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Singularity Analysis for Heavy-Tailed Random Variables

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Abstract We propose a novel complex-analytic method for sums of i.i.d. random variables that are heavy-tailed and integer-valued. The method combines singularity analysis, Lindelöf integrals, and bivariate saddle points. As an application, we prove three theorems on precise large and moderate deviations which provide a local variant of a result by Nagaev (Transactions of the sixth Prague conference on information theory, statistical decision functions, random processes, Academia, Prague, 1973). The theorems generalize five theorems by Nagaev (Litov Mat Sb 8:553–579, 1968) on stretched exponential laws $p(k) = c \exp(-k^\alpha)$ and apply to logarithmic hazard functions $c \exp(-(\log k)^\beta)$, $\beta > 2$; they cover the big-jump domain as well as the small steps domain. The analytic proof is complemented by clear probabilistic heuristics. Critical sequences are determined with a non-convex variational problem.

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1 Introduction

The motivation of the present article is twofold. First, we present a new analytic method
for the investigation of large powers of generating functions of sequences that satisfy
some analyticity and log-convexity conditions. The method is explained and developed
for probability generating functions but it has potentially broader applications and is
motivated by techniques commonly used in analytic combinatorics [9]. Specifically,
we show that methods akin to singularity analysis can be pushed beyond the realm of
functions amenable to singularity analysis in the sense of [9, Chapter VI.1].

Second, we explore consequences for probabilistic limit laws and prove three the-
orems on precise large and moderate deviations for sums of independent identically
distributed (i.i.d.) random variables that are heavy-tailed [5] and integer-valued. The
theorems generalize results on stretched exponential laws by Nagaev [12] which have
recently attracted interest in the context of the zero-range process [2]. They are close in
spirit to results by Nagaev [14], however with more concrete conditions on the domain
of validity of the theorems, and provide deviations results "on the whole axis" [17]. Our
assumptions are more restrictive than one may wish from a probabilistic perspective;
in return, they allow for sharp results and may provide a helpful class of explicit refer-
ence examples. For example, we prove that one of the bounds of the (local) big-jump
domain for logarithmic hazard functions derived in [3] is sharp.

The analytic proof of the theorems is complemented by clear probabilistic heuris-
tics. Our results cover different regimes: a small-steps or moderate deviations regime,
where a classical variant of a local central limit theorem with corrections expressed
with the Cramér series holds [10], and a big-jump regime where the large deviation
is realized by making one out of the n variables large. In the language of statistical
mechanics and the zero-range process, they correspond to supersaturated gas and a
condensed phase [2]. The critical scales that distinguish between regimes are defined
with the help of a non-convex variational problem which encodes competing proba-
bilistic effects. The variational problem has been analyzed before [12,17]; our strong
assumptions on the probability weights allow for a detailed analysis. Our results are
further facilitated by the non-negativity of the random variables, which dispenses us
from dealing with left tails.

The study of combinatorial generating functions shares much in common with the
study of probability generating functions; in fact in many instances they coincide or
run parallel as is the case for more recent investigations in the area of random combi-
natorial structures [1]. From the viewpoint of complex function theory, the key here
involves relating asymptotic questions to questions about the nature of the singularities
of generating functions viewed as more global analytic objects. In most of the suc-

cessful applications of this *singularity analysis* to coefficient asymptotics a bridge is provided by the realization that the series in question satisfies some global algebraic or differential equation. Generating functions for which this is the case are referred to as *holonomic*. Pushing beyond this class in a systematic way requires new ideas and one of the most promising of these is the use of Lindelöf integrals. Lindelöf introduced these classically [11] as a means to constructively carry out analytic continuations of function elements (series) in a fairly general setting. In more recent times, his construction has begun to be used to study non-holonomic combinatorial generating functions [8]. The generating functions for heavy-tailed distributions studied in this paper are of non-holonomic type and our methods of studying them show a new application of Lindelöf's construction that has novel connections to other areas of analytic asymptotic analysis such as bivariate steepest descent. In future work, we hope to build on the present article in a way that broadens the application of harmonic analysis and complex function theory to problems of asymptotic analysis in both probability and combinatorics, such as applying the theory of Hardy spaces and Riemann-Hilbert analysis and extensions of Tauberian theorems as originally envisioned by Paley and Wiener [7].

Our proof shares some features with [14], where cumulative distribution functions are approximate Laplace transforms and approximating moment generating function admit analytic extensions. Contour integrals that appear in inversion formulas are deformed and analyzed by Gaussian approximation—our proof details in Sect. 5.4 follow [14]. There are, however, key differences: we need not deal with approximation errors because of stronger analyticity assumptions, and our detailed analysis of the underlying variational problem allows us to formulate more concrete conditions for our theorems.

The remainder of this article is organized as follows. In Sect. 2, we formulate our main results and discuss applications to stretched exponential weights $c \exp(-k^\alpha))$ and weights $c \exp(-(\log x)^\beta)$ with logarithmic hazard functions. In Sect. 3, we explain the proof strategy in five steps, which are treated in detail in the remaining sections. Steps 1 and 2 concern analytic extensions and notably use the Lindelöf and Bromwich integrals (Sect. 4). Steps 4 and 5 analyze the critical points of a bivariate function and deal with the Gaussian approximation to a double integral (Sect. 5). The pivotal Step 3 connects the contour integral and the bivariate double integral; it leads to the full proof of our theorems that can be found in Sect. 6.

2 Results

We use the notation $a_n \sim b_n$ if $a_n = (1 + o(1))b_n$ and $a_n \ll b_n$ if $a_n = o(b_n)$.

