

CRITICAL TEMPERATURE OF HEISENBERG MODELS ON REGULAR TREES, VIA RANDOM LOOPS

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ABSTRACT. We compute the critical temperature of a family of quantum spin systems on regular trees of large degree. The systems include the spin- $\frac{1}{2}$ XXZ model and the spin-1 nematic model. Our formula is conjectured to be valid for large-dimensional cubic lattices. Our method of proof uses a probabilistic representation in terms of random loops.

1. INTRODUCTION AND MAIN RESULT

The main goal of this study is to predict an expression for the critical temperature of a family of quantum spin systems on the cubic lattice \mathbb{Z}^ν that holds asymptotically for large dimension ν . More precisely, we propose the first two terms in the expansion in powers of ν^{-1} . The family of quantum spin systems includes the spin $\frac{1}{2}$ ferromagnetic and antiferromagnetic Heisenberg models and the XXZ model. We also consider spin 1 quantum nematic systems. Our results are expected to be exact but they are not rigorous on \mathbb{Z}^ν . In fact we do not perform calculations with the cubic lattice but we consider the model on regular trees with d descendants; we obtain the first two terms of the critical inverse temperature in powers of d^{-1} . For trees our computations are completely rigorous. We conjecture that our expression applies to \mathbb{Z}^ν when taking $d = 2\nu - 1$.

We use a random loop representation. We describe it in Section 1.1. The consequences for the quantum models are discussed in Section 1.2.

1.1. Random-loop model. The present model of random loops is defined for arbitrary finite graphs and we consider here regular trees. Let T denote an infinite rooted tree where each vertex has $d \geq 2$ offsprings, and write ρ for its root. For $m \geq 0$ let T_m denote the subtree of T consisting of the first m generations (ρ being generation zero). Write V_m and E_m for the vertex- and edge-sets of T_m .

Let $\mathbb{P}_m(\cdot)$ denote a probability measure governing a collection $\omega = (\omega_{xy} : xy \in E_m)$ of independent Poisson processes on the interval $[0, 1]$, indexed by the edge-set E_m , each having rate β (the inverse-temperature). We refer to ω as a collection of *links*, and to ω_{xy} as the links *supported by* the edge xy . Thus, disjoint sub-intervals $I, J \subseteq [0, 1]$ independently receive uniformly placed links, their number being Poisson-distributed with mean $\beta|I|$ and $\beta|J|$, respectively. We write $\mathbb{E}_m[\cdot]$ for expectation under $\mathbb{P}_m(\cdot)$.

A given link is assigned to be a *cross* with probability u , otherwise a *double-bar*, independently between different links. The collection of links then decomposes $T_m \times [0, 1]$ into a collection of disjoint *loops* in a natural way. Rather than giving a formal definition here, we refer to Fig. 1.

The total number of loops is denoted $\ell = \ell(\omega)$. We actually work with a weighted version of $\mathbb{P}_m(\cdot)$, denoted $\mathbb{P}_m^{(\theta)}(\cdot)$ with a positive parameter θ . This is the probability measure whose

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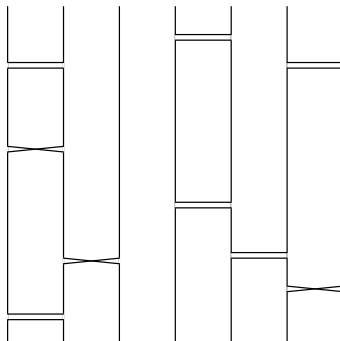


FIGURE 1. Random loops coming from a configuration ω of crosses and bars, in the case when the underlying graph is a line with seven vertices. To each vertex correspond a vertical line segment which is a copy of the interval $[0, 1]$. Loops reverse direction when traversing a double-bar, maintain direction when traversing a cross, and proceed periodically in the vertical direction. In this example there are $\ell(\omega) = 4$ loops.

expectation operator $\mathbb{E}_m^{(\theta)}[\cdot]$ is given by

$$\mathbb{E}_m^{(\theta)}[X] = \frac{\mathbb{E}_m[X\theta^{\ell(\omega)}]}{\mathbb{E}_m[\theta^{\ell(\omega)}]}. \quad (1.1)$$

Note that $\mathbb{P}_m^{(1)} = \mathbb{P}_m$.

All loops are small when β is small, but it is expected that there exists β_c , that depends on the parameter θ and the outdegree d , such that infinite loops occur with positive probability for $\beta > \beta_c$. Our main result is a formula for β_c ; it is asymptotic in the outdegree $d \rightarrow \infty$, namely

$$\frac{\beta_c}{\theta} = \frac{1}{d} + \frac{1 - \theta u(1 - u) - \frac{1}{6}\theta^2(1 - u)^2}{d^2} + o(d^{-2}), \quad (1.2)$$

and we can prove that there are infinite loops for $\beta > \beta_c$ in the vicinity of β_c . For a more precise statement, see Theorem 1.1 below.

The first study of this model on trees is due to Angel [2], who established the presence of long loops for a range of parameters β ; he only considered the case $u = 1$ and $\theta = 1$. Angel's results were extended by Hammond [9, 10]; he gave a precise characterisation of the critical parameter β_c . The formula (1.2) was established in [6] in the case $\theta = 1$ — our study following a suggestion of Hammond. Another, independent, extension to $\theta \neq 1$ has been proposed by Betz, Ehlert, and Lees [4].

1.2. Quantum spin systems. Let (Λ, \mathcal{E}) denote a graph, with Λ the set of vertices and \mathcal{E} the set of edges. The relevant graphs here are regular trees, and finite subsets of \mathbb{Z}^d with nearest-neighbour edges. The spin- $\frac{1}{2}$ systems have Hilbert space $\mathcal{H}_\Lambda = \otimes_{x \in \Lambda} \mathbb{C}^2$ and the hamiltonian is

$$H_\Lambda = -2 \sum_{\{x,y\} \in \mathcal{E}} \left(S_x^{(1)} S_y^{(1)} + S_x^{(2)} S_y^{(2)} + \Delta S_x^{(3)} S_y^{(3)} \right), \quad (1.3)$$

where $S_x^{(i)}$, $i = 1, 2, 3$ denotes the i th spin operator at site $x \in \Lambda$. Here, $\Delta \in [-1, 1]$ is a parameter.

As was progressively understood in [13, 1, 15], this quantum system is represented by the model of random loops with $\theta = 2$ and $u = \frac{1}{2}(1 + \Delta)$. Indeed, the quantum two-point

correlation function is given by loop correlations,

$$\langle S_x^{(1)} S_y^{(1)} \rangle := \frac{\text{tr} (S_x^{(1)} S_y^{(1)} e^{-\beta H_\Lambda})}{\text{tr} (e^{-\beta H_\Lambda})} = \frac{1}{4} \mathbb{P}_\Lambda^{(\theta=2)}(x \leftrightarrow y), \quad (1.4)$$

where $\{x \leftrightarrow y\}$ is the event that $(x, 0)$ and $(y, 0)$ belong to the same loop. It follows that magnetic long-range order is related to the occurrence of large loops.

