^{(1)}$.

(a) *If $\gamma \ge 2$, we have $\mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_1) \ge \mathtt{cst.}\, n^{-\beta}\big(X_n^{(1)}\big)^{(\rho_n-\nu_n)n}$.*

(b) *If $1 < \gamma < 2$, then, for all ω_n such that $n^{1/\gamma} \ll \omega_n \ll n\delta_n$, we have $\mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_1) \ge \mathtt{cst.}\, n^{-\beta}\big(X_n^{(1)}\big)^{(\rho_n-\nu_n)n+\omega_n}$.*

(ii) *If $\gamma < 1$, then $\mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_1) = (1+o(1))\,\alpha_1\alpha_2(\rho-\rho^\star)^{-\beta-\gamma}\Gamma(\gamma+1)n^{1-\beta-\gamma}$.*
If $\gamma = 1$, then $\mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_1) = \Theta_{\mathtt{P}}(n^{-\beta})$.

Proof. First note that, by definition of the events $\mathcal{E}_{1,i}^*$, $\mathcal{E}_1$ and $\mathcal{D}_{i,j}$,

$$\sum_{i=1}^n \mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_{1,i}^*) - \sum_{i\neq j} \mathbb{P}_{\boldsymbol{X}}(\mathcal{D}_{i,j}) \le \mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_1) \le \sum_{i=1}^n \mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_{1,i}^*).$$

Our aim is to prove that $\sum_{i\neq j} \mathbb{P}_{\boldsymbol{X}}(\mathcal{D}_{i,j})$ is negligible with respect to $\sum_{i=1}^n \mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_{1,i}^*)$. In view of Lemmas 4.1 and 4.2, we have

$$\sum_{i=1}^n \mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_{1,i}^*) \approx \alpha_2(\rho-\rho^\star)^{-\beta} n^{-\beta} \sum_{i=1}^n \frac{X_i^{(\rho_n-\nu_n\mp\delta_n)n}}{\Phi(X_i)}(1+o(1)),$$

where the $o(1)$-term is independent of i, and

$$\sum_{i,j:i\neq j}\mathbb{P}_X(\mathcal{D}_{i,j})\leq \mathtt{cst}.n^{-2\beta}\sum_{i,j:i\neq j}X_i^{(\rho_n-\nu_n-\delta_n)n}\sum_{k>\varepsilon n}X_j^k.$$

It is thus enough to prove that the ratio

$$\Delta_n:=\frac{n^{-\beta}\sum_{i\neq j}X_i^{(\rho_n-\nu_n-\delta_n)n}\sum_{k>\varepsilon n}X_j^k}{\sum_{i=1}^n X_i^{(\rho_n-\nu_n+\delta_n)n}}$$

tends to zero in P-probability, as $n\uparrow\infty$.

(i) **Assume** $\gamma>1$. In this case, using Lemma A.3(i) then Lemma A.1, we have

$$\begin{aligned}\Delta_n&\leq \mathtt{cst}.n^{-\beta}\frac{\big(X_n^{(1)}\big)^{(\rho_n-\nu_n-\delta_n)n}\sum_{k>\varepsilon n}\big(X_n^{(1)}\big)^kV_k^{(n)}}{(X_n^{(1)})^{(\rho_n-\nu_n+\delta_n)n}}=O_{\mathrm{P}}\big(n^{1/\gamma-\beta}\big)(X_n^{(1)})^{(\varepsilon-2\delta_n)n}\\&=O_{\mathrm{P}}\big(n^{1/\gamma-\beta}\big),\end{aligned}$$

which tends to 0 when $n\uparrow\infty$. We now prove that $\sum_{i=1}^n\mathbb{P}_X(\mathcal{E}_{1,i}^*)\sim\mathbb{P}_X(\mathcal{E}_{1,J_n}^*)$. It is enough to prove that

$$\Lambda_n:=\frac{\sum_{i\neq J_n}\mathbb{P}_X(\mathcal{E}_{1,i}^*)}{\mathbb{P}_X(\mathcal{E}_{1,J_n}^*)}\to 0\quad\text{as } n\uparrow\infty.$$

We have, in view of Lemmas 4.1 and A.1, for sufficiently large n,

$$\begin{aligned}\Lambda_n&\leq p_0^{-1}\frac{n^{-\beta}\sum_{i\neq J_n}X_i^{(\rho_n-\nu_n-\delta_n)n}}{n^{-\beta}(X_n^{(1)})^{(\rho_n-\nu_n+\delta_n)n}}(1+o(1))\leq p_0^{-1}\frac{n(X_n^{(2)})^{(\rho_n-\nu_n-\delta_n)n}}{(X_n^{(1)})^{(\rho_n-\nu_n+\delta_n)n}}(1+o(1))\\&\leq\mathtt{cst}.n\Big(\frac{X_n^{(2)}}{X_n^{(1)}}\Big)^{(\rho_n-\nu_n-\delta_n)n}\big(X_n^{(1)}\big)^{-2\delta_n n}\leq\mathtt{cst}.n\big(1-\Theta_{\mathrm{P}}(n^{-1/\gamma})\big)^{(\rho_n-\nu_n-3\delta_n)n}.\end{aligned}$$

This implies $\Lambda_n\leq\mathtt{cst}.n\exp\big(-(\rho_n-\nu_n-3\delta_n)\Theta_{\mathrm{P}}(n^{1-1/\gamma})\big)\to 0$, as $n\uparrow\infty$, concluding the proof of (i).

(a) **Assume** $\gamma\geq 2$. We have $\mathbb{P}_X(\mathcal{E}_1)\sim\mathbb{P}_X(\mathcal{E}_{1,J_n}^*)$, where J_n is the index of the largest fitness. Moreover,

$$\begin{aligned}\mathbb{P}_X(\mathcal{E}_{1,J_n}^*)&=\sum_{|k-(\rho_n-\nu_n)n|\leq\delta_n n}\frac{p_k\big(X_n^{(1)}\big)^k}{\Phi\big(X_n^{(1)}\big)}\mathbb{P}_X\Big(\sum_{\substack{i=1\\ i\neq J_n}}^n Q_i=m-k\Big)\\&\geq\sum_{\substack{(\rho_n-\nu_n-\delta_n)n\\ \leq k\leq(\rho_n-\nu_n)n}}p_k\big(X_n^{(1)}\big)^k\,\mathbb{P}_X\Big(\sum_{\substack{i=1\\ i\neq J_n}}^n Q_i=m-k\Big)\\&\geq\mathtt{cst}.n^{-\beta}\big(X_n^{(1)}\big)^{(\rho_n-\nu_n)n}\,\mathbb{P}_X\Big(0\leq\sum_{\substack{i=1\\ i\neq J_n}}^n Q_i-\nu_n n\leq\delta_n n\Big).\end{aligned}$$

Recall that $Q_{J_n}/n\delta_n$ goes to zero in $\mathbb{P}_X$-probability, and hence, by the grand canonical central limit theorem with a normal limit, the probability above goes to 1/2. Therefore we get $\mathbb{P}_X(\mathcal{E}^*_{1,J_n}) \geq \texttt{cst}.n^{-\beta}\big(X_n^{(1)}\big)^{(\rho_n-\nu_n)n}$.

(b) **Assume** $1 < \gamma < 2$. Let $n^{1/\gamma} \ll \omega_n \leq n\delta_n$, then, as above

$$\mathbb{P}_X(\mathcal{E}^*_{1,J_n}) \geq \texttt{cst}.\, n^{-\beta}\big(X_n^{(1)}\big)^{(\rho_n-\nu_n)n+\omega_n}\,\mathbb{P}_X\Big(-\omega_n \leq \sum_{\substack{i=1\\ i\neq J_n}}^{n} Q_i - \nu_n n \leq \delta_n n\Big).$$

Note that $n^{\kappa} \leq n^{1/\gamma} \ll \omega_n$ where $\kappa = \max\{\frac{1}{2}, \frac{1}{\beta+\gamma-1}\}$. Thus, in view of Proposition 3.3, we have

$$\mathbb{P}_X\Big(-\omega_n \leq \sum_{\substack{i=1\\ i\neq J_n}}^{n} Q_i - \nu_n n \leq \delta_n n\Big) \to 1,$$

when n goes to infinity, implying the statement.