2.1 Preliminaries

In order to formulate the results, we need to introduce critical sequences deduced from a variational problem and the Cramér series. Let $X, X_1, X_2, \ldots$ be independent,

identically distributed random variables with values in $\mathbb{N}$ and law

$$\mathbb{P}(X = k) = p(k) = \exp(-q(k)) \quad (k \in \mathbb{N}) \tag{2.1}$$

for some sequence $(q(k))_{k \in \mathbb{N}}$. We assume that X is heavy-tailed and has moments of all orders, i.e., the generating function

$$G(z) = \sum_{k=1}^{\infty} p(k) z^k \quad (|z| \leq 1) \tag{2.2}$$

has radius of convergence 1 and $\mathbb{E}[X^m] = \sum_{k=1}^{\infty} k^m p(k) < \infty$ for all $m \in \mathbb{N}$. Let μ and σ^2 be the expectation and variance of X. Set $S_n = X_1 + \cdots + X_n$. We are interested in the asymptotic behavior of $\mathbb{P}(S_n = \mu n + N_n)$ when $n, N_n \to \infty$ with $N_n \gg \sqrt{n}$. The following assumption is similar to conditions considered by Nagaev [14].

Assumption 2.1 For some $a > 0$, the sequence $(q(k))_{k \in \mathbb{N} \cap (a, \infty)}$ extends to a smooth function $q : (a, \infty) \to \mathbb{R}$ which has the following properties:

(i) $q' > 0$, $q'' < 0$, and $q''' > 0$.
(ii) $\lim_{x \to \infty} x q'(x)/(\log x) = \infty$.
(iii) $c_1 \frac{q'(x)}{x} \leq |q''(x)| \leq c_2 \frac{q'(x)}{x}$ for some constants $c_1, c_2 > 0$.
(iv) $c_3 \frac{|q''(x)|}{x} \leq q'''(x) \leq c_4 \frac{|q''(x)|}{x}$ for some constants $c_3, c_4 > 0$.
(v) $q'(x) \leq \alpha \frac{q(x)}{x}$ for some $\alpha \in (0, 1)$.

Assumption 2.1 allows for an easy analysis of an auxiliary variational problem, which is essential to the formulation of our main results. Let us collect a few elementary consequences. Under Assumption 2.1, q is concave on (a, ∞) and $p = \exp(-q)$ is log-convex. Moreover, $\lim_{x \to \infty} x^2 q''(x)/\log x = -\infty$ and for $y > x > a$, using

$$\frac{q'(y)}{q'(x)} = \exp\left(-\int_x^y \frac{|q''(u)|}{q'(u)} du\right), \tag{2.3}$$

we estimate

$$\left(\frac{y}{x}\right)^{-c_2} \leq \frac{q'(y)}{q'(x)} \leq \left(\frac{y}{x}\right)^{-c_1} \leq 1 \tag{2.4}$$

Similarly, for $y > x > a$,

$$\left(\frac{y}{x}\right)^{c_3} \leq \frac{q''(y)}{q''(x)} \leq \left(\frac{y}{x}\right)^{c_4}. \tag{2.5}$$

Since $G(z) = \sum_k z^k \exp(-q(k))$ has radius of convergence 1, we also know that

$$\lim_{x \to \infty} q'(x) = \lim_{x \to \infty} q''(x) = \lim_{x \to \infty} q'''(x) = 0. \tag{2.6}$$

Indeed by Assumption 2.1, q' is eventually decreasing and the limit $\ell := \lim_{x \to \infty} q'(x)$ exists in $\mathbb{R} \cup \{-\infty\}$. Then, $\ell = \lim_{x \to \infty} q(x)/x$ and $G(z)$ has radius of convergence

$\exp(\ell) = 1$, whence $\ell = 0$. Assumption 2.1(iii) and (iv) leads to the statements on higher order derivatives. Assumption 2.1(v) implies $q(x) = O(x^\alpha)$ as $x \to \infty$.

Our method of proof requires two more analyticity assumptions.

Assumption 2.2 There exists $b \geq 0$ such that $(p(n))_{n \in \mathbb{N} \cap [b, \infty)}$ extends to a function $p(\zeta)$ that is continuous on a closed half-plane $\mathrm{Re}\,\zeta \geq b$, analytic on the open half-plane $\mathrm{Re}\,\zeta > b$, and in addition satisfies

(i) For every $\varepsilon \in (0, \pi)$, some $C_\varepsilon > 0$, and all ζ, we have $|p(\zeta)| \leq C_\varepsilon \exp(\varepsilon|\zeta|)$.
(ii) $\int_{-\infty}^{\infty} |(b + \mathrm{i}s)^k p(b + \mathrm{i}s)| \mathrm{d}s < \infty$ for all $k \in \mathbb{N}$.

Moreover, $p(x) = \exp(-q(x))$ for all $x \geq \max(a, b)$ with a, $q(x)$ as in Assumption 2.1.

Assumption 2.3 Let $p(\zeta) = \exp(-q(\zeta))$ be the analytic extension from Assumption 2.2, defined in $\mathrm{Re}\,\zeta \geq b$. Then, $q(\zeta) = -\mathrm{Log}\,\zeta$, defined with the principal branch of the logarithm is analytic as well, and the following holds:

(i) For $r > 0$ large, let $z_r = b + \mathrm{i}\sqrt{r^2 - b^2}$. Then, as $r \to \infty$,

$$\left| \int_{\mathrm{Re}\,\zeta = b,\, |\zeta| \geq r} \exp(-\mathrm{Re}\,q(\zeta)) \mathrm{d}\zeta \right| \leq \exp(-\mathrm{Re}\,q(z_r) + O(\log r)).$$

(ii) $\mathrm{Im}\,(\zeta q'(\zeta)) \leq \mathrm{Im}\,(\zeta q'(r))$ for all large r and all ζ with $\mathrm{Im}\,\zeta \geq 0$ and $|\zeta| = r$.
(iii) $|q'''(\zeta)| \leq C|q''(\zeta)/\zeta|$ for some $C > 0$ and all ζ.