On \mathbb{Z}^3 , the critical inverse temperature has been computed numerically; it was found that

$$\beta_c^{(\nu=3)}(\Delta) = \begin{cases} 0.59 & \text{if } \Delta = 1; \text{ Troyer et.al. [14]} \\ 0.496 & \text{if } \Delta = 0; \text{ Wessel, private communication in [3]} \\ 0.53 & \text{if } \Delta = -1; \text{ Sandvik [11]} \end{cases} \quad (1.5)$$

For large ν , the lattice \mathbb{Z}^ν behaves like a tree of outdegree $d = 2\nu - 1$. Our formula (1.2) gives

$$\beta_c^{(\nu)}(\Delta) = \frac{1}{\nu} + \frac{1}{\nu^2} \left[1 - \frac{1}{6} (1 - \Delta)(2 + \Delta) \right] + o\left(\frac{1}{\nu^2}\right). \quad (1.6)$$

With $\nu = 3$, the values for $\Delta = 1, 0, -1$ are $\frac{4}{9}, \frac{11}{27}, \frac{11}{27}$ respectively. They corroborate the numerical values (1.5) to some extent. Of course, the formula (1.6) gets more accurate in high dimensions.

In the case of spin-1 systems, the Hilbert space is $\mathcal{H}_\Lambda = \otimes_{x \in \Lambda} \mathbb{C}^3$ and the hamiltonian is

$$H_\Lambda = - \sum_{\{x,y\} \in \mathcal{E}} \left(u \vec{S}_x \cdot S_y + (\vec{S}_x \cdot S_y)^2 \right), \quad (1.7)$$

See [15]. The phase diagram of this model was uncovered in [7]. For $0 < u < 1$ the system displays nematic long-range order at low temperatures (if $d \geq 3$; also in the ground state when $d = 2$). This was rigorously proved in [12, 15]. The corresponding loop model has parameter $\theta = 3$, and the same u as in (1.7). Loop correlations are related to nematic long-range order, namely

$$\langle A_x A_y \rangle = \frac{2}{9} \mathbb{P}_\Lambda^{(\theta=3)}(x \leftrightarrow y), \quad (1.8)$$

with $A_x = (S_x^{(3)})^2 - \frac{2}{3}$. We are not aware of numerical calculations of the critical inverse temperature β_c for this model on \mathbb{Z}^3 . With $\theta = 3$ and $d = 2\nu - 1$, the formula (1.2) gives

$$\beta_c^{(\nu)}(u) = \frac{3}{2\nu} + \frac{3}{2\nu^2} \left[1 - \frac{3}{4} (1 - u^2) \right] + o\left(\frac{1}{\nu^2}\right). \quad (1.9)$$

1.3. Main result. In the rest of this paper we deal only with the probabilistic model of random loops defined above, and we allow θ to be any (fixed) positive real number. Our main result is that, as the distance between x and y goes to ∞ , the two-point function vanishes or stays positive, according to whether β is smaller or larger than β_c given above. Motivated by (1.4) and (1.8) we consider

$$\sigma_m = \mathbb{P}_m^{(\theta)}(\rho \leftrightarrow m), \quad (1.10)$$

that is, σ_m is the $\mathbb{P}_m^{(\theta)}$ -probability that $(\rho, 0)$ belongs to a loop which reaches some vertex in generation m in T_m .

Throughout this paper we work with β of the form

$$\beta/\theta = \frac{1}{d} + \frac{\alpha}{d^2}, \quad \text{where } |\alpha| \leq \alpha_0 \quad (1.11)$$

for some fixed but arbitrary $\alpha_0 > 0$. All error terms $O(\cdot)$, $o(\cdot)$ and constants may depend on α_0 but are otherwise uniform in α .

Theorem 1.1. *Consider β of the form (1.11), and write*

$$\alpha_* = \alpha_*(\theta, u) = 1 - \theta u(1 - u) - \frac{1}{6}\theta^2(1 - u)^2.$$

For any $\delta > 0$ there exists $d_0 = d_0(\theta, u, \alpha_, \varepsilon)$ such that for $d \geq d_0$ we have:*

- *if $\alpha \leq \alpha_* - \delta$ then $\lim_{m \rightarrow \infty} \sigma_m = 0$;*
- *if $\alpha \geq \alpha_* + \delta$ then $\liminf_{m \rightarrow \infty} \sigma_m > 0$.*

Let us remark that for $\theta = 1$ the result was shown in our previous work [6]. The arguments presented here are strengthened versions of those arguments.

2. PROOF OF THE MAIN RESULT

In this article the indicator function of an event A is written $\mathbb{1}_A$ or $\mathbb{1}\{A\}$. The partition function for the loop model in T_m is written $Z_m = \mathbb{E}_m[\theta^\ell]$. For convenience we also define

$$z_m = e^{-d\beta(1-1/\theta)} \frac{\theta Z_{m-1}^d}{Z_m}. \quad (2.1)$$

For given $m \geq 1$ and $\varepsilon > 0$ we define

$$\tilde{\sigma}_m = \sigma_m \wedge \sigma_{m-1} \wedge \left(\frac{\varepsilon}{d}\right). \quad (2.2)$$

(A priori we need not have $\sigma_m \leq \sigma_{m-1}$ since they are computed using different measures.) In this section we will prove the following recursion-inequalities.

Proposition 2.1. *For all $m \geq 1$ we have*

$$\sigma_m \geq \tilde{\sigma}_{m-1} + \frac{\tilde{\sigma}_{m-1}}{d}(\alpha - \alpha_*) - \frac{1}{2}\tilde{\sigma}_{m-1}^2 + O(d^{-3}), \quad (2.3)$$

and

$$\sigma_m \leq (\sigma_{m-1} \vee \sigma_{m-2}) \left[1 + \frac{1}{d}(\alpha - \alpha_*) + O(d^{-2})\right]. \quad (2.4)$$

Here the $O(d^{-3})$ and $O(d^{-2})$ are uniform in m .

Our main result follows easily:

Proof of Thm 1.1. First suppose $\alpha > \alpha_*$. Since $\sigma_0 = 1$ and σ_1 is bounded below by a positive constant uniformly in d , if $\varepsilon > 0$ is small enough and d large enough then (2.3) gives by induction that $\sigma_m \geq \tilde{\sigma}_m \geq \varepsilon/d$ for all $m \geq 1$.

Now suppose $\alpha < \alpha_*$. For d large enough the factor in square brackets in (2.4) is strictly smaller than 1. This easily gives that σ_m decays to 0 exponentially fast. \square

2.1. Preliminary calculations. Let us first introduce some notations and prove some facts that will be used for establishing both bounds in Prop 2.1.

Write A_1 for the event that, for each child x of ρ , there is at most one link between ρ and x . Write A_2 for the event that: (i) there is a unique child x of ρ with exactly 2 links between ρ and x , (ii) for all siblings x' of x there is at most one link between ρ and x' , and (iii) for all children y of x there is at most one link between x and y . See Fig. 2.