(ii) **Assume** $\gamma \leq 1$ **and** $\beta+\gamma > 2$. We have, in view of Lemma A.3(ii) and (iii), that $\sum_{j=1}^n X_j^{\varepsilon n}$ is of order $\Theta_{\mathrm{P}}(n^{1-\gamma})$ if $\gamma < 1$, and of order $o(\log n)$ if $\gamma = 1$. Therefore,

$$\Delta_n \leq \texttt{cst}.\begin{cases} n^{-\beta}\dfrac{[(\rho_n-\nu_n-\delta_n)n]^{-\gamma}\sum_{k>\varepsilon n} nk^{-\gamma}U_k^{(n)}}{[(\rho_n-\nu_n+\delta_n)n]^{-\gamma}} = O_{\mathrm{P}}(n^{2-\beta-\gamma}) & \text{if } \gamma < 1,\\[2ex] n^{1-\beta}\dfrac{\sum_{i=1}^n X_i^{(\rho_n-\nu_n-\delta_n)n}\sum_{j=1}^n X_j^{\varepsilon n}}{\sum_{i=1}^n X_i^{(\rho_n-\nu_n+\delta_n)n}} = o(n^{1-\beta}\log^2 n) & \text{if } \gamma = 1.\end{cases}$$

Hence $\Delta_n \to 0$ in P-probability if $\gamma < 1$ and $\beta+\gamma > 2$, or if $\gamma = 1$. Moreover, we have

$$\mathbb{P}_X(\mathcal{E}_1) = \mathbb{P}_X\left(\bigcup_{i=1}^n \mathcal{E}^*_{1,i}\right)(1+o(1)) \approx \alpha_2(\rho-\rho^\star)^{-\beta}n^{-\beta}\sum_{i=1}^n \frac{X_i^{(\rho_n-\nu_n\mp\delta_n)n}}{\Phi(X_i)}(1+o(1))$$

using Lemma A.3(ii) if $\gamma < 1$, and Lemma A.3(iii) if $\gamma = 1$ concludes the proof. □

Lemma 4.4. *If $\beta+\gamma > 2$, then with high* P*-probability,* $\mathbb{P}_X(\mathcal{E}_2) \ll \mathbb{P}_X(\mathcal{E}_1)$.

Proof. Note that $\mathbb{P}_X(\mathcal{E}_2) = \sum_{i\neq j}\mathbb{P}_X(\mathcal{D}_{i,j})$, and we have already shown in the proof of Lemma 4.3 that this sum is negligible in front of $\mathbb{P}_X(\mathcal{E}_1)$. □

Lemma 4.5. *If $\beta+\gamma > 2$, then, with high* P*-probability,* $\mathbb{P}_X(\mathcal{E}_4) \ll \mathbb{P}_X(\mathcal{E}_1)$.

Proof. We define the truncated variables $\bar{Q}_i := Q_i\mathbb{1}\{Q_i \leq \varepsilon n\}$ and $\bar{S}_n = \sum_{i=1}^n \bar{Q}_i$. As $\mathcal{E}_4 \subset \{\bar{S}_n = m\}$, we have

$$\mathbb{P}_X(\mathcal{E}_4) \leq \mathrm{e}^{-sm}\mathbb{E}_X\big[\mathrm{e}^{s\bar{S}_n}\big] = \mathrm{e}^{-sm}\prod_{i=1}^n \mathbb{E}_X\big[\mathrm{e}^{s\bar{Q}_i}\big], \quad \text{for every } s > 0.$$

There exist two constants $K_1, K_2 > 0$, such that

$$\mathbb{E}_X \mathrm{e}^{s\bar{Q}_i} \le 1 + s\mathbb{E}_X \bar{Q}_i + \sum_{k=1}^{\varepsilon n} \frac{p_k X_i^k}{\Phi(X_i)} (\mathrm{e}^{sk} - 1 - sk)$$
$$\le 1 + s\mathbb{E}_X Q_i + K_1 \sum_{k=1}^{2\beta/s} k^{-\beta} X_i^k (sk)^2 + K_2 \sum_{k=2\beta/s}^{\varepsilon n} k^{-\beta} X_i^k \mathrm{e}^{sk}.$$

Allowing s to depend on n, we define, for any sequence (s_n), the quantities

$$S_n^{(1)} := \sum_{i=1}^{n} \sum_{k=1}^{2\beta/s_n} k^{-\beta} X_i^k (s_n k)^2, \quad \text{and} \quad S_n^{(2)} := \sum_{i=1}^{n} \sum_{k=2\beta/s_n}^{\varepsilon n} k^{-\beta} X_i^k \mathrm{e}^{s_n k}.$$

We then have

$$\mathbb{P}_X(\mathcal{E}_4) \le \exp\big(-s_n m + s_n n \nu_n + K_1 S_n^{(1)} + K_2 S_n^{(2)}\big).$$

(i) **The case $\gamma \le 1$ and $beta + \gamma > 2$.** We fix $s_n := a\frac{\log n}{n}$, where $a = \frac{\beta+\gamma}{\rho-\rho^\star}$. We first prove that

$$S_n^{(1)} = o(n s_n) \quad \text{as } n \uparrow \infty.$$

In view of Lemma A.3(ii) and (iii), using that $2\beta/s_n = o(n^{1/\gamma})$, we have

$$S_n^{(1)} = s_n^2 \sum_{k=1}^{2\beta/s_n} k^{2-\beta} \sum_{i=1}^{n} X_i^k = n s_n^2 \sum_{k=1}^{2\beta/s_n} k^{2-\beta-\gamma} U_k^{(n)} \le \mathtt{cst.}\, n s_n^2 \sum_{k=1}^{2\beta/s_n} k^{2-\beta-\gamma},$$

from which we infer that $S_n^{(1)} = o(n s_n)$. Next, we prove that

$$S_n^{(2)} = o(n s_n) \quad \text{as } n \uparrow \infty.$$

Denote by $u_k := k^{-\beta} X_i^k \mathrm{e}^{s_n k}$. Observe that, for all $k \ge \frac{2\beta}{s_n}$, we have $\frac{u_k}{u_{k+1}} \le \frac{\mathrm{e}^{-s_n/2}}{X_i}$, and thus

$$u_k \le \left(\frac{\mathrm{e}^{-s_n/2}}{X_i}\right)^{\lfloor \varepsilon n \rfloor - k} u_{\lfloor \varepsilon n \rfloor}.$$

This implies that

$$\sum_{k=2\beta/s_n}^{\varepsilon n} k^{-\beta} X_i^k \mathrm{e}^{s_n k} \le \lfloor \varepsilon n \rfloor^{-\beta} \mathrm{e}^{s_n \lfloor \varepsilon n \rfloor} \sum_{k=2\beta/s_n}^{\varepsilon n} X_i^k \left(\mathrm{e}^{-s_n/2}\right)^{\lfloor \varepsilon n \rfloor - k} \le \mathtt{cst.} n^{a\varepsilon - \beta} \frac{X_i^{2\beta/s_n}}{1 - \mathrm{e}^{-s_n/2}}.$$

Using $1 - \mathrm{e}^{-s_n/2} > s_n/4$ for n large enough, and Lemma A.3(ii) in conjunction with $2\beta/s_n \ll n^{1/\gamma}$, we get

$$S_n^{(2)} \le \mathtt{cst.} \frac{n^{a\varepsilon-\beta}}{s_n} \sum_{i=1}^{n} X_i^{2\beta/s_n} = O_{\mathrm{P}}\big(n^{1+a\varepsilon-\beta} s_n^{\gamma-1}\big),$$

and, since $a\varepsilon < \beta + \gamma - 2$, this implies $S_n^{(2)} = o(n s_n)$ as required. Summarising, we have shown that

$$\mathbb{P}_X(\mathcal{E}_4) \le \exp(-s_n m + s_n n \nu_n + o(s_n n)) = n^{-a(\rho-\rho^\star+o(1))}.$$

Recall that $\mathbb{P}_X(\mathcal{E}_1) = n^{1-\beta-\gamma}(1+o(1))$. As $a(\rho-\rho^\star) > \beta+\gamma-1$, we get that $\mathbb{P}_X(\mathcal{E}_4) \ll \mathbb{P}_X(\mathcal{E}_1)$, as $n \uparrow \infty$.