Assumption 2.3 enters the proof of Theorem 4.4 only.

Variational Problem and Critical Scale

Assumption 2.1 is tailored to the analysis of an auxiliary variational problem (see also [12,17]), motivated by the following heuristics. For subexponential random variables, the typical large deviations behavior is realized by making one out of the n variables large,

$$\mathbb{P}(S_n = \mu n + N_n) \approx n \mathbb{P}(X_n = N_n - k_n) \mathbb{P}(S_{n-1} = \mu n + k_n) \qquad (2.7)$$

with a yet to be determined optimal k_n. Assuming that a normal approximation for the second factor is justified, we get

$$\mathbb{P}(S_n = \mu n + N_n) \approx \exp\left(-q(N_n - k_n) - \frac{k_n^2}{2n\sigma^2}\right) \qquad (2.8)$$

where we have neglected prefactors n and $1/\sqrt{2\pi n\sigma^2}$ [see Eq. (2.11) below for a more refined heuristics]. The optimal k_n is then determined by minimizing the term in the exponential. Thus, we are led to the minimization of

$$f_n(x) = q(x) + \frac{(N_n - x)^2}{2n\sigma^2}. \qquad (2.9)$$

As illustrated in Fig. 1, the nature of the variational problem changes with N_n. Define $x_n^* > 0$ and N_n^* by

$$q''(x_n^*) = -\frac{1}{n\sigma^2}, \quad N_n^* = x_n^* + n\sigma^2 q'(x_n^*). \tag{2.10}$$

For sufficiently large n, the inflection point x_n^* is uniquely defined because of the monotonicity from Assumption 2.1 and Eq. (2.6), moreover $x_n^* \to \infty$. The quantity N_n^* is defined in such a way that the tangent to the curve $y = q'(x)$ at $x = x_n^*$ has equation $y = (N_n^* - x)/(n\sigma^2)$, see Fig. 3.

The next two lemmas characterize the minimization of f_n; they are proven in Sect. 5.1. The first lemma relates the critical points of f_n to the location of N_n compared to N_n^*.

Lemma 2.4 *For sufficiently large n, the following holds true:*

(a) *If $N_n < N_n^*$, then $f_n' > 0$ on (a, ∞).*
(b) *If $N_n = N_n^*$, then f_n' has the unique zero x_n^*, moreover $f_n'(x) \geq 0$ with equality if and only if $x = x_n^*$.*
(c) *If $N_n > N_n^*$ and $\limsup_{n\to\infty} N_n/(n\sigma^2) < \lim_{x\searrow a} q'(x)$, then f_n has exactly two critical points x_n and x_n', which satisfy $x_n' < x_n^* < x_n < N_n$ and*

$$f_n(x_n') = \max_{(a, x_n^*)} f_n, \quad f_n(x_n) = \min_{(x_n^*, \infty)} f_n.$$

(d) *If $N_n > N_n^*$ and $\liminf_{n\to\infty} N_n/(n\sigma^2) > \lim_{x\searrow a} q'(x)$, then f_n has a unique critical point x_n. It satisfies $x_n \in (x_n^*, N_n)$ and is a global minimizer.*

For $N_n > N_n^*$, the function f_n may have two local minimizers: a and x_n, and we may wonder which one is the global minimizer. The answer depends on the location of N_n compared to a new critical sequence N_n^{**}. Concrete examples are given in Sects. 2.3 and 2.4.

Lemma 2.5 *For n sufficiently large, there is a uniquely defined $N_n^{**} > N_n^*$ such that:*

(a) *If $N_n^* < N_n < N_n^{**}$, then $f_n(a) < f_n(x_n)$.*
(b) *If $N_n = N_n^{**}$, then $f_n(a) = f_n(x_n)$.*
(c) *If $N_n > N_n^{**}$, then $f_n(x_n) < f_n(a)$.*

In general it may not be straightforward to determine N_n^{**} exactly, but it is simple to find a lower bound: if $a = 0$ and $q(N_n) < N_n^2/(2n\sigma^2)$, then $N_n > N_n^{**}$. This lower bound corresponds, roughly, to the sequence $\Lambda(n)$ in [14].

The sequences introduced up to now are ordered as follows.

Lemma 2.6 *As $n \to \infty$, we have*

$$\sqrt{n} \ll x_n^* < N_n^* < N_n^{**} = O(n^{1/(2-\alpha)}) \ll n,$$

and for some constants $C, \delta > 0$
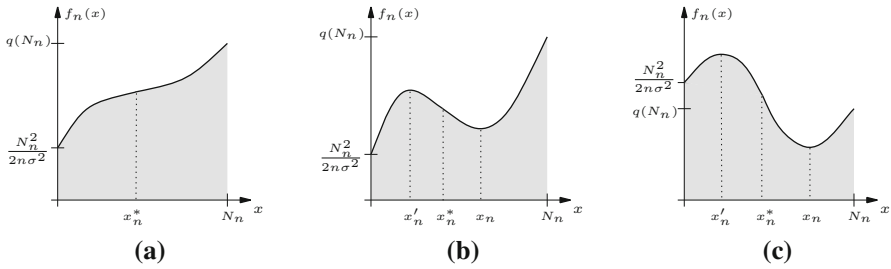
$$(1 + \delta)x_n^* \leq N_n^* \leq Cx_n^*.$$

Fig. 1 Minimization of $f_n(x) = q(x) + (N_n - x)^2/(2n\sigma^2)$ and illustration of Lemmas 2.4 and 2.5 for weights $q : (0, \infty) \to \mathbb{R}$ with $q(0) = 0$. For $N_n > N_n^*$, $f_n(x)$ has two critical points x_n' and x_n separated by an inflection point x_n^*. The global minimum is reached either at $x = x_n$ or at $x = 0$. **a** $N_n < N_n^*$. **b** $N_n^* < N_n < N_n^{**}$. **c** $N_n > N_n^{**}$

The lemma is proven in Sect. 5.1. Lemmas 2.4 and 2.5, together with the heuristics described above, suggest that for $N_n > N_n^{**}$, the unlikely event $S_n = n\mu + N_n$ is realized by making one component of the order of x_n. One may wonder how far x_n is from swallowing all of the overshoot N_n.