Let $\zeta_m = 1 - \sigma_m$ and let B_m^ρ be the event that $(\rho, 0)$ does *not* belong to a loop which reaches generation m in T_m , thus $\mathbb{P}_m^{(\theta)}(B_m^\rho) = \zeta_m$. Clearly we have that

$$\zeta_m = \mathbb{P}_m^{(\theta)}(B_m^\rho) = \mathbb{P}_m^{(\theta)}(B_m^\rho \cap A_1) + \mathbb{P}_m^{(\theta)}(B_m^\rho \cap A_2) + \mathbb{P}_m^{(\theta)}(B_m^\rho \setminus (A_1 \cup A_2)). \quad (2.5)$$

Let us enumerate the children of ρ by $i = 1, \dots, d$ and let ℓ_i denote the number of loops in the restriction of ω to the subtree rooted at child i . On the event A_1 , and if there are k links from ρ , the number ℓ of loops satisfies

$$\ell = \sum_{i=1}^d \ell_i - k + 1. \quad (2.6)$$

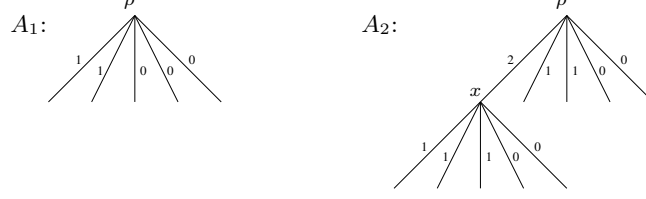


FIGURE 2. Illustrations of the two events A_1 and A_2 . Numbers on edges indicate the number of links.

To see this, one may imagine that the k links to ρ are put in last, one at a time. Each such link then merges some loop in the corresponding subtree with a loop visiting ρ . It follows that

$$\begin{aligned} \mathbb{E}_m[\theta^\ell \mathbb{1}_{A_1}] &= \sum_{k=0}^d \theta^{-k+1} \mathbb{E}_m[\theta^{\sum_i \ell_i} \mathbb{1}_{A_1} \mathbb{1}\{k \text{ links at } \rho\}] \\ &= \theta Z_{m-1}^d (e^{-\beta} (1 + \beta/\theta))^d \end{aligned} \quad (2.7)$$

and hence (recalling z_m from (2.1))

$$\mathbb{P}_m^{(\theta)}(A_1) = z_m (e^{-\beta/\theta} (1 + \frac{\beta}{\theta}))^d. \quad (2.8)$$

Similarly, since the k children with links would need to be blocked from reaching distance $m-1$, we also have

$$\mathbb{P}_m^{(\theta)}(A_1 \cap B_m^\rho) = z_m (e^{-\beta/\theta} (1 + \zeta_{m-1} \frac{\beta}{\theta}))^d. \quad (2.9)$$

For the event A_2 , we decompose it as $A_2 = A_2^{\text{mix}} \cup A_2^{\text{same}}$, according as the 2 links from ρ to x are different sorts (crosses/double-bars) or the same (Fig 3). If we look at the restriction of ω to the link ρx only (i.e., at $\omega_{\rho x}$) then it has two loops on A_2^{same} and a single loop on A_2^{mix} . Let us number the children of x together with the children of ρ excepting x by $i = 1, \dots, 2d-1$. Then we see that

$$\ell = \sum_{i=1}^{2d-1} \ell_i - k + \begin{cases} 1 & \text{on } A_2^{\text{mix}}, \\ 2 & \text{on } A_2^{\text{same}}, \end{cases} \quad (2.10)$$

where k denotes the total number of 1-links at ρ and at x . Let Λ denote the loop in $\omega_{\rho x}$ containing $(\rho, 0)$, and let $\Lambda_\rho = \Lambda \cap (\{\rho\} \times [0, 1])$ and $\Lambda_x = \Lambda \cap (\{x\} \times [0, 1])$ denote the parts of Λ at ρ and at x , respectively. There is a random variable X such that Λ_ρ and Λ_x have respective lengths X and $1-X$ in the case of two crosses; X and X in the case of two double-bars; and 1 in the case of a mixture. One may check that X has mean $\frac{2}{3}$ (to be precise, $\mathbb{E}_m[X | A_2] = \frac{2}{3}$).

For B_m^ρ to happen, children of ρ which link to Λ need to be blocked from distance $m-1$ and children of x which link to Λ need to be blocked from distance $m-2$; the remaining children of ρ and x do not need to be blocked. A calculation therefore gives

$$\mathbb{E}_m[\theta^\ell \mathbb{1}_{B_m^\rho} \mathbb{1}_{A_2^{\text{mix}}}] = \theta Z_{m-1}^{d-1} Z_{m-2}^d \frac{d\beta^2 e^{-\beta}}{2} 2u(1-u) (e^{-\beta} (1 + \frac{\beta}{\theta} \zeta_{m-1}))^{d-1} (e^{-\beta} (1 + \frac{\beta}{\theta} \zeta_{m-2}))^d \quad (2.11)$$

and

$$\begin{aligned} \mathbb{E}_m[\theta^\ell \mathbb{1}_{B_m^\rho} \mathbb{1}_{A_2^{\text{same}}}] &= \theta^2 Z_{m-1}^{d-1} Z_{m-2}^d \frac{d\beta^2 e^{-\beta}}{2} (u^2 + (1-u)^2) \\ &\quad \mathbb{E}[(e^{-\beta} (1 + \frac{\beta}{\theta} \zeta_{m-1} |\Lambda_\rho| + \frac{\beta}{\theta} (1 - |\Lambda_\rho|)))^{d-1} (e^{-\beta} (1 + \frac{\beta}{\theta} \zeta_{m-2} |\Lambda_x| + \frac{\beta}{\theta} (1 - |\Lambda_x|)))^d]. \end{aligned} \quad (2.12)$$

Here we have simply written $\mathbb{E}[\cdot]$ for $\mathbb{E}_m[\cdot | A_2]$, this expectation is over the choice of crosses or double-bars and over the lengths $|\Lambda_\rho|$ and $|\Lambda_x|$ only.

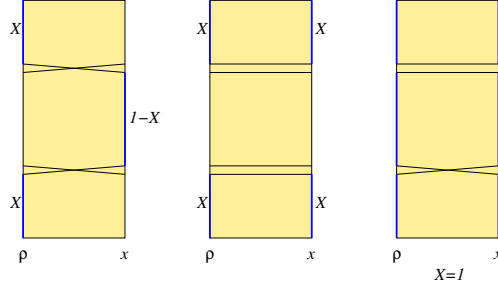


FIGURE 3. Illustration of the possibilities for $\omega_{\rho x}$ on the event A_2 . On A_2^{same} there are two loops, one of which contains $(\rho, 0)$; on A_2^{mix} only one. The latter is thus more advantageous for long connections. The random variable X has mean $\frac{2}{3}$.