(ii) **The case** $\gamma > 1$. In this case, choose $s_n = -\log X_n^{(1)} + \frac{a\log n}{n} = \Theta_{\mathrm{P}}(n^{-1/\gamma})$ for some positive a satisfying

$$\frac{\beta}{\rho-\rho^\star} < a < \frac{\beta-1}{\varepsilon\gamma}.$$

We now show that

$$S_n^{(1)} = o(1).$$

We have

$$S_n^{(1)} \leq \texttt{cst.}\, s_n^2 \sum_{k=1}^{\frac{2\beta}{s_n}} k^{2-\beta} \sum_{i=1}^{n} X_i^k = \texttt{cst.}\, s_n^2 \left(\sum_{k=1}^{\frac{2\beta}{s_n \log n}} k^{2-\beta} \sum_{i=1}^{n} X_i^k + \sum_{k=\frac{2\beta}{s_n\log n}}^{\frac{2\beta}{s_n}} k^{2-\beta} \sum_{i=1}^{n} X_i^k \right).$$

Using the notation of Lemma A.3(ii), we get

$$S_n^{(1)} \leq \texttt{cst.}\, s_n^2 \left(\sum_{k=1}^{\frac{2\beta}{s_n \log n}} n k^{2-\beta-\gamma} U_k^{(n)} + \sum_{k=\frac{2\beta}{s_n\log n}}^{\frac{2\beta}{s_n}} k^{2-\beta} \sum_{i=1}^{n} X_i^{\frac{2\beta}{s_n\log n}} \right).$$

There exists an integer M_n such that $\max\{U_k^{(n)} : k \in \{1, \ldots, 2\beta/s_n \log n\}\} = U_{M_n}^{(n)}$. Using Lemma A.3(ii) in conjunction with $M_n \leq 2\beta/s_n \log n \ll n^{1/\gamma}$, we get that $U_{M_n}^{(n)} \sim \alpha_1 \Gamma(\gamma+1)$. Thus, using again Lemma A.3(ii) for the second term of the sum, we get

$$S_n^{(1)} \leq \texttt{cst.}\, n s_n^2 \left(\sum_{k=1}^{\frac{2\beta}{s_n \log n}} k^{2-\beta-\gamma} + \left(\frac{2\beta}{s_n \log n}\right)^{-\gamma} \sum_{k=\frac{2\beta}{s_n\log n}}^{\frac{2\beta}{s_n}} k^{2-\beta} \right).$$

Starting from this, a simple calculation gives $S_n^{(1)} = o(1)$, as claimed. We now show that

$$S_n^{(2)} = o(1).$$

To this end, recall the definition of s_n, then split the sum and estimate

$$\begin{aligned} S_n^{(2)} &= \sum_{k=2\beta/s_n}^{\varepsilon n} k^{-\beta} \mathrm{e}^{s_n k} \sum_{i=1}^{n} X_i^k = \sum_{k=2\beta/s_n}^{\varepsilon n} k^{-\beta} \mathrm{e}^{ak\frac{\log n}{n}} \sum_{i=1}^{n} \left(\frac{X_i}{X_n^{(1)}}\right)^k \\ &\leq V_{2\beta/s_n}^{(n)} \mathrm{e}^{a n^{1/\gamma - 1}\log n} \sum_{k=2\beta/s_n}^{n^{1/\gamma}\log^2 n} k^{-\beta} + \mathrm{e}^{a\varepsilon \log n} \sum_{k=n^{1/\gamma}\log^2 n}^{\varepsilon n} k^{-\beta} V_k^{(n)}, \end{aligned}$$

using the notation and result of Lemma A.3(i). Using again Lemma A.3(i), for all $k \geq n^{1/\gamma}\log^2 n$, we have $V_k^{(n)} \leq V_{n^{1/\gamma}\log^2 n}^{(n)}$ and the right hand side converges to one. Using also Lemma A.3(iii) we get,

$$\begin{aligned} S_n^{(2)} &\leq o(\log n) \left(\frac{2\beta}{s_n}\right)^{1-\beta} + \texttt{cst.} n^{a\varepsilon} \left(n^{1/\gamma}\log^2 n\right)^{1-\beta} \\ &\leq o\left(n^{\frac{1-\beta}{\gamma}} \log n\right) + o\left(n^{a\varepsilon + \frac{1-\beta}{\gamma}}\right) = o(1). \end{aligned}$$

To complete the proof recall that

$$\mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_4) \le \exp\left(-(\rho_n - \nu_n)ns_n + o(1)\right) = n^{-a(\rho-\rho^\star)+o(1)}\left(X_n^{(1)}\right)^{(\rho_n-\nu_n)n},$$

and $\mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_1) \ge \mathtt{cst}.n^{-\beta}(X_n^{(1)})^{(\rho_n-\nu_n+\delta_n)n}$. Therefore,

$$\frac{\mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_4)}{\mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_1)} \le n^{\beta-a(\rho-\rho^\star)+o(1)}\left(X_n^{(1)}\right)^{-\delta_n n}.$$

Since $X_n^{(1)} = 1 - \Theta_{\mathrm{P}}(n^{-1/\gamma})$, we have that $(X_n^{(1)})^{-\delta_n n} = \exp(\Theta_{\mathrm{P}}(\delta_n n^{1-1/\gamma}))$. If $1 < \gamma < 2$, we have that $\delta_n n^{1-1/\gamma} \to 0$, which implies $\mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_4) \ll \mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_1)$ by choice of a. If $\gamma \ge 2$, we conclude the proof using the better bound for $\mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_1)$, which was proved in Lemma 4.3(ia). □

Lemma 4.6. *If $\beta + \gamma > 2$, then, with high* P*-probability, $\mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_3) \ll \mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_1)$.*

Proof. (i) **The case $\gamma \le 1$ and $\beta + \gamma > 2$.** In this case $\mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_1) = \Theta_{\mathrm{P}}(n^{1-\beta-\gamma})$. We get

$$\begin{aligned}\mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_3) &\le \sum_{j=1}^n \sum_{\substack{|k-(m-\nu_n n)|>\delta_n n\\ k>\varepsilon n}} \frac{p_k X_j^k}{\Phi(X_j)} \mathbb{P}_{\boldsymbol{X}}\Big(\sum_{i\neq j} Q_i = m-k\Big)\\ &\le \mathtt{cst}.(\varepsilon n)^{-\beta} \sum_{j=1}^n X_j^{\varepsilon n}\, \mathbb{P}_{\boldsymbol{X}}(|S_{n-1}^{(j)} - \nu_n n| > \delta_n n) = o\left(n^{1-\beta-\gamma}\right),\end{aligned}$$

in view of Lemma A.3(ii) and Lemma 3.2.