Lemma 2.7 *Suppose Assumption 2.1(i) holds true. Let $N_n > N_n^*$. Then,*

$$N_n - N_n^* \le x_n \le N_n, \quad n\sigma^2 f_n''(x_n) = 1 - n\sigma^2 |q''(x_n)| = 1 + O\left(\frac{N_n^*}{N_n}\right).$$

In particular, for $N_n \gg N_n^*$, we have $x_n \sim N_n$ and $n\sigma^2 f''(x_n) \to 1$. The lemma is proven in Sect. 5.1. The information on the second derivative enters a refined heuristics: we make the ansatz that conditional on the unlikely event $S_n = n\mu + N_n$, there is one large component of size x_n, but the size is not deterministic. Instead, there are fluctuations around x_n. This yields

$$\mathbb{P}(S_n = n\mu + N_n) \approx n \sum_\ell \mathbb{P}(X_1 = x_n + \ell)\mathbb{P}(S_{n-1} = N_n - x_n - \ell)$$

$$\approx n \sum_\ell \frac{\exp(-f_n(x_n + \ell))}{\sqrt{2\pi n\sigma^2}} \approx n\, \mathrm{e}^{-f_n(x_n)} \sum_\ell \frac{\exp(-f_n''(x_n)\ell^2/2)}{\sqrt{2\pi n\sigma^2}}$$

$$\approx \frac{n \exp(-f_n(x_n))}{\sqrt{1 - n\sigma^2 |q''(x_n)|}}. \tag{2.11}$$

Theorem 2.11 below confirms the heuristics for large N_n, up to correction terms both in the prefactor and in the exponential.

Cumulants and Cramér Series

The heuristics together with Lemma 2.7 suggest that the optimal $k_n = N_n - x_n$ in Eq. (2.8) is of order up to $N_n^* \gg \sqrt{n}$. At this scale, the normal approximation fails and requires correction terms. The latter are usually expressed with the Cramér series [10],

whose definition we briefly recall. Let $\varphi(t)$ be the cumulant generating function of X,

$$\varphi(t) = \log \mathbb{E}[e^{tX}] = \log G(e^t) \quad (\operatorname{Re} t \leq 0). \tag{2.12}$$

Note $\varphi(0) = 0$. As $t \to 0$, $\varphi(t)$ can be approximated to arbitrary order

$$\varphi(t) = \sum_{j=1}^{r} \kappa_j \frac{t^j}{j!} + O(t^{r+1}) \tag{2.13}$$

with finite and real expansion coefficients $\kappa_j \in \mathbb{R}$, the cumulants, see Sect. 4. Notice that $\kappa_1 = \mu$ is the expectation and $\kappa_2 = \sigma^2 > 0$ is the variance of X.

Definition 2.8 Let $t(\tau) = \frac{1}{\sigma^2}\tau + \sum_{j\geq 2} a_j \tau^j$ be the formal power series obtained by inverting

$$\sigma^2 t(\tau) + \sum_{j\geq 2} \kappa_{j+1} \frac{t(\tau)^j}{j!} = \tau.$$

The *Cramér series* $\sum_{j\geq 0} \lambda_j \tau^j$ is the formal power series defined by composing the expansion of $t(\tau)$ with the expansion of $(\mu + \tau)t - \varphi(t)$,

$$(\mu + \tau)t(\tau) - \sum_{j\geq 1} \kappa_j \frac{t(\tau)^j}{j!} = -\frac{\tau^2}{2\sigma^2} + \tau^3 \sum_{j\geq 0} \lambda_j \tau^j.$$

Equivalently, the Cramér series is the left-sided Taylor expansion of the Legendre transform φ^* at μ: let $\varphi^*(x) := \sup_{t\leq 0}(tx - \varphi(t))$. Then, as $\tau \nearrow 0$,

$$\varphi^*(\mu + \tau) = -\frac{\tau^2}{2\sigma^2} + \tau^3 \sum_{j=0}^{r} \lambda_j \tau^j + O(\tau^{r+4}) \tag{2.14}$$

to arbitrarily high order r.

Remark 1 For $t > 0$, $\log[\sum_{k\geq 1} p(k) \exp(kt)]$ is infinite and the standard convention is to set $\varphi(t) = \infty$; then $\varphi^*(\mu + \tau) \equiv 0$ for $\tau \geq 0$ and Eq. (2.14) no longer applies. We adopt a different convention, however, for which $\varphi(t)$ is smooth in a neighborhood of 0 (see Theorem 4.2), though it becomes complex-valued, and Eq. (2.14) applies to $(\operatorname{Re}\varphi)^*$ for positive τ as well.