At this point, let us mention the following asymptotics, which will be useful several times: if $\sigma = O(d^{-1})$ and $x \in \mathbb{R}$ then we have

$$\left(e^{-\beta/\theta} \left(1 + \frac{\beta}{\theta} - \sigma x \frac{\beta}{\theta}\right)\right)^d = 1 - \frac{1}{d}(1/2 + x\sigma d) + \frac{1}{d^2}(1/3 - \alpha + x\sigma d - \alpha x\sigma d + \frac{1}{2}(1/2 + x\sigma d)^2) + O(d^{-3}). \quad (2.13)$$

Replacing ζ_{m-1} and ζ_{m-2} in (2.11) and (2.12) by 1, and adding the equations together, we see that

$$\begin{aligned} \mathbb{P}_m^{(\theta)}(A_2) &= z_m z_{m-1} \frac{d\beta^2 e^{-\beta/\theta}}{2\theta} \left(e^{-\beta/\theta} \left(1 + \frac{\beta}{\theta}\right)\right)^{2d-1} (2u(1-u) + \theta(u^2 + (1-u)^2)) \\ &= z_m z_{m-1} \frac{\theta}{2d} \left[1 - \frac{1}{d}\right] (2u(1-u) + \theta(u^2 + (1-u)^2) + O(d^{-2})). \end{aligned} \quad (2.14)$$

For the last step we used (2.13) to first order, and that

$$\frac{d\beta^2 e^{-\beta/\theta}}{2\theta} = \frac{\theta}{2d} + O(d^{-2}). \quad (2.15)$$

2.2. Stochastic domination. In some estimates we will want to approximate the complicated measure $\mathbb{P}_m^{(\theta)}(\cdot)$, which involves counting loops, by some simpler measure. For this we use *stochastic domination*. Let us define $\beta^+ = \max\{\beta\theta, \beta/\theta\}$, thus $\beta^+ = \beta\theta$ if $\theta > 1$ and $\beta^+ = \beta/\theta$ otherwise. Also let us define \mathbb{E}_m^+ in the same way as \mathbb{E}_m but with β replaced by β^+ ; thus the links form independent Poisson processes with rate β^+ . We say that an event A is *increasing* if it cannot be destroyed by adding more links; examples of increasing events include A_1^c and A_2^c where A_1 and A_2 are as defined above. Stochastic domination tells us that

$$A \text{ increasing} \quad \Rightarrow \quad \mathbb{P}_m^{(\theta)}(A) \leq \mathbb{P}_m^+(A). \quad (2.16)$$

See the proof of Lemma 3.1 in [5] for an argument and [8] for more information about stochastic domination. An immediate consequence is that there is some constant $c > 0$ such that

$$\mathbb{P}_m^{(\theta)}(A_1^c \cap A_2^c) \leq c/d^2 \quad \text{for all } m, d \geq 1. \quad (2.17)$$

We now deduce some information about the asymptotic behaviour of the numbers $z_m = e^{-d\beta(1-1/\theta)} \theta Z_{m-1}^d / Z_m$. We write

$$q = q(\theta, u) = \frac{\theta}{2} (2u(1-u) + \theta(u^2 + (1-u)^2)) \quad (2.18)$$

$$r = r(\theta, u) = 2\theta u(1-u) + \frac{1}{2}\theta^2(u^2 + \frac{4}{3}(1-u)^2) \quad (2.19)$$

so that $\alpha_* = 1 + q - r$.

Proposition 2.2. *There is a constant C and there are functions $\varepsilon_m^{(j)}(d)$ satisfying*

$$|\varepsilon_m^{(j)}(d)| \leq C/d^2 \text{ for all } m, d \geq 1, \quad (2.20)$$

such that

$$z_m = 1 - \frac{1}{d}(q - 1/2) + \varepsilon_m^{(1)}(d) \quad (2.21)$$

and

$$z_m \left(1 + \frac{1}{d}(q - 1/2) + \varepsilon_m^{(2)}(d)\right) = 1 - \varepsilon_m^{(3)}(d). \quad (2.22)$$

Proof. Note that (2.21) and (2.22) are equivalent, hence one may proceed by induction on m , proving (2.22) with the induction hypothesis provided by (2.21). For the base case $m = 1$ one may establish (2.21) directly, using

$$Z_1 = \theta^{d+1} e^{-d\beta(1-1/\theta)} \left[\left(e^{-\beta/\theta} \left(1 + \frac{\beta}{\theta}\right) \right)^d + q(\theta, u) \frac{d\beta^2 e^{-\beta/\theta}}{\theta^2} \left(e^{-\beta/\theta} \left(1 + \frac{\beta}{\theta}\right) \right)^{d-1} + \varepsilon_1(d) \right], \quad (2.23)$$

where $0 \leq \varepsilon_1(d) \leq e^{d\beta(1-1/\theta)} \mathbb{P}_1(A_1^c \cap A_2^c)$ satisfies (2.20).

For $m > 1$, write $\varepsilon_m^{(3)}(d) = \mathbb{P}_m^{(\theta)}(A_1^c \cap A_2^c)$, this satisfies (2.20) by (2.17). From (2.8) and (2.14) we have

$$\begin{aligned} 1 &= \mathbb{P}_m^{(\theta)}(A_1) + \mathbb{P}_m^{(\theta)}(A_2) + \mathbb{P}_m^{(\theta)}(A_1^c \cap A_2^c) = \\ &= z_m \left(e^{-\beta/\theta} \left(1 + \frac{\beta}{\theta}\right) \right)^d + z_m z_{m-1} \frac{d\beta^2 e^{-\beta/\theta}}{\theta^2} \left(e^{-\beta/\theta} \left(1 + \frac{\beta}{\theta}\right) \right)^{2d-1} q(\theta, u) + \varepsilon_m^{(3)}(d) \end{aligned} \quad (2.24)$$

Hence, using (2.13) and (2.15),

$$1 - \varepsilon_m^{(3)}(d) = z_m \left[1 - \frac{1}{2d} + z_{m-1} \frac{1}{d} \left(1 - \frac{1}{d}\right) q + \varepsilon^{(4)}(d) \right] \quad (2.25)$$

for a function $\varepsilon^{(4)}(d)$ not depending on m but otherwise satisfying the bounds (2.20). Using the induction hypothesis we get

$$1 - \varepsilon_m^{(3)}(d) = z_m \left[1 + \frac{1}{d} \left(q - \frac{1}{2} \right) + \varepsilon_m^{(2)}(d) \right], \quad (2.26)$$

where

$$\varepsilon_m^{(2)}(d) = \varepsilon^{(4)}(d) - \frac{q}{d^2} \left(q + \frac{1}{2} \right) + \frac{q}{d^3} \left(q - \frac{1}{2} \right) + \frac{q}{d} \left(1 - \frac{1}{d} \right) \varepsilon_{m-1}^{(1)}(d)$$

is easily seen to satisfy (2.20). \square

Remark 2.3. *From the proposition it follows that*

$$z_m z_{m-1} = 1 + O(d^{-1}) \quad (2.27)$$

where the $O(\cdot)$ is uniform in m .