(ii) **The case $\gamma > 1$.** We decompose the event $\mathcal{E}_3 \subset \mathcal{E}_{3,1} \cup \bigcup_{j=1}^n \mathcal{E}_{3,2}^{(j)}$ where

$$\mathcal{E}_{3,1} = \{S_n = m\ ;\ \exists i, j \in \{1,\dots,n\} \text{ such that } Q_i \ge m - \nu_n n + \delta_n n \text{ and } Q_j > \varepsilon n\}$$

and

$$\mathcal{E}_{3,2}^{(j)} = \{S_n = m \text{ and } Q_i < m - \nu_n n - \delta_n n\ \forall i \in \{1,\dots,n\} \text{ and } Q_j > \varepsilon n\},$$
for $j \in \{1,\dots,n\}$.

Note that, in view of Lemma A.3(i) and our choice of δ_n,

$$\begin{aligned}\mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_{3,1}) &\le \sum_{i=1}^n \sum_{k\ge m-\nu_n n+\delta_n n} \frac{p_k X_i^k}{\Phi(X_i)} \mathbb{P}_{\boldsymbol{X}}\Big(\sum_{\substack{j=1\\ j\neq i}}^n Q_j = m-k\Big)\\ &\le \mathtt{cst}.n^{-\beta} \sum_{i=1}^n X_i^{(\rho_n-\nu_n+\delta_n)n} \mathbb{P}_{\boldsymbol{X}}\Big(\sum_{\substack{j=1\\ j\neq i}}^n Q_j - \nu_n n \le -\delta_n n\Big)\\ &= o\big(n^{-\beta}\big)\big(X_n^{(1)}\big)^{(\rho_n-\nu_n+\delta_n)n}.\end{aligned}$$

Recalling the lower bound in Lemma 4.3(i) we get $\mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_{3,1}) \ll \mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_1)$.

We now focus on the estimate for the events $\mathcal{E}_{3,2}^{(j)}$. We first deal with the summand $j = J_n$, the index of the site carrying the largest fitness. Abbreviate $c_n := \rho_n - \nu_n - \delta_n$ and denote, for $k > \varepsilon n$,

$$\mathcal{E}_{3,2}^k = \Big\{\forall i \neq J_n\ Q_i < c_n n \text{ and } \sum_{i\neq J_n} Q_i = m-k\Big\}.$$

Then, letting $s_n = -\log X_n^{(2)}$ and $\bar{Q}_i = Q_i \mathbb{1}\{Q_i < c_n n\}$, we get from Markov's inequality

$$\mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_{3,2}^k) \le \mathrm{e}^{-(m-k)s_n} \prod_{i \neq J_n} \mathbb{E}_{\boldsymbol{X}}\left[\mathrm{e}^{s_n \bar{Q}_i}\right].$$

Observe that, for all $i \neq J_n$, we have

$$\begin{aligned}\mathbb{E}_{\boldsymbol{X}}\left[\mathrm{e}^{s_n \bar{Q}_i}\right] &\le 1 + s_n\, \mathbb{E}_{\boldsymbol{X}} \bar{Q}_i + \sum_{\ell < c_n n} \frac{p_\ell X_i^\ell}{\Phi(X_i)} (\mathrm{e}^{s_n \ell} - 1 - s_n \ell) \\ &\le 1 + s_n\, \mathbb{E}_{\boldsymbol{X}} Q_i + K_1 \sum_{\ell \le 2\beta/s_n} \ell^{-\beta} X_i^\ell (s_n \ell)^2 + K_2 \sum_{2\beta/s_n < \ell < c_n n} \ell^{-\beta} X_i^\ell \mathrm{e}^{s_n \ell},\end{aligned}$$

where K_1 and K_2 are two positive constants that do not depend on i. Thus,

$$\mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_{3,2}^k) \le \exp\left(-(m - k - \nu_n n)s_n + K_1 S_n^{(1)} + K_2 S_n^{(2)}\right),$$

where

$$S_n^{(1)} := \sum_{i \neq J_n} \sum_{\ell \le 2\beta/s_n} \ell^{-\beta} X_i^\ell (s_n \ell)^2 \quad \text{and} \quad S_n^{(2)} := \sum_{i \neq J_n} \sum_{2\beta/s_n < \ell < c_n n} \ell^{-\beta} X_i^\ell \mathrm{e}^{s_n \ell}.$$

Note that $S_n^{(1)}$ and $S_n^{(2)}$ are independent of k. We have already encountered $S_n^{(1)}$ in the proof of Lemma 4.5, and proved that $S_n^{(1)} = o(1)$. The sum $S_n^{(2)}$ is slightly different than the one studied in the proof of Lemma 4.5, but the same calculation yields $S_n^{(2)} = O_{\mathrm{P}}(n^{(1-\beta)/\gamma}) = o(1)$. Summarising, we see that

$$\mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_{3,2}^k) \le \exp\bigl(-(m - k - \nu_n n)s_n + o(1)\bigr) = \bigl(X_n^{(2)}\bigr)^{(\rho_n - \nu_n)n - k}(1 + o(1)),$$

where the $o(1)$-term does not depend on k. Thus,

$$\begin{aligned}\mathbb{P}_{\boldsymbol{X}}\bigl(\mathcal{E}_{3,2}^{(J_n)}\bigr) &= \sum_{\varepsilon n < k < c_n n} \mathbb{P}_{\boldsymbol{X}}\bigl(Q_{J_n} = k\bigr)\, \mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_{3,2}^k) \\ &\le \mathtt{cst.}\bigl(X_n^{(2)}\bigr)^{(\rho_n - \nu_n)n} \sum_{\varepsilon n < k < c_n n} k^{-\beta} \left(\frac{X_n^{(1)}}{X_n^{(2)}}\right)^k \\ &\le O_{\mathrm{P}}\bigl(n^{1/\gamma - \beta}\bigr)\bigl(X_n^{(2)}\bigr)^{(\rho_n - \nu_n)n} \left(\frac{X_n^{(1)}}{X_n^{(2)}}\right)^{c_n n},\end{aligned}$$

where we have used that $\frac{X_n^{(1)}}{X_n^{(2)}} = 1 + \Theta_{\mathrm{P}}(n^{-1/\gamma})$ by Lemma A.1.

Now assume that $\gamma > 2$. Then, in view of the lower bound proved in Lemma 4.3(ia), we have

$$\frac{\mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_{3,2}^{(J_n)})}{\mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_1)} \le O_{\mathrm{P}}\bigl(n^{1/\gamma}\bigr)\left(\frac{X_n^{(2)}}{X_n^{(1)}}\right)^{\delta_n n} \le O_{\mathrm{P}}\bigl(n^{1/\gamma}\bigr)\mathrm{e}^{-\Theta_{\mathrm{P}}(\delta_n n^{1-1/\gamma})} = o(1),$$

because $\delta_n n^{1-1/\gamma} \to \infty$. If $1 < \gamma \le 2$, we use the lower bound proved in Lemma 4.3(ib) for $\delta_n n \gg \omega_n \gg n^{1/\gamma}$, and get

$$\frac{\mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_{3,2}^{(J_n)})}{\mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_1)} \le O_{\mathrm{P}}\bigl(n^{1/\gamma}\bigr)\left(\frac{X_n^{(2)}}{X_n^{(1)}}\right)^{\delta_n n}\bigl(X_n^{(1)}\bigr)^{-\omega_n} \le O_{\mathrm{P}}\bigl(n^{1/\gamma}\bigr)\mathrm{e}^{-\Theta_{\mathrm{P}}(\delta_n n^{1-1/\gamma}) + \Theta_{\mathrm{P}}(\omega_n n^{-1/\gamma})} = o(1).$$