2.2 Main Theorems

Set $f_{n0} = f_n$ and for $r \geq 1$,

$$f_{nr}(x) = q(x) + \frac{(N_n - x)^2}{2n\sigma^2} - \frac{(N_n - x)^3}{n^2} \sum_{j=0}^{r-1} \lambda_j \left(\frac{N_n - x}{n}\right)^j. \tag{2.15}$$

Remember the minimization of $f_n(x)$ and the critical scales $\sqrt{n} \ll N_n^* < N_n^{**} = O(n^{1/(2-\alpha)}) \ll n$ illustrated in Fig. 1. In Proposition 2.12 below, we check that the properties of f_n carry over to f_{nr}. The following theorems provide a local variant of a large deviations theorem by Nagaev [14], see also [15, Theorem 2.1]. Local results have been provided before, see [3] and the references therein. The principal difference, apart from the local character of the theorems, is that our detailed investigation of the variational problem and notably Lemma 2.7 allows us to formulate conditions directly in terms of N_n, whereas S. V. Nagaev's criteria included an indirect condition on the sign of some second derivative.

Theorem 2.9 *Let* $N_n \to \infty$ *with* $\sqrt{n} \ll N_n \le (1 + o(1))N_n^*$. *Pick* r *large enough so that* $n(N_n/n)^r \to 0$. *Then,*

$$\mathbb{P}(S_n = \mu n + N_n) \sim \frac{1}{\sqrt{2\pi\sigma^2 n}} \exp\left(-\frac{N_n^2}{2n\sigma^2} + \frac{N_n^3}{n^2} \sum_{j=0}^{r-1} \lambda_j \left(\frac{N_n}{n}\right)^j\right).$$

Theorem 2.10 *Let* $N_n \to \infty$ *with* $\liminf N_n/N_n^* > 1$ *and* $N_n = O(n^{1/[2-\alpha]})$. *Pick* r *large enough so that* $n(N_n^*/n)^r \to 0$. *Then,*

$$\mathbb{P}(S_n = \mu n + N_n) = (1 + o(1))\frac{1}{\sqrt{2\pi\sigma^2 n}} \exp\left(-\frac{N_n^2}{2n\sigma^2} + \frac{N_n^3}{n^2} \sum_{j=0}^{r-1} \lambda_j \left(\frac{N_n}{n}\right)^j\right)$$

$$+ (1 + o(1))\frac{n}{\sqrt{1 - n\sigma^2|q''(x_{nr})|}} \exp\left(-f_{nr}(x_{nr})\right)$$

with $x_{nr} = N_n + O(N_n^*)$, *the largest solution of* $f_{nr}'(x_{nr}) = 0$.

Lemma 2.5 suggests that for $N_n \ll N_n^{**}$, the first contribution dominates and for $N_n \gg N_n^{**}$ the second contribution wins, but one has to be careful because of the factors n and $1/\sqrt{2\pi n\sigma^2}$ as well as the Cramér corrections; a detailed evaluation is best left to concrete examples (see, however, Corollary 2.13 below).

Theorem 2.11 *Let* $N_n \to \infty$ *with* $N_n \gg n^{1/(2-\alpha)}$. *Pick* r *large enough so that* $n(N_n^*/n)^r \to 0$. *Then,*

$$\mathbb{P}(S_n = \mu n + N_n) \sim n \exp\left(-f_{nr}(x_{nr})\right)$$

with $x_{nr} = N_n + O(N_n^*)$, *the largest solution of* $f_{nr}'(x_{nr}) = 0$.

In practice, one may prefer not to deal with the Cramér corrections or the variational problem, and the following proposition is helpful.

Proposition 2.12 *Suppose that* $\liminf_{n\to\infty} N_n/N_n^* > 1$. *Fix* $r \in \mathbb{N}_0$. *Then, for sufficiently large* n, x_{nr} *is the maximizer of* f_{nr} *restricted to* (x_n^*, N_n) *and the unique zero*

of f'_{nr} in that interval. Moreover $1 - n\sigma^2|q''(x_{nr})| = 1 + O(N_n^*/N_n)$ stays bounded away from zero and

$$x_{nr} = N_n - (1 + o(1))n\sigma^2 q'(x_{nr}) = N_n + O(N_n^*),$$

$$f_{nr}(x_{nr}) = q(x_{nr}) + \frac{1}{2}(1 + o(1))n\sigma^2 q'(x_{nr}) = q(N_n)\Big(1 + O\Big(\frac{N_n^*}{N_n}\Big)\Big).$$

The proposition is proven in Sect. 5.1. For $N_n \gg N_n^*$, we obtain $x_{nr} \sim N_n$, $q'(x_{nr}) \sim q'(N_n)$, and $q(x_{nr}) = q(N_n) - (1 + o(1))n\sigma^2 q'(N_n)$, hence

$$f_{nr}(x_{nr}) = q(N_n) - \frac{1}{2}(1 + o(1))n\sigma^2 q'(N_n)^2. \tag{2.16}$$

Now suppose in addition that $\liminf \frac{N_n^2}{2n\sigma^2}/q(N_n) > 1$. Then, in Theorem 2.10, the first summand is of order $\exp(-(1+o(1))N_n^2/(2n\sigma^2)$, the second of order $\exp(-(1+o(1))q(N_n))$, so the first contribution is negligible and the validity of Theorem 2.11 extends accordingly, since $1 - n\sigma^2|q''(x_{nr})| \to 1$ for $N_n \gg N_n^*$. Eq. (2.16) now yields the following corollary.

Corollary 2.13 *Take $N_n \to \infty$ with $N_n \gg N_n^*$ and $\liminf \frac{N_n^2}{2n\sigma^2}/q(N_n) > 1$. Then*

$$\mathbb{P}(S_n = n\mu + N_n) \sim n\,e^{-q(N_n)} = n\mathbb{P}(X = N_n)$$

if and only if $\sqrt{n\sigma^2}q'(N_n) \to 0$.