We now turn to the details of the proof of Prop. 2.1.

2.3. Proof of the lower bound (2.3). We have from (2.5) that

$$\sigma_m \geq \mathbb{P}_m^{(\theta)}(A_1) - \mathbb{P}_m^{(\theta)}(B_m^\rho \cap A_1) + \mathbb{P}_m^{(\theta)}(A_2) - \mathbb{P}_m^{(\theta)}(B_m^\rho \cap A_2) \quad (2.28)$$

Consider first the terms involving A_1 . From (2.8) and (2.9) and using (2.13) as well as Prop. 2.2 we get

$$\begin{aligned} \mathbb{P}_m^{(\theta)}(A_1) - \mathbb{P}_m^{(\theta)}(B_m^\rho \cap A_1) &\geq z_m \left(\tilde{\sigma}_{m-1} - \frac{\tilde{\sigma}_{m-1}}{d} (3/2 - \alpha) - \frac{1}{2} \tilde{\sigma}_{m-1}^2 + O(d^{-3}) \right) \\ &= \left(1 - \frac{1}{d} (q - 1/2) \right) \left(\tilde{\sigma}_{m-1} - \frac{\tilde{\sigma}_{m-1}}{d} (3/2 - \alpha) - \frac{1}{2} \tilde{\sigma}_{m-1}^2 \right) + O(d^{-3}) \\ &= \tilde{\sigma}_{m-1} + \frac{\tilde{\sigma}_{m-1}}{d} (\alpha - q - 1) - \frac{1}{2} \tilde{\sigma}_{m-1}^2 + O(d^{-3}). \end{aligned} \quad (2.29)$$

Now consider the terms involving A_2 . Using that $\zeta_{m-1}, \zeta_{m-2} \leq 1 - \tilde{\sigma}_{m-1}$, as well as the asymptotics (2.13) to order d^{-1} , we deduce from (2.11) that

$$\begin{aligned} \mathbb{P}_m^{(\theta)}(B_m^\rho \cap A_2^{\text{mix}}) &\leq z_m z_{m-1} \frac{d\beta^2 e^{-\beta/\theta}}{2\theta} 2u(1-u) \left(e^{-\beta/\theta} \left(1 + \frac{\beta}{\theta} - \tilde{\sigma}_{m-1} \frac{\beta}{\theta} \right) \right)^{2d-1} \\ &= z_m z_{m-1} \frac{\theta}{2d} \left[\left(1 - \frac{1}{d} \right) 2u(1-u) - \tilde{\sigma}_{m-1} 4u(1-u) + O(d^{-2}) \right] \end{aligned} \quad (2.30)$$

and from (2.12) that

$$\begin{aligned} \mathbb{P}_m^{(\theta)}(B_m^\rho \cap A_2^{\text{same}}) &\leq z_m z_{m-1} \frac{d\beta^2 e^{-\beta/\theta}}{2} (u^2 + (1-u)^2) \\ &\quad \mathbb{E} \left[\left(e^{-\beta/\theta} \left(1 + \frac{\beta}{\theta} - \tilde{\sigma}_{m-1} |\Lambda_\rho| \frac{\beta}{\theta} \right) \right)^{d-1} \left(e^{-\beta/\theta} \left(1 + \frac{\beta}{\theta} - \tilde{\sigma}_{m-1} |\Lambda_x| \frac{\beta}{\theta} \right) \right)^d \right] \\ &= z_m z_{m-1} \frac{\theta^2}{2d} (u^2 + (1-u)^2) \left[\left(1 - \frac{1}{d} \right) - \tilde{\sigma}_{m-1} \mathbb{E}(|\Lambda_\rho| + |\Lambda_x|) + O(d^{-2}) \right] \\ &= z_m z_{m-1} \frac{\theta^2}{2d} \left[\left(1 - \frac{1}{d} \right) (u^2 + (1-u)^2) - \tilde{\sigma}_{m-1} \left(u^2 + \frac{4}{3}(1-u)^2 \right) + O(d^{-2}) \right]. \end{aligned} \quad (2.31)$$

(See also Fig. 3.) Using also (2.14) and (2.27) we get

$$\begin{aligned} &\mathbb{P}_m^{(\theta)}(A_2) - \mathbb{P}_m^{(\theta)}(B_m^\rho \cap A_2) \\ &\geq z_m z_{m-1} \frac{\theta}{2d} \left\{ 2u(1-u) \left(\left[1 - \frac{1}{d} \right] - \left[1 - \frac{1}{d} - 2\tilde{\sigma}_{m-1} \right] \right) \right. \\ &\quad \left. + \theta(u^2 + (1-u)^2) \left(\left[1 - \frac{1}{d} \right] - \left[1 - \frac{1}{d} - \tilde{\sigma}_{m-1} \frac{u^2 + \frac{4}{3}(1-u)^2}{u^2 + (1-u)^2} \right] \right) + O(d^{-2}) \right\} \\ &= r(\theta, u) \frac{\tilde{\sigma}_{m-1}}{d} + O(d^{-3}), \end{aligned} \quad (2.32)$$

where r is defined in (2.19). Putting this together in (2.28) gives

$$\sigma_m \geq \tilde{\sigma}_{m-1} + \frac{\tilde{\sigma}_{m-1}}{d} (\alpha - [1 + q - r]) - \frac{1}{2} \tilde{\sigma}_{m-1}^2 + O(d^{-3}). \quad (2.33)$$

Since $\alpha_* = 1 + q - r$ this gives (2.3). \square

2.4. Proof of the upper bound (2.4). Write Σ_m^ρ for the complement of B_m^ρ , so that $\sigma_m = \mathbb{P}_m^{(\theta)}(\Sigma_m^\rho)$. Clearly

$$\sigma_m = \mathbb{P}_m^{(\theta)}(A_1 \cap \Sigma_m^\rho) + \mathbb{P}_m^{(\theta)}(A_2 \cap \Sigma_m^\rho) + \mathbb{P}_m^{(\theta)}(A_1^c \cap A_2^c \cap \Sigma_m^\rho). \quad (2.34)$$

The following will be proved at the end of this section:

Proposition 2.4. *For all d large enough there is a constant C such that*

$$\mathbb{P}_m^{(\theta)}(A_1^c \cap A_2^c \cap \Sigma_m^\rho) \leq \frac{C}{d^2} (\sigma_{m-1} \vee \sigma_{m-2}). \quad (2.35)$$

Before proving this we show how to deduce (2.4). We have from (2.8) and (2.9) that

$$\begin{aligned} \mathbb{P}_m^{(\theta)}(A_1 \cap \Sigma_m^\rho) &= z_m \left\{ \left(e^{-\beta/\theta} \left(1 + \frac{\beta}{\theta} \right) \right)^d - \left(e^{-\beta/\theta} \left(1 + \frac{\beta}{\theta} - \frac{\beta}{\theta} \sigma_{m-1} \right) \right)^d \right\} \\ &= z_m \left(e^{-\beta/\theta} \left(1 + \frac{\beta}{\theta} \right) \right)^d \left\{ 1 - \left(1 - \frac{\beta \sigma_{m-1}}{1 + \beta/\theta} \right)^d \right\} \\ &\leq z_m \left(e^{-\beta/\theta} \left(1 + \frac{\beta}{\theta} \right) \right)^d \frac{d\beta}{\theta} \left(1 + \frac{\beta}{\theta} \right)^{-1} \sigma_{m-1}. \end{aligned} \quad (2.36)$$

In the last step we used the concavity of the function $f(x) = 1 - (1-x)^d$ to bound $f(x) \leq x f'(0)$.