It remains to investigate the other summands, corresponding to $j \neq J_n$. The same argument as above, with X_j playing the role of $X_n^{(1)}$ and $X_n^{(1)}$ playing the role of $X_n^{(2)}$, yields

$$\begin{aligned}\mathbb{P}_{\boldsymbol{X}}\big(\mathcal{E}_{3,2}^{(j)}\big) &\leq \mathtt{cst.}\big(X_n^{(1)}\big)^{(\rho_n-\nu_n)n} \sum_{\varepsilon n<k<c_n n} k^{-\beta}\Big(\frac{X_n^{(2)}}{X_n^{(1)}}\Big)^k \\ &\leq O_{\mathrm{P}}\big(n^{1/\gamma-\beta}\big)\big(X_n^{(1)}\big)^{(\rho_n-\nu_n)n}\Big(\frac{X_n^{(2)}}{X_n^{(1)}}\Big)^{\varepsilon n}.\end{aligned}$$

In the case $\gamma > 2$ we can use Lemma 4.3(ia) and Lemma A.1 again and get

$$\frac{\sum_{j\neq J_n}\mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_{3,2}^{(j)})}{\mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_1)} \leq O_{\mathrm{P}}\big(n^{1+1/\gamma}\big)\Big(\frac{X_n^{(2)}}{X_n^{(1)}}\Big)^{\varepsilon n} \leq O_{\mathrm{P}}\big(n^{1+1/\gamma}\big)\mathrm{e}^{-\Theta_{\mathrm{P}}(\varepsilon n^{1-1/\gamma})} = o(1).$$

If $1 < \gamma \leq 2$, we use again Lemma 4.3(ib) with $\delta_n n \gg \omega_n \gg n^{1/\gamma}$, and get

$$\begin{aligned}\frac{\sum_{j\neq J_n}\mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_{3,2}^{(j)})}{\mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_1)} &\leq O_{\mathrm{P}}\big(n^{1+1/\gamma}\big)\big(X_n^{(1)}\big)^{-\omega_n}\Big(\frac{X_n^{(2)}}{X_n^{(1)}}\Big)^{\varepsilon n} \\ &\leq O_{\mathrm{P}}\big(n^{1+1/\gamma}\big)\mathrm{e}^{-\Theta_{\mathrm{P}}(\varepsilon n^{1-1/\gamma})+\Theta_{\mathrm{P}}(\omega_n n^{-1/\gamma})} = o(1),\end{aligned}$$

as required to prove the claim. □

Proof of Theorem 2.1. We have proved through Lemmas 4.4–4.6 that, if $\beta + \gamma \geq 2$, we have $\mathbb{P}_{\boldsymbol{X}}(S_n = m) = (1 + o(1))\,\mathbb{P}_{\boldsymbol{X}}(\mathcal{E}_1)$, as $n \uparrow \infty$ and $m/n \to \rho > \rho^\star$. This proves Theorem 2.1. □

5. Intermediate symmetry-breaking and the Gamma law

Proof of Theorem 2.2. (i) **The case** $\gamma > 1$. We have shown that $\mathbb{P}_{\boldsymbol{X}}(\mathcal{E}^*_{1,J_n} \mid S_n = m) \to 1$ when $n \uparrow \infty$. Thus, with high probability, the condensate is located at index J_n and its rank is by definition one.

(ii) **The case** $\gamma < 1$. Let $a, b > 0$. Then, by Lemma 4.1,

$$\begin{aligned}\mathbb{P}_{\boldsymbol{X}}\Big(\Big(\frac{K_n}{n^{1-\gamma}}\Big)^{1/\gamma} \in [a,b] \text{ and } S_n = m\Big) &= \mathbb{P}_{\boldsymbol{X}}\big(a^\gamma n^{1-\gamma} \leq K_n \leq b^\gamma n^{1-\gamma} \text{ and } S_n = m\big) \\ &= (1+o(1)) \sum_{\substack{i \text{ such that} \\ X_n^{(\lceil a^\gamma n^{1-\gamma}\rceil)} \leq X_i \leq X_n^{(\lfloor b^\gamma n^{1-\gamma}\rfloor)}}} \mathbb{P}_{\boldsymbol{X}}(\mathcal{E}^*_{1,i}) \\ &\approx (1+o(1))\alpha_2(\rho-\rho^\star)^{-\beta} n^{-\beta} \sum_{i=a^\gamma n^{1-\gamma}}^{b^\gamma n^{1-\gamma}} \frac{\big(X_n^{(i)}\big)^{(\rho_n-\nu_n\mp\delta_n)n}}{\Phi\big(X_n^{(i)}\big)} \\ &\approx (1+o(1))\alpha_2(\rho-\rho^\star)^{-\beta} n^{-\beta} \int_{a^\gamma n^{1-\gamma}}^{b^\gamma n^{1-\gamma}+1} \frac{\big(X_n^{(\lfloor x\rfloor)}\big)^{(\rho_n-\nu_n\mp\delta_n)n}}{\Phi\big(X_n^{(\lfloor x\rfloor)}\big)}\,dx \\ &\approx (1+o(1))\alpha_2(\rho-\rho^\star)^{-\beta} n^{1-\beta-\gamma} \int_a^{b+o(1)} \frac{\big(X_n^{(\lfloor y^\gamma n^{1-\gamma}\rfloor)}\big)^{(\rho_n-\nu_n\mp\delta_n)n}}{\Phi\big(X_n^{(\lfloor y^\gamma n^{1-\gamma}\rfloor)}\big)}\,\gamma y^{\gamma-1}\,dy.\end{aligned}$$

Note that, in view of Assumption (RVμ), $\mathtt{E}\big[n^{\gamma-1}\big|\{i\colon X_i \geq 1 - x/n\}\big|\big] \sim \alpha_1 x^\gamma$ and $\mathtt{Var}\big[n^{\gamma-1}\big|\{i\colon X_i \geq 1 - x/n\}\big|\big] = o(1)$ when $n \uparrow \infty$. Hence, by Chebyshev's inequality, for all $x \geq 0$, in P-probability,

$$n^{\gamma-1}\Big|\Big\{i : X_i \geq 1 - {}^{x}\!/\!{}_{n}\Big\}\Big| \to \alpha_1 x^\gamma.$$

Thus, in P-probability, $X_n^{(\lfloor y^\gamma n^{1-\gamma}\rfloor)} \sim 1 - \frac{y}{n\alpha_1{}^{1/\gamma}}$ as $n \uparrow \infty$, which implies

$$\begin{aligned}
&\mathbb{P}_X\Big(\Big(\frac{K_n}{n^{1-\gamma}}\Big)^{1/\gamma} \in [a,b] \text{ and } S_n = m\Big)\\
&\quad\approx (1+o(1))\,\alpha_2(\rho-\rho^\star)^{-\beta} n^{1-\beta-\gamma}\int_a^{b+o(1)} \frac{\mathrm{e}^{-(\rho_n-\nu_n\mp\delta_n)y\alpha_1^{-1/\gamma}}}{\Phi\big(1-y\alpha_1^{-1/\gamma}n^{-1}\big)}\,\gamma y^{\gamma-1}\,dy\\
&\quad= (1+o(1))\,\alpha_2\gamma(\rho-\rho^\star)^{-\beta} n^{1-\beta-\gamma}\int_a^{b} \exp\big(-(\rho-\rho^\star)\alpha_1^{-1/\gamma}y\big)\,y^{\gamma-1}\,dy.
\end{aligned}$$