In concrete examples, Theorem 2.9 should allow us to extend the domain of validity of the corollary to $N_n \gg N_n^{**}$. The condition $\sqrt{n\sigma^2}q'(N_n) \to 0$ is closely related to the *insensitivity* scale discussed by Denisov, Dieker and Shneer [3], as

$$\frac{p(N_n \pm \sqrt{n\sigma^2})}{p(N_n)} \to 1 \Leftrightarrow \sqrt{n\sigma^2}q'(N_n) \to 0. \tag{2.17}$$

Remark 2 (Big-jump vs. small-steps) The domain where $\mathbb{P}(S_n = n\mu + N_n) \sim n\mathbb{P}(X = N_n)$ is sometimes called *big-jump domain*. Think of S_n as the position of a random walker with step size distribution $p(k)$. In the situation of Corollary 2.13, the unlikely event that the walker has travelled a distance $\mu n + N_n$ much larger than the expected distance μn is realized by one big-jump of size N_n. Finding the boundary of the big-jump domain is an active field of research [3].

The interpretation of Theorem 2.9, in contrast, is that the moderate overshoot N_n is achieved by a collective effort: all steps tend to stay small, though each stretches a little beyond its expected value μ. For stretched exponential variables, this interpretation is made rigorous in [12] and [2].

Remark 3 (Condensation in the zero-range process) In the zero-range process, the random variables $X_1, \dots, X_n$ model the number of particles at lattice sites $j = 1, \dots, n$,

with S_n the total number of particles, and μ is a critical density. Theorem 2.9 corresponds to supersaturated gas. In Corollary 2.13, the particle excess N_n is absorbed by a condensate, i.e., one large occupation number. In Theorem 2.11, the particle excess is shared by a condensate of size $x_{nr} < N_n$ and supersaturated gas. See [2] and [6, Section 7].

We conclude with an equivalent but more intrinsic formulation of Theorem 2.11. In Sect. 4, we shall see that $G(z)$ extends to a function that is analytic in the slit plane $\mathbb{C} \setminus [1, \infty)$, and in addition the limit $G(e^t) = \lim_{\varepsilon \searrow 0} G(e^t + i\varepsilon)$ exists for all $t \geq 0$. So the cumulant generating function $\varphi(t) = \log G(e^t)$ extends to a function that is well-defined and smooth in a neighborhood of the origin in $\mathbb{R}$, and Eq. (2.13) stays valid for small positive t. We define

$$\Phi_n(t, \zeta) = -q(\zeta) + n \operatorname{Re} \varphi(t) - (\mu n + N_n)t + t\zeta$$

$$= -q(\zeta) + n \sum_{j=2}^{r-1} \kappa_j \frac{t^j}{j!} - t(N_n - \zeta) + O(t^r). \tag{2.18}$$

The asymptotic expansion holds for every order r. For $N_n \gg N_n^*$ and sufficiently large n, the bivariate function $\Phi_n(t, \zeta)$ has exactly two critical points in $(0, \frac{N_n}{n\sigma^2}) \times (a, \infty)$. We label them as (t_n, ζ_n) and (t_n', ζ_n') with $t_n < t_n'$. Then, in the situation of Theorem 2.11, we have

$$\mathbb{P}(S_n = n\mu + N_n) \sim n \, e^{\Phi_n(t_n, \zeta_n)}. \tag{2.19}$$

It is in this form that we prove the theorem. Let us explain how to recover the expression in terms of f_{nr}. We may solve for $\nabla \Phi_n(t, \zeta) = 0$ in two steps: (1) use $\partial_t \Phi_n(t, \zeta) = 0$ to express $t = t(\zeta)$ as a function of ζ, (2) plug the expression into $\partial_\zeta \Phi_n(t, \zeta)$ to obtain an equation for ζ. This latter step breaks into the following two stages:

(2a) substitute the expression of $t = t(\zeta)$ into the expression for $\Phi_n(t, \zeta)$ so as to obtain a function $\Phi_n(t(\zeta), \zeta)$ of ζ only;
(2b) set the derivative of $\Phi_n(t(\zeta), \zeta)$ with respect to ζ to zero,

which is valid since

$$\frac{d}{d\zeta} \Phi_n(t(\zeta), \zeta) = \frac{\partial \Phi_n}{\partial \zeta}(t(\zeta), \zeta) + \frac{\partial \Phi_n}{\partial t}(t(\zeta), \zeta) \frac{dt}{d\zeta}(t) = \frac{\partial \Phi_n}{\partial \zeta}(t(\zeta), \zeta). \tag{2.20}$$

By the definition of the Cramér series, step (2a) gives

$$\Phi_n(t(\zeta), \zeta) = -q(\zeta) - \frac{(N_n - \zeta)^2}{2n\sigma^2} + \frac{(N_n - \zeta)^3}{n^2} \sum_{j \geq 0} \lambda_j \left(\frac{N_n - \zeta}{n} \right)^j \tag{2.21}$$

Truncation of the asymptotic expansion on the right-hand side gives precisely the function $-f_{nr}(\zeta)$. For r large enough, step (2b) shows that we may approximate

$$\Phi_n(t_n, \zeta_n) = -f_{nr}(x_{nr}) + o(1), \quad n\sigma^2 q''(\zeta_n) = n\sigma^2 q''(x_{nr}) + o(1) \tag{2.22}$$

hence the equivalence of Eq. (2.19) with the expression from Theorem 2.11.

2.3 Application to Stretched Exponential Laws

Here, we explain how to recover five theorems by Nagaev [12] for stretched exponential variables. Let $\alpha \in (0, 1)$, $c > 0$, and

$$p(k) = c \exp(-k^\alpha), \quad q(k) = k^\alpha - \log c \quad (k \in \mathbb{N}). \tag{2.23}$$

We need not check Assumption 2.3 since Theorem 4.4 for stretched exponential weights has already been proven in [10, Theorem 2.4.6].