Similarly using (2.11) and concavity of $f(x, y) = 1 - (1 - x)^{d-1}(1 - y)^d$ (for $d \geq 3$),

$$\begin{aligned} \mathbb{P}_m^{(\theta)}(\Sigma_m^\rho \cap A_2^{\text{mix}}) &= z_m z_{m-1} \frac{d\beta^2 e^{-\beta/\theta}}{2} \frac{2}{\theta} u(1-u) (e^{-\beta/\theta} (1 + \frac{\beta}{\theta}))^{2d-1} \left\{ 1 - \left(1 - \frac{\frac{\beta}{\theta} \sigma_{m-1}}{1 + \frac{\beta}{\theta}} \right)^{d-1} \left(1 - \frac{\frac{\beta}{\theta} \sigma_{m-2}}{1 + \frac{\beta}{\theta}} \right)^d \right\} \\ &\leq z_m z_{m-1} \frac{d\beta^2 e^{-\beta/\theta}}{2} \frac{2}{\theta} u(1-u) (e^{-\beta/\theta} (1 + \frac{\beta}{\theta}))^{2d-1} \frac{\beta}{\theta} (1 + \frac{\beta}{\theta})^{-1} \{(d-1)\sigma_{m-1} + d\sigma_{m-2}\} \\ &\leq z_m z_{m-1} \frac{d\beta^2 e^{-\beta/\theta}}{2} (e^{-\beta/\theta} (1 + \frac{\beta}{\theta}))^{2d-1} \frac{d\beta}{\theta} (1 + \frac{\beta}{\theta})^{-1} \frac{2}{\theta} u(1-u) (\sigma_{m-1} + \sigma_{m-2}). \end{aligned} \quad (2.37)$$

The same argument applied to (2.12) gives (with the notation \mathbb{E} used there)

$$\begin{aligned} \mathbb{P}_m^{(\theta)}(\Sigma_m^\rho \cap A_2^{\text{same}}) &= z_m z_{m-1} \frac{d\beta^2 e^{-\beta/\theta}}{2} (u^2 + (1-u)^2) (e^{-\beta/\theta} (1 + \frac{\beta}{\theta}))^{2d-1} \\ &\quad \cdot \mathbb{E} \left[1 - \left(1 - \frac{\frac{\beta}{\theta} \sigma_{m-1} |\Lambda_\rho|}{1 + \frac{\beta}{\theta}} \right)^{d-1} \left(1 - \frac{\frac{\beta}{\theta} \sigma_{m-2} |\Lambda_x|}{1 + \frac{\beta}{\theta}} \right)^d \right] \\ &\leq z_m z_{m-1} \frac{d\beta^2 e^{-\beta/\theta}}{2} (u^2 + (1-u)^2) (e^{-\beta/\theta} (1 + \frac{\beta}{\theta}))^{2d-1} \frac{\beta}{\theta} (1 + \frac{\beta}{\theta})^{-1} \\ &\quad \cdot \{(d-1)\sigma_{m-1} \mathbb{E}|\Lambda_\rho| + d\sigma_{m-2} \mathbb{E}|\Lambda_x|\} \\ &\leq z_m z_{m-1} \frac{d\beta^2 e^{-\beta/\theta}}{2} (e^{-\beta/\theta} (1 + \frac{\beta}{\theta}))^{2d-1} \frac{d\beta}{\theta} (1 + \frac{\beta}{\theta})^{-1} \\ &\quad \cdot \{\sigma_{m-1} \frac{2}{3} (u^2 + (1-u)^2) + \sigma_{m-2} (\frac{1}{3} u^2 + \frac{2}{3} (1-u)^2)\} \end{aligned} \quad (2.38)$$

From Prop. 2.2, (2.13) and $\frac{d\beta}{\theta} (1 + \frac{\beta}{\theta})^{-1} = 1 + \frac{\alpha-1}{d} + O(d^{-2})$ we see that the right-hand-side of (2.36) satisfies

$$z_m (e^{-\beta/\theta} (1 + \frac{\beta}{\theta}))^d \frac{d\beta}{\theta} (1 + \frac{\beta}{\theta})^{-1} \sigma_{m-1} = \left(1 + \frac{\alpha - (1+q)}{d} + O(d^{-2}) \right) \sigma_{m-1}, \quad (2.39)$$

where $q = q(\theta, u)$ was defined in (2.18). Similarly, using (2.15) and (2.27), in the right-hand-sides of (2.37) and (2.38) we have the factors

$$z_m z_{m-1} \frac{d\beta^2 e^{-\beta/\theta}}{2} (e^{-\beta/\theta} (1 + \frac{\beta}{\theta}))^{2d-1} \frac{d\beta}{\theta} (1 + \frac{\beta}{\theta})^{-1} = \frac{\theta^2}{2d} (1 + O(d^{-1})). \quad (2.40)$$

Hence, bounding also σ_{m-1} and σ_{m-2} by their maximum, we have that

$$\begin{aligned} \sigma_m &\leq (\sigma_{m-1} \vee \sigma_{m-2}) \left[1 + \frac{\alpha - (1+q)}{d} + \frac{\theta^2}{2d} \left(\frac{4}{\theta} u(1-u) + \frac{2}{3} (u^2 + (1-u)^2) + \frac{1}{3} u^2 + \frac{2}{3} (1-u)^2 \right) + O(d^{-2}) \right] \\ &\quad + \mathbb{P}_m^{(\theta)}(A_1^c \cap A_2^c \cap \Sigma_m^\rho) \\ &= (\sigma_{m-1} \vee \sigma_{m-2}) \left[1 + \frac{\alpha - (1+q-r)}{d} + O(d^{-2}) \right] + \mathbb{P}_m^{(\theta)}(A_1^c \cap A_2^c \cap \Sigma_m^\rho), \end{aligned} \quad (2.41)$$

where $r = r(\theta, u)$ was defined in (2.19). In the above, all $O(d^{-2})$ terms are uniform in m . Since $1 + q - r = \alpha_*$ we see that (2.4) follows once we prove Prop. 2.4.

In the following argument we will examine the subtree \tilde{T} of T_m which contains the root and is spanned by edges supporting at least two links. In \tilde{T} , the loop-structure is very complicated and we will not attempt to keep track of it. Instead we use that \tilde{T} is likely to be small, and that a loop exiting it must do so across an edge supporting exactly one link, which is a simpler situation to analyze. The details are quite technical.