Now recall from Lemma 4.3(ii) that $\mathbb{P}_X(S_n = m) = (1+o(1))\,\alpha_1\alpha_2(\rho-\rho^\star)^{-\beta-\gamma}\Gamma(\gamma+1)n^{1-\beta-\gamma}$, to obtain

$$\begin{aligned}
&\mathbb{P}_X\Big(\Big(\frac{K_n}{n^{1-\gamma}}\Big)^{1/\gamma} \in [a,b]\,\Big|\,S_n = m\Big)\\
&\quad= (1+o(1))\,\frac{(\rho-\rho^\star)^\gamma}{\alpha_1\Gamma(\gamma)}\int_a^b \exp\big(-(\rho-\rho^\star)\alpha_1^{-1/\gamma}y\big)\,y^{\gamma-1}\,dy,
\end{aligned}$$

which concludes the proof of Theorem 2.2. □

Proof of Theorem 2.3. Fix $u > 0$, $\Delta > 0$ and calculate

$$\begin{aligned}
\mathbb{P}_X\big(n(1-F_n) \leq u \text{ and } S_n = m\big) &= \mathbb{P}_X\big(F_n \geq 1 - {}^{u}\!/\!{}_{n} \text{ and } S_n = m\big)\\
&= (1+o(1))\sum_{\substack{i \text{ such that}\\ X_i \geq 1-u/n}} \mathbb{P}_X\big(\mathcal{E}^*_{1,i}\big),
\end{aligned}$$

since we have shown that $\mathbb{P}_X(\bigcup_{i=1}^n \mathcal{E}^*_{1,i} \mid S_n = m) \to 1$ when $n \uparrow \infty$. Thus,

$$\begin{aligned}
&\mathbb{P}_X\big(n(1-F_n) \leq u \text{ and } S_n = m\big)\\
&\quad\approx \sum_{k=0}^{\frac{u}{\Delta}-1}\ \sum_{\substack{i \text{ such that}\\ X_i\in[1-\Delta\frac{k+1}{n},1-\Delta\frac{k}{n})}} \alpha_2(\rho-\rho^\star)^{-\beta}n^{-\beta}\,\frac{X_i^{(\rho_n-\nu_n\mp\delta_n)n}}{\Phi(X_i)},
\end{aligned}$$

in view of Lemma 4.1. It implies

$$\begin{aligned}
&\mathbb{P}_X\big(n(1-F_n) \leq u \text{ and } S_n = m\big)\\
&\quad\geq \sum_{k=0}^{\frac{u}{\Delta}-1}\ \sum_{\substack{i \text{ such that}\\ X_i\in[1-\Delta\frac{k+1}{n},1-\Delta\frac{k}{n})}} \alpha_2(\rho-\rho^\star)^{-\beta}n^{-\beta}\left(1-\frac{\Delta(k+1)}{n}\right)^{(\rho_n-\nu_n+\delta_n)n}(1+o(1))\\
&\quad\geq \sum_{k=0}^{\frac{u}{\Delta}-1} N_k(n)\,\alpha_2(\rho-\rho^\star)^{-\beta}n^{-\beta}\left(1-\frac{\Delta(k+1)}{n}\right)^{(\rho_n-\nu_n+\delta_n)n}(1+o(1)),
\end{aligned}$$

where $N_k(n) = \Big|\Big\{i : X_i \in [1-\Delta\frac{k+1}{n}, 1-\Delta\frac{k}{n})\Big\}\Big|$. Estimating the expectation and variance of $N_k(n)$ and applying Chebyshev's inequality gives, in P-probability, $n^{\gamma-1}N_k(n) \to$

$\alpha_1 \Delta^\gamma((k+1)^\gamma - k^\gamma)$, if $n \uparrow \infty$ and $0 \le k < \frac{u}{\Delta}$. Thus,

$$\begin{aligned}\mathbb{P}_X\big(n(1-F_n) \le u \text{ and } S_n = m\big) &\ge \alpha_1\alpha_2(\rho-\rho^\star)^{-\beta} n^{1-\beta-\gamma}\,\Delta^\gamma \sum_{k=0}^{\frac{u}{\Delta}-1}\big((k+1)^\gamma - k^\gamma\big)\\ &\quad\times\, \mathrm{e}^{-(\rho_n-\nu_n+\delta_n)\Delta(k+1)}(1+o(1))\\ &\ge \alpha_1\alpha_2\gamma(\rho-\rho^\star)^{-\beta} n^{1-\beta-\gamma}\mathrm{e}^{-(\rho-\rho^\star)\Delta}\\ &\quad\times \int_0^u x^{\gamma-1}\mathrm{e}^{-(\rho_n-\nu_n+\delta_n)x}dx\ (1+o(1)),\end{aligned}$$

because the function $x \mapsto x^{\gamma-1}\mathrm{e}^{-(\rho_n-\nu_n+\delta_n)x}$ is decreasing on $(0,\infty)$. Recall that, as $\gamma < 1$, we have $\mathbb{P}_X(S_n = m) = \alpha_1\alpha_2(\rho-\rho^\star)^{-\beta-\gamma}\Gamma(\gamma+1)n^{1-\beta-\gamma}(1+o(1))$. Together, this implies

$$\liminf_{n\to\infty} \mathbb{P}_X\big(n(1-F_n) \le u \big| S_n = m\big) \ge \frac{(\rho-\rho^\star)^\gamma}{\Gamma(\gamma)}\mathrm{e}^{-(\rho-\rho^\star)\Delta}\int_0^u x^{\gamma-1}\mathrm{e}^{-(\rho-\rho^\star)x}dx,$$

and letting $\Delta \downarrow 0$ concludes the proof. □

6. Fluctuations of the condensate in the weak disorder case

In this section, we prove Theorem 2.4. It follows by combining Proposition 3.3 with the following result.

Proposition 6.1. *Let $2-\beta < \gamma < 1$ and $\rho > \rho^\star$. For all $u \in \mathbb{R}$ there exists $u_n \downarrow 0$ such that, with high* P*-probability as $n \uparrow \infty$, we have*

$$\mathbb{P}_X\Big(\frac{Q_n^{(1)} - m + n\nu_n}{n^\kappa} \le u \,\Big|\, S_n = m\Big) \approx (1+o(1))\,\mathbb{P}_X\left(\frac{\nu_n n - \sum_{i=1}^n Q_i}{n^\kappa} \le u \pm u_n\right),$$

where $\kappa = \frac{1}{2}$, if $\beta+\gamma \ge 3$, and $\kappa = \frac{1}{\beta+\gamma-1}$ otherwise.

Proof. By Lemma 4.3 we have

$$\mathbb{P}_X\Big(\frac{Q_n^{(1)} - m + n\nu_n}{n^\kappa} \le u \,\Big|\, S_n = m\Big) \sim \frac{\sum_{i=1}^n \mathbb{P}_X(\mathcal{E}^*_{1,i} \cap \{Q_n^{(1)} - (\rho_n-\nu_n)n \le un^\kappa\})}{\sum_{i=1}^n \mathbb{P}_X(\mathcal{E}^*_{1,i})}.$$

The right hand side can be written as

$$\begin{aligned}&\frac{\sum_{i=1}^n \sum_{\substack{-\delta_n n \le k-n(\rho_n-\nu_n)\\ \le un^\kappa}} \mathbb{P}_X(Q_i = k)\mathbb{P}_X\Big(\sum_{j\ne i} Q_j = m-k\Big)}{\sum_{i=1}^n \sum_{|k-n(\rho_n-\nu_n)|\le\delta_n n} \mathbb{P}_X(Q_i = k)\mathbb{P}_X\Big(\sum_{j\ne i} Q_j = m-k\Big)}\\ &\approx \frac{\sum_{i=1}^n \frac{X_i^{n(\rho_n-\nu_n)\mp\delta_n n}}{\Phi(X_i)}\mathbb{P}_X\Big(n\nu_n - un^\kappa \le \sum_{j\ne i} Q_j \le n\nu_n + \delta_n n\Big)}{\sum_{i=1}^n \frac{X_i^{n(\rho_n-\nu_n)\pm\delta_n n}}{\Phi(X_i)}\mathbb{P}_X\Big(n\nu_n - \delta_n n \le \sum_{j\ne i} Q_j \le n\nu_n + \delta_n n\Big)}(1+o(1)). \qquad (8)\end{aligned}$$