Lemma 2.14 *The probability weights* (2.23) *satisfy Assumptions* 2.1 *and* 2.2, *and we have*

$$x_n^* = \left[\alpha(1-\alpha)n\sigma^2\right]^{1/(2-\alpha)}, \quad N_n^* = \frac{2-\alpha}{1-\alpha}x_n^*, \quad N_n^{**} = C_\alpha(n\sigma^2)^{1/(2-\alpha)}$$

with $C_\alpha = (2-\alpha)(2-2\alpha)^{-(1-\alpha)/(2-\alpha)}$. *Moreover,* $\sqrt{n\sigma^2}q'(N_n) \to 0$ *if and only if* $N_n \gg n^{-1/(2-2\alpha)}$.

The proof of the lemma is sketched in "Appendix A". The critical scale $n^{-1/(2-\alpha)}$ is explained by a simple scaling relation: for $N_n = kn^{1/(2-\alpha)}$, we have

$$f_n\left(yn^{1/(2-\alpha)}\right) = n^{\alpha/(2-\alpha)}\left(y^\alpha + \frac{(k-y)^2}{2\sigma^2}\right) - \log c. \tag{2.24}$$

A careful examination of the expressions in Theorem 2.10 shows that the first summand dominates if $N_n \le (1-\delta)N_n^{**}$ while the second dominates if $N_n \ge (1+\delta)N_n^{**}$ for some $\delta > 0$. In particular, Theorem 2.9 extends to $N_n \le (1-\delta)N_n^{**}$, which corresponds to Theorem 1 in [12]. Theorem 2.10 for $N_n \sim N_n^{**}$ is Theorem 4 in [12]. For $N_n \ge (1+\delta)N_n^{**}$, we have

$$\mathbb{P}(S_n = n\mu + N_n) \sim \frac{1}{\sqrt{1 - n\sigma^2 q''(x_{nr})}} e^{-f_{nr}(x_{nr})}. \tag{2.25}$$

This regime can be divided into three cases:

(a) $N_n \ll n^{-1/(2-2\alpha)}$ corresponds to Theorem 2 in [12].
(b) When N_n is of the order of $n^{-1/(2-2\alpha)}$, the corrections from the Cramér series are irrelevant and
$$\mathbb{P}(S_n = \mu n + N_n) \sim n \exp\left(-f_n(x_n)\right), \tag{2.26}$$
i.e., we may choose $r = 0$. This corresponds to Theorem 6 from the erratum [13], replacing Theorem 5 in the original article [12]. The statement actually extends

to $N_n \gg n^{-1/(3-3\alpha)}$. Indeed, $q'(x_{nr}) \sim (N_n - x_{nr})/(n\sigma^2) = o(N_n/n)$ yields $N_n - x_{nr} = (1 + o(1))n\sigma^2\alpha N_n^{\alpha-1}$ and

$$f_{nr}(x_{nr}) = f_n(x_{nr}) + O\big(nN_n^{-(3-3\alpha)}\big), \tag{2.27}$$

and one can check that $f_n(x_{nr}) = f_n(x_n) + o(1)$ if $N_n \gg n^{-1/(3-3\alpha)}$.
(c) $N_n \gg n^{-1/(2-2\alpha)}$ corresponds to Theorem 3 in [12] and our Corollary 2.13.

2.4 Application to Logarithmic Hazard Functions

Here we specialize to

$$p(k) = c \exp\big(-(\log k)^\beta\big), \quad q(k) = -\log c + (\log k)^\beta \tag{2.28}$$

with $\beta > 2$.[1]

Lemma 2.15 *The weights* (2.28) *satisfy Assumptions* 2.1–2.3. *Moreover*

$$N_n^* \sim 2x_n^* \sim 2\sqrt{2^{1-\beta}\beta n\sigma^2(\log n)^{\beta-1}}, \quad N_n^{**} \sim \sqrt{2n\sigma^2(\log n)^\beta},$$

and $\sqrt{n\sigma^2}q'(N_n) \to 0$ *if and only if* $N_n \gg \sqrt{n}(\log n)^{\beta-1}$.

The lemma is proven "Appendix A". Notice that unlike the stretched exponential case (Lemma 2.14), N_n^{**} is much larger than N_n^*. The scaling relation (2.24) is modified as follows: for $N_n = k\sqrt{n\sigma^2(\log n)^{\beta-1}} \sim kx_n^*$, we have

$$f_n(yx_n^*) = (\log x_n^*)^\beta + \beta(\log n)^{\beta-1}\Big(\log y + \frac{(k-y)^2}{2}\Big) + o\big((\log n)^{\beta-1}\big) \tag{2.29}$$

and $(\log x_n^*)^\beta \sim (\log n)^\beta \sim (\log N_n)^\beta$.
    Our results may now be applied to obtain a sharp boundary for the big-jump domain.

Theorem 2.16 *Let* $p(k)$ *be as in Eq.* (2.28) *with* $\beta > 2$ *and* $N_n \gg \sqrt{n}$. *Then* $\mathbb{P}(S_n = \mu n + N_n) \sim n\mathbb{P}(X = N_n)$ *if and only if* $N_n \gg \sqrt{n}(\log n)^{\beta-1}$.

The "if" part of the theorem is actually a special case of [3, Theorem 8.2] and as such not new. The "only if" part shows that the boundary derived in [3] is in fact sharp.