Proof of Prop. 2.4. We begin by defining \tilde{T} carefully: we let \tilde{T} be the (random) subtree of T_m containing

- (1) the root ρ
- (2) any vertex in generation 1 with ≥ 2 links to ρ ,

- (3) in general, any vertex in generation k with ≥ 2 links to some vertex of \tilde{T} in generation $k-1$.

Note that $A_1^c \cap A_2^c$ is precisely the event that \tilde{T} has at least two edges. Let $V_k(\tilde{T})$ denote the set of vertices in \tilde{T} in generation k . For x a vertex of \tilde{T} , $x \notin V_m(\tilde{T})$, let d_x denote its number of descendants *not in* \tilde{T} . Thus x has d_x outgoing edges carrying only 0 or 1 link of ω . For $0 \leq k \leq m-1$ we let \mathcal{E}_k denote the set of outgoing links from generation k (to generation $k+1$) which carry precisely 1 link.

Note that if the loop of $(\rho, 0)$ reaches generation m then either it reaches generation m within \tilde{T} , or it passes some link of $\cup_{k=0}^{m-1} \mathcal{E}_k$. Let us by convention set $\sigma_{-1} = 1$ and $|\mathcal{E}_m| = |V_m(\tilde{T})|$. We claim that

$$\mathbb{P}_m^{(\theta)}(A_1^c \cap A_2^c \cap \Sigma_m^\rho) \leq \sum_{k=0}^m \sigma_{m-k-1} \mathbb{E}_m^{(\theta)}[|\mathcal{E}_k| \mathbb{1}_{A_1^c \cap A_2^c}]. \quad (2.42)$$

Intuitively, this is because if the loop exits \tilde{T} through some edge in \mathcal{E}_k , then it has distance $m-k-1$ left to go to reach the m :th generation of T_m . A detailed justification of (2.42) requires dealing with the dependencies caused by the factor θ^ℓ .

To do this, let us introduce the following notation. First, let $\tilde{\omega}$ denote the restriction of ω to \tilde{T} . Next, let $\partial^+ \tilde{T}$ denote the set of vertices $y \in T_m \setminus \tilde{T}$ whose parent belongs to \tilde{T} , and write ω_y for the restriction of ω to the subtree rooted at y . For simplicity, in the rest of this proof we simply write \mathbb{E} for \mathbb{E}_m . We will make use of the fact that, given $\tilde{\omega}$, the random collections $(\mathcal{E}_j)_{j=0}^{m-1}$ and $(\omega_y)_{y \in \partial^+ \tilde{T}}$ are conditionally independent under \mathbb{E} . This implies that for three functions

$$F_1(\tilde{\omega}), \quad F_2(\tilde{\omega}, (\mathcal{E}_j)_{j=0}^{m-1}), \quad F_3(\tilde{\omega}, (\omega_y)_{y \in \partial^+ \tilde{T}}) \quad (2.43)$$

we have

$$\begin{aligned} & \mathbb{E}[F_1(\tilde{\omega}) F_2(\tilde{\omega}, (\mathcal{E}_j)_{j=0}^{m-1}) F_3(\tilde{\omega}, (\omega_y)_{y \in \partial^+ \tilde{T}})] \\ &= \mathbb{E}[F_1(\tilde{\omega}) \mathbb{E}[F_2(\tilde{\omega}, (\mathcal{E}_j)_{j=0}^{m-1}) \mid \tilde{\omega}] \mathbb{E}[F_3(\tilde{\omega}, (\omega_y)_{y \in \partial^+ \tilde{T}}) \mid \tilde{\omega}]] \end{aligned} \quad (2.44)$$

Note that we have the decomposition

$$\ell = \tilde{\ell} + \sum_{j=0}^{m-1} \left[\sum_{x \in V_j(\tilde{T})} \sum_{i=1}^{d_x} \ell_i^{(x)} - |\mathcal{E}_j| \right], \quad (2.45)$$

where $\tilde{\ell}$ denotes the number of loops in the configuration $\tilde{\omega}$, and $\ell_i^{(x)}$ denotes the number of loops in the subtree rooted at the i :th descendant of x not belonging to \tilde{T} (in some numbering of these descendants). Hence

$$\theta^\ell = \theta^{\tilde{\ell}} \left(\prod_{j=0}^{m-1} \theta^{-|\mathcal{E}_j|} \right) \left(\prod_{j=0}^{m-1} \prod_{x \in V_j(\tilde{T})} \prod_{i=1}^{d_x} \theta^{\ell_i^{(x)}} \right) \quad (2.46)$$

is a factorization into three functions as in (2.43). Turning to (2.42), we have

$$\mathbb{P}_m^{(\theta)}(A_1^c \cap A_2^c \cap \Sigma_m^\rho) \leq \mathbb{E}_m^{(\theta)}[|V_m(\tilde{T})| \mathbb{1}_{A_1^c \cap A_2^c}] + \sum_e \sum_{k=0}^{m-1} \mathbb{P}_m^{(\theta)}(A_1^c \cap A_2^c \cap \{e \in \mathcal{E}_k\} \cap \{(e^+, t^+) \leftrightarrow m\}), \quad (2.47)$$

where the first sum is over all edges e of T_m , and $\{(e^+, t^+) \leftrightarrow m\}$ denotes the event that the further (from ρ) endpoint (e^+, t^+) of the unique link at e lies in a loop of ω_{e^+} reaching the

m :th generation of T_m . Applying (2.44) and (2.46) we have

$$\begin{aligned} & \mathbb{P}_m^{(\theta)}(A_1^c \cap A_2^c \cap \{e \in \mathcal{E}_k\} \cap \{(e^+, t^+) \leftrightarrow m\}) \\ &= \frac{1}{Z_m} \mathbb{E} \left[\mathbb{1}_{A_1^c \cap A_2^c} \theta^{\tilde{\ell}} \mathbb{E} \left[\mathbb{1}_{\{e \in \mathcal{E}_k\}} \prod_{j=0}^{m-1} \theta^{-|\mathcal{E}_j|} \mid \check{T} \right] \sigma_{m-k-1} \prod_{j=0}^{m-1} \prod_{x \in V_j(\check{T})} Z_{m-j-1}^{d_x} \right]. \end{aligned} \quad (2.48)$$

Taking out the factor σ_{m-k-1} , applying (2.44) again in reverse, and putting back into (2.47), we obtain (2.42).