Note that $\max_{i=1..n} \mathbb{P}_{\boldsymbol{X}}(Q_i \geq a_n) \to 0$ for any $a_n \uparrow \infty$. Hence we can find $u_n \downarrow 0$ such that, for all $i \in \{1, \dots, n\}$,

$$\mathbb{P}_{\boldsymbol{X}}\Big(n\nu_n - un^\kappa \leq \sum_{j\neq i} Q_j \leq n\nu_n + \delta_n n\Big)$$
$$\approx (1+o(1))\,\mathbb{P}_{\boldsymbol{X}}\Big(n\nu_n - (u \pm u_n)n^\kappa \leq \sum_{j=1}^{n} Q_j \leq n\nu_n + \delta_n n \pm \frac{1}{2}\delta_n n\Big),$$

where the $o(1)$-term can be chosen independently of i. Therefore, using the choice of δ_n and a similar bound for the probability in the denominator, we see that (8) is

$$\approx (1+o(1)) \frac{\sum_{i=1}^{n} \frac{X_i^{n(\rho_n-\nu_n)\mp\delta_n n}}{\Phi(X_i)}}{\sum_{i=1}^{n} \frac{X_i^{n(\rho_n-\nu_n)\pm\delta_n n}}{\Phi(X_i)}}\, \mathbb{P}_{\boldsymbol{X}}\Big(n\nu_n - (u \pm u_n)n^\kappa \leq \sum_{j=1}^{n} Q_j\Big).$$

In view of Lemma A.3(ii), using that $\gamma < 1$, we get that

$$\frac{\sum_{i=1}^{n} \frac{X_i^{n(\rho_n-\nu_n)\mp\delta_n n}}{\Phi(X_i)}}{\sum_{i=1}^{n} \frac{X_i^{n(\rho_n-\nu_n)\pm\delta_n n}}{\Phi(X_i)}} \approx \left(1 \pm \frac{2\delta_n}{\rho_n - \nu_n}\right)^{-\gamma} (1+o(1)) = (1+o(1)).$$

This proves the statement. □

7. Further comments and open questions

Fluctuations in the presence of strong disorder.

Our result on quenched fluctuations in the size of the condensate, Theorem 2.4, is restricted to the weak disorder regime $\gamma < 1$. We now give some hints how fluctuations could be treated in the strong disorder case. We do not provide details since the focus of the paper is on the weak disorder case.

In the case $1 \leq \gamma < 2$ the assumption $n^{1/\gamma} = o(\delta_n n)$ made in the proof of Theorem 2.1, and used to prove Lemma 4.6 makes δ_n too large to control precisely the fluctuations of the size of the condensate. We believe that with some extra effort this assumption can be dropped and Theorem 2.4 can be extended verbatim to this regime.

When $\gamma \geq 2$ more significant changes to the statement proof of Theorem 2.4 are needed. It turns out that due to the large fluctuations of the fitness values in this regime the random variables $\sum_{i=1}^{n} Q_i$ in the grand canonical framework are not a sufficiently good approximation of the size of the condensate in the canonical framework. A solution to this problem comes from renormalising the fitnesses by their maximum. More precisely, for any n, let $\bar{X}_{i,n} = X_i/X_n^{(1)}$, for all $i \in \{1, \dots, n\}$. Note that the renormalised fitnesses $(\bar{X}_{1,n}, \dots, \bar{X}_{n,n})$ are no longer independent random variables, but

$$\lim_{n\to\infty} \sup_{1\leq i\leq n} \bar{X}_{i,n}/X_i = 1, \quad \text{in P-probability}.$$

Defining the random variables $\bar{Q}_{1,n}, \dots, \bar{Q}_{n,n}$ by

$$\mathbb{P}_X(\bar{Q}_{i,n} = k) = \frac{p_k \bar{X}_{i,n}^k}{\Phi(\bar{X}_{i,n})}, \quad \text{for all } k \in \mathbb{N},$$

it is straightforward to see from Eq. (1) that the law of $(\bar{Q}_{1,n}, \dots, \bar{Q}_{n,n})$ conditional on $\sum_{i=1}^n \bar{Q}_{i,n} = m$ is equal to the law of $(Q_1, \dots, Q_n)$ under $P_{m,n}$. Analysing this ensemble would permit to prove that, if $\rho > \rho^\star$, we have in quenched distribution,

$$\frac{Q_n^{(1)} - (m - \bar{\nu}_n n)}{n^{1/2}} \to W,$$

where W is a normally distributed random variable, and

$$\bar{\nu}_n := \frac{1}{n} \sum_{\substack{i=1 \\ i \neq J_n}}^{n} \mathbb{E}_X \bar{Q}_{i,n},$$

where $J_n \in \{1, \dots, n\}$ is the index with $X_{J_n} = X_n^{(1)}$. Note that the size of the condensate is approximated by $m - \bar{\nu}_n n$ and not by $m - \nu_n n$ as in Theorem 2.4. If $\gamma \geq 2$ this makes a difference. Indeed, by a Taylor expansion of the function $x \mapsto {}^{x\Phi'(x)}/_{\Phi(x)}$, using that $X_n^{(1)} = 1 - \Theta_{\mathrm{P}}(n^{-1/\gamma})$, one can see that, in P-probability, the scaled difference $\sqrt{n}\,(\nu_n - \bar{\nu}_n)$ tends to zero when $\gamma < 2$ but does not tend to zero when $\gamma \geq 2$.

Shape of the bulk

In the homogeneous zero-range process, it is known that if one removes the site containing the condensate, then the distribution of the configuration is a critical zero-range process (with $\rho = \rho^\star$) with occupation numbers being i.i.d. (see for example Janson [13] or Armendàriz and Loulakis [3]). We believe that a similar result should still hold in our random environment framework, where instead of i.i.d. random variables, one would have independent random variables depending on the random environment. Note that such a result would imply our fluctuation results as a corollary (see Theorem 2.4) using standard central limit theorem for sums of independent random variables; our efforts towards proving this stronger result have been unsuccessful so far.

Behaviour at criticality

In the present article, we assume that the density of particles $\rho_n := {}^m/_n \to \rho > \rho^\star$ when $n \uparrow \infty$. It would be interesting to *zoom into the transition window*, assuming that ρ_n behaves like $\rho_n = \rho^\star + \varepsilon_n$ for some $\varepsilon_n \downarrow 0$. How does the phase transition manifest itself at criticality?

Strong excess of particles

In another direction, it could be of interest to understand how the system behaves when the average number of particles in the grand canonical model is no longer of order $\rho^\star n$, but of order ρn^η where $\eta > 1$. Under which condition on β, γ, η do we have condensation? Where is the condensation happening? What is the size of the condensate? It would be particularly interesting to know whether an intermediate symmetry-breaking regime also appears in this framework when the disorder is weak (i.e. when $\gamma < 1$).

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Appendix. Random variables near their essential supremum

This appendix is devoted to asymptotic properties of a random variable X with distribution μ on $[0, 1]$, which satisfies (RVμ). We denote by $(X_i)_{i\in\mathbb{N}}$ an i.i.d. sequence of random variables with the same distribution as X. Let $(X_n^{(1)}, \dots, X_n^{(n)})$ be the order statistics of $(X_1, \dots, X_n)$. In some results we additionally refer to a continuous function $\Psi\colon [0,1] \to (0,\infty)$ such that $\Psi(1) = 1$.