We proceed by bounding the expectations

$$\mathbb{E}_m^{(\theta)}[|\mathcal{E}_k| \mathbb{1}_{A_1^c \cap A_2^c}] = \frac{1}{Z_m} \mathbb{E}[\theta^{\tilde{\ell}} |\mathcal{E}_k| \mathbb{1}_{A_1^c \cap A_2^c}]. \quad (2.49)$$

Arguing as above we get:

$$\mathbb{E}[\theta^{\tilde{\ell}} |\mathcal{E}_k| \mathbb{1}_{A_1^c \cap A_2^c}] = \mathbb{E} \left[\mathbb{1}_{A_1^c \cap A_2^c} \theta^{\tilde{\ell}} \prod_{j=0}^{m-1} \prod_{x \in V_j(\check{T})} Z_{m-j-1}^{d_x} \mathbb{E} \left[|\mathcal{E}_k| \prod_{j=0}^{m-1} \theta^{-|\mathcal{E}_j|} \mid \check{T} \right] \right]. \quad (2.50)$$

The $|\mathcal{E}_j|$ are conditionally independent given \check{T} , hence

$$\mathbb{E} \left[|\mathcal{E}_k| \prod_{j=0}^{m-1} \theta^{-|\mathcal{E}_j|} \mid \check{T} \right] = \mathbb{E} \left[|\mathcal{E}_k| \theta^{-|\mathcal{E}_k|} \mid \check{T} \right] \prod_{j \neq k} \mathbb{E} \left[\theta^{-|\mathcal{E}_j|} \mid \check{T} \right]. \quad (2.51)$$

Let $p_i = e^{-\beta} \beta^i / i!$ denote the probabilities of a Poisson(β) random variable. Direct computation gives

$$\mathbb{E} \left[\theta^{-|\mathcal{E}_k|} \mid \check{T} \right] = \prod_{x \in V_k(\check{T})} \left(\frac{p_0 + p_1/\theta}{p_0 + p_1} \right)^{d_x} \quad (2.52)$$

and (e.g. by differentiating the previous expression)

$$\mathbb{E} \left[|\mathcal{E}_k| \theta^{-|\mathcal{E}_k|} \mid \check{T} \right] = \frac{p_1/\theta}{p_0 + p_1/\theta} \left(\sum_{x \in V_k(\check{T})} d_x \right) \prod_{x \in V_k(\check{T})} \left(\frac{p_0 + p_1/\theta}{p_0 + p_1} \right)^{d_x - 1}. \quad (2.53)$$

Hence

$$\frac{\mathbb{E} \left[|\mathcal{E}_k| \theta^{-|\mathcal{E}_k|} \mid \check{T} \right]}{\mathbb{E} \left[\theta^{-|\mathcal{E}_k|} \mid \check{T} \right]} \leq \frac{dp_1/\theta}{p_0 + p_1/\theta} |V_k(\check{T})|, \quad (2.54)$$

and

$$\mathbb{E}[\theta^{\tilde{\ell}} |\mathcal{E}_k| \mathbb{1}_{A_1^c \cap A_2^c}] \leq \frac{dp_1/\theta}{p_0 + p_1/\theta} \mathbb{E} \left[|V_k(\check{T})| \mathbb{1}_{A_1^c \cap A_2^c} \theta^{\tilde{\ell}} \prod_{j=0}^{m-1} \prod_{x \in V_j(\check{T})} Z_{m-j-1}^{d_x} \mathbb{E} \left[\prod_{j=0}^{m-1} \theta^{-|\mathcal{E}_j|} \mid \check{T} \right] \right]. \quad (2.55)$$

Applying (2.44) in reverse it follows that

$$\mathbb{E}_m^{(\theta)}[|\mathcal{E}_k| \mathbb{1}_{A_1^c \cap A_2^c}] \leq \frac{dp_1/\theta}{p_0 + p_1/\theta} \mathbb{E}_m^{(\theta)}[|V_k(\check{T})| \mathbb{1}_{A_1^c \cap A_2^c}]. \quad (2.56)$$

We bound the last expectation using stochastic domination. Indeed, both $|V_k(\check{T})|$ and $\mathbb{1}_{A_1^c \cap A_2^c}$ are increasing functions of ω . Hence from (2.16)

$$\mathbb{E}_m^{(\theta)}[|V_k(\check{T})| \mathbb{1}_{A_1^c \cap A_2^c}] \leq \mathbb{E}^+ [|V_k(\check{T})| \mathbb{1}_{A_1^c \cap A_2^c}]. \quad (2.57)$$

Write $p_i^+ = e^{-\beta^+} (\beta^+)^i / i!$ for the Poisson probabilities with parameter β^+ , and $p_{\geq i}^+ = p_i^+ + p_{i+1}^+ + \dots$. By a recursive computation using independence we see that

$$\mathbb{E}^+ |V_k(\check{T})| = (dp_{\geq 2}^+) \mathbb{E}^+ |V_{k-1}(\check{T})| = \dots = (dp_{\geq 2}^+)^k. \quad (2.58)$$

We also have

$$|V_k(\check{T})| \mathbb{1}_{A_1} = \delta_{k,0} \mathbb{1}_{A_1}, \quad |V_k(\check{T})| \mathbb{1}_{A_2} = (\delta_{k,0} + \delta_{k,1}) \mathbb{1}_{A_2}. \quad (2.59)$$

Using (2.42) we find that

$$\begin{aligned}
\mathbb{P}_m^{(\theta)}(A_1^c \cap A_2^c \cap \Sigma_m^\rho) &\leq \frac{dp_1/\theta}{p_0 + p_1/\theta} \sum_{k=0}^m \sigma_{m-k-1} \mathbb{E}^+ [|V_k(\tilde{T})| \mathbb{1}_{A_1^c \cap A_2^c}] \\
&= \frac{dp_1/\theta}{p_0 + p_1/\theta} \left(\sigma_{m-1} (1 - \mathbb{P}^+(A_1 \cup A_2)) + \sigma_{m-2} (dp_{\geq 2}^+ - \mathbb{P}^+(A_2)) \right. \\
&\quad \left. + \sum_{k=2}^m \sigma_{m-k-1} (dp_{\geq 2}^+)^k \right) \\
&\leq c_0 \left(\sigma_{m-1} \frac{c_1}{d^2} + \sigma_{m-2} \frac{c_2}{d^2} + \sum_{k=2}^m \sigma_{m-k-1} \left(\frac{c_3}{d} \right)^k \right),
\end{aligned} \tag{2.60}$$

for constants c_0, \dots, c_3 uniform in d . Now we use that there is some $c_4 > 0$, uniform in d , such that $\sigma_{m-1} \leq c_4 \sigma_m$ for all $m \geq 0$. (This can be seen e.g. by considering the event that A_1 occurs and that (x, t_x) lies in a loop reaching generation m in its subtree, where x is some fixed child of ρ and t_x is the ‘time’ of the incoming link from ρ . This gives $\sigma_m \geq \sigma_{m-1} \mathbb{P}_m^{(\theta)}(A_1)$.) It follows that

$$\mathbb{P}_m^{(\theta)}(A_1^c \cap A_2^c \cap \Sigma_m^\rho) \leq \frac{C'}{d^2} \left(\sigma_{m-1} + \sigma_{m-2} + \sigma_{m-2} \sum_{k=2}^{\infty} \left(\frac{c_3 c_4}{d} \right)^{k-2} \right). \tag{2.61}$$

The last sum converges if d is large enough, and this establishes Prop 2.4. \square

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