This first lemma is a classical result for regularly varying random variables:

Lemma A.1 (*See [15, Chapter 0.4]*). *We have, in probability as $n \uparrow \infty$,*

$$1 - X_n^{(1)} = \Theta_{\mathrm{P}}\big(n^{-1/\gamma}\big) \quad \text{and} \quad 1 - \frac{X_n^{(2)}}{X_n^{(1)}} = \Theta_{\mathrm{P}}\big(n^{-1/\gamma}\big).$$

Lemma A.2. *As $r \uparrow \infty$, we have $\mathrm{E}\left(\frac{X^r}{\Psi(X)}\right) \sim \alpha_1 \Gamma(\gamma+1)\, r^{-\gamma}$.*

Proof. First note that

$$\mathrm{E}\left[\frac{X^r}{\Psi(X)}\right] = \mathrm{E}\left[\frac{X^r}{\Psi(X)}\, \mathbb{1}\{X > 1 - 2\gamma \log r/r\}\right] + \mathrm{E}\left[\frac{X^r}{\Psi(X)}\, \mathbb{1}\{X \le 1 - 2\gamma \log r/r\}\right].$$

The second term of the above sum verifies

$$\mathrm{E}\left[\frac{X^r}{\Psi(X)}\, \mathbb{1}\{X \le 1 - 2\gamma \log r/r\}\right] \le r^{-2\gamma}\mathrm{E}\left[\frac{1}{\Psi(X)}\right] \le \frac{1}{p_0} r^{-2\gamma},$$

since Ψ is bounded from below by some $p_0 > 0$ on $[0, 1]$. The fact that Ψ is continuous in 1 gives that

$$\mathrm{E}\left[\frac{X^r}{\Psi(X)}\, \mathbb{1}\{X > 1 - 2\gamma \log r/r\}\right] = (1+o(1))\, \mathrm{E}[X^r\, \mathbb{1}\{X > 1 - 2\gamma \log r/r\}].$$

By Fubini's theorem, and in view of Assumption (RVμ), we have

$$\begin{aligned}
\mathrm{E}[X^r\, \mathbb{1}\{X > 1 - 2\gamma \log r/r\}] &= \int_0^1 \mathrm{P}(X^r\, \mathbb{1}\{X > 1 - 2\gamma \log r/r\} \ge x)\, dx \\
&= \int_0^{(1-2\gamma\log r/r)^r} \mathrm{P}(X > 1 - 2\gamma \log r/r)\, dx \\
&\quad + \int_{(1-2\gamma\log r/r)^r}^1 \mathrm{P}(X^r \ge x)\, dx. \\
&= (1+o(1))\, r^{-2\gamma} \mu(1 - 2\gamma \log r/r, 1)
\end{aligned}$$

$$+\int_{(1-2\gamma\log r/r)^r}^1 \mu(x^{1/r},1)\,dx$$

$$= o(r^{-\gamma}) + \int_0^{2\gamma\log r} \mu(1-z/r,1)(1-z/r)^{r-1}\,dz,$$

by the change of variables $r(1-x^{1/r}) = z$, $dx = -(1-z/r)^{r-1}dz$. Thus by Eq. (RVμ), we get

$$\mathtt{E}[X^r\,\mathbb{1}\{X > 1-2\gamma\log r/r\}] = \alpha_1 r^{-\gamma}\int_0^{2\gamma\log r} z^\gamma(1-z/r)^{r-1}dz + o(r^{-\gamma})$$
$$= (\alpha_1+o(1))\,r^{-\gamma}\int_0^\infty z^\gamma \mathtt{e}^{-z}dz,$$

which concludes the proof. □

Note that $\mathtt{E}X^r \sim \alpha_1\Gamma(\gamma+1)\,r^{-\gamma}$ as $r\uparrow\infty$, by choosing $\Psi(x)=1$ for all $x\in[0,1]$.

Lemma A.3. (i) *For all $n\geq 1$ and $k\geq 0$, let*

$$V_k^{(n)} := \frac{\sum_{i=1}^n X_i^k}{(X_n^{(1)})^k}.$$

The sequence $(V_k^{(n)})_{k\geq 0}$ is non-increasing for all integer n.
Let $(s_n)_{n\geq 1}$ be a sequence of positive reals, such that $s_n \gg n^{1/\gamma}\log n$. Then,

$$\lim_{n\to\infty} V_{s_n}^{(n)} = 1 \quad \textit{in } \mathtt{P}\textit{-probability}.$$

(ii) *For all $n\geq 1$ and for all $k\geq 0$, let $U_0^{(n)} := 0$ and*

$$U_k^{(n)} := \frac{k^\gamma}{n}\sum_{i=1}^n \frac{X_i^k}{\Psi(X_i)}.$$

Let $(s_n)_{n\geq 1}$ be a sequence of positive reals, such that $s_n \ll n^{1/\gamma}$. Then,

$$\lim_{n\to\infty} U_{s_n}^{(n)} = \alpha_1\Gamma(\gamma+1) \quad \textit{in } \mathtt{P}\textit{-probability}.$$

(iii) *For all constants $c>0$, the sequence $(\sum_{i=1}^n X_i^{cn^{1/\gamma}})_{n\geq 1}$ is tight.*

Proof. (i) Fix $n\geq 1$, then, for all $k\geq 0$,

$$\frac{V_k^{(n)}}{V_{k+1}^{(n)}} = X_n^{(1)}\,\frac{\sum_{i=1}^n X_i^k}{\sum_{i=1}^n X_i^{k+1}} \geq 1,$$

using that $X_i \leq X_n^{(1)}$ for all $i\in\{1,\ldots,n\}$. Now, observe that, $\sum_{i=2}^n (X_n^{(i)})^{s_n} \leq n(X_n^{(2)})^{s_n}$, which implies that

$$\frac{\sum_{i=2}^n (X_n^{(i)})^{s_n}}{(X_n^{(1)})^{s_n}} \leq n\Big(\frac{X_n^{(2)}}{X_n^{(1)}}\Big)^{s_n} = n\big(1-\Theta_{\mathtt{P}}\big(n^{-1/\gamma}\big)\big)^{s_n} = o(1),$$

which concludes the proof of (i).

(ii) We have, as $n \to \infty$, in view of Lemma A.2,

$$\mathbb{E}\left[\frac{s_n^{\gamma}}{n}\sum_{i=1}^{n}\frac{X_i^{s_n}}{\Psi(X_i)}\right] = s_n^{\gamma}\mathbb{E}\left[\frac{X_i^{s_n}}{\Psi(X_i)}\right] \sim \alpha_1 \Gamma(1+\gamma).$$

Moreover, applying Lemma A.2 again and denoting by p_0 the positive lower bound of Ψ on $[0, 1]$,

$$\mathtt{Var}\left[\frac{s_n^{\gamma}}{n}\sum_{i=1}^{n}\frac{X_i^{s_n}}{\Psi(X_i)}\right] = \frac{s_n^{2\gamma}}{n}\mathtt{Var}\left[\frac{X^{s_n}}{\Psi(X)}\right] \leq \frac{s_n^{2\gamma}}{p_0^2 n}\mathbb{E}X^{2s_n} = o(1).$$

The statement now follows by Chebyshev's inequality.

(iii) Note that, as $n \uparrow \infty$, in view of Lemma A.2,

$$\mathbb{E}\left[\sum_{i=1}^{n} X_i^{cn^{1/\gamma}}\right] = n\mathbb{E}\big[X^{n^{1/\gamma}}\big] = c^{-\gamma}\alpha_1\Gamma(1+\gamma).$$

Similarly, $\mathtt{Var}\left(\sum_{i=1}^{n} X_i^{cn^{1/\gamma}}\right) = n\mathtt{Var}\big(X^{cn^{1/\gamma}}\big) = O(1)$, which implies the result by Chebyshev's inequality. □

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