Proof Theorem 2.16 Let $I_n := \sqrt{n\sigma^2}(\log n)^{\beta-1}$ and notice $I_n \gg N_n^{**}$ for $\beta > 2$. Suppose that $N_n \gg I_n$. Then we have, in particular, $N_n \gg N_n^{**}$. Write $N_n = \alpha_n N_n^{**}$

---

[1] For $\beta \in (1, 2]$, $xq'(x) = \beta(\log x)^{\beta-1} \to \infty$ but the stronger condition $xq'(x)/\log x \to \infty$ from Assumption 2.1(ii) fails. We suspect that this restriction is technical and could be lifted with more detailed estimates, but a proof or disproof is beyond this article's scope.

with $\alpha_n \to \infty$. Then,

$$\frac{N_n^2/(2n\sigma^2)}{(\log N_n)^\beta} = \alpha_n^2 \Big(1 + o(1)\frac{\log \alpha_n}{\log n}\Big)^{-\beta} \geq \frac{\alpha_n^2}{(\log \alpha_n)^\beta} \to \infty \qquad (2.30)$$

and it follows from Corollary 2.13 that $\mathbb{P}(S_n = n\mu + N_n) \sim n\mathbb{P}(X = n)$, hence $N_n \gg N_n^{**}$ is indeed a sufficient condition. In order to check that it is necessary, we treat the case $N_n = O(I_n)$ with Theorems 2.9 and 2.10.

Case 1 $N_n^* \ll N_n = O(I_n)$. In Theorem 2.10 we obtain a lower bound by neglecting the first contribution and estimating $1 - n\sigma^2|q''(x_{nr})| \leq 1$. Combining with Proposition 2.12, we find

$$\mathbb{P}(S_n = n\mu + N_n) \geq n\exp(-f_{nr}(x_{nr}) + o(1))$$
$$\geq n\exp\Big(-q(N_n) + (1 + o(1))n\sigma^2 q'(N_n)^2 + o(1)\Big) \; (2.31)$$

hence in view of $N_n = O(I_n)$ and Lemma 2.15

$$\liminf_{n\to\infty} \frac{\mathbb{P}(S_n = n\mu + N_n)}{n\mathbb{P}(X = N_n)} \geq \liminf_{n\to\infty} \exp\big((1 + o(1)n\sigma^2 q'(N_n)^2)\big) > 1. \quad (2.32)$$

Case 2 $N_n = O(N_n^*)$. Then, we have in particular $N_n = o(N_n^{**})$. Write $N_n = \alpha_n N_n^{**}$ with $\alpha_n \to 0$, then

$$\frac{N_n^2}{2n\sigma^2(\log N_n)^\beta} \leq \frac{\alpha_n^2(\log n)^\beta}{(\log \sqrt{n})^\beta} \to 0 \qquad (2.33)$$

and Theorems 2.9 and 2.10 show

$$\log \frac{\mathbb{P}(S_n = n\mu + N_n)}{n\mathbb{P}(X = N_n)} \geq -\frac{N_n^2}{2n\sigma^2} + (\log N_n)^\beta + O(\log n) \to \infty. \qquad (2.34)$$

$\square$

3 Proof Strategy

Here, we explain the strategy for the proof of Theorems 2.9–2.11. We focus on the case $N_n = o(n)$ and Theorem 2.10. Set $m = \mu n + N_n$. We start from the observation that $\mathbb{P}(S_n = m)$ is equal to $[z^m] G(z)^m$, the coefficient of z^m in the expansion of $G(z)^m$, which in turn is given by contour integrals

$$[z^m]G(z)^n = \frac{1}{2\pi i} \oint \frac{G(z)^n}{z^m} \frac{dz}{z} = \frac{1}{2\pi i} \int_0^{2\pi i} e^{n\varphi(t) - mt} \, dt. \qquad (3.1)$$

The contour integral can be taken over any circle centered at the origin with radius $r \leq 1$. A steepest descent ansatz would look for a point z_n such that $z_n G'(z_n) = $
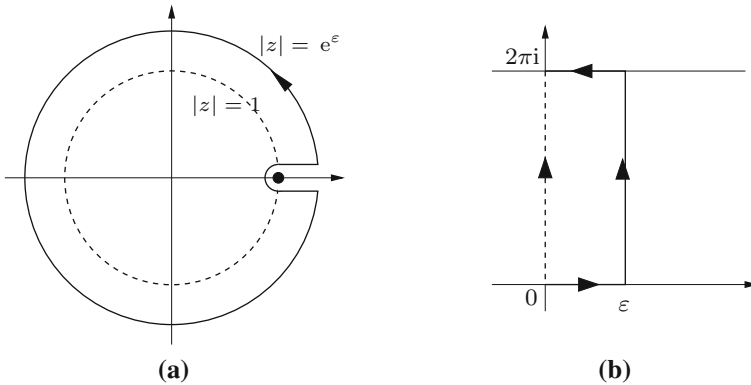
Fig. 2 Contour integrals in the z-plane (**a**) and in the t-plane (**b**). Dotted lines: Eq. (3.1). Solid lines: deformed contours in Eq. (3.3). Recall that $z = e^t$. Later $\varepsilon = \eta_n$ will be chosen in a judicious way. For $N_n \ll N_n^{**}$, the dominant contribution should come from the horizontal pieces of the deformed contour in the t-plane. For $N_n \gg N_n^{**}$, the dominant contribution should instead be from the vertical line

$m/n = \mu + N_n/n$, or η_n with $\varphi'(\eta_n) = \mu + N_n/n$, and then integrate over $|z| = |z_n|$ (or Re $t = $ Re η_n). However, in the regime $m/n > G'(1) = \mu$ that we investigate there is no such point, and instead, we follow an approach that is in the spirit of singularity analysis [9] but with several novel ingredients. Crucially, the generating function $G(z)$ *does not* fall into the class of functions which Flajolet and Sedgwick call "amenable to singularity analysis" [9] [Chapter VI].

Step 1: Analytic extension to slit plane Observe that $G(z)$ has an analytic extension to the slit plane $\mathbb{C}\backslash[1, \infty)$. This is proven with the help of the Lindelöf integral [8,11], see Proposition 4.1. The key ingredient here is that $p(\zeta)$ is analytic in a complex half-plane containing the integers $k \in \mathbb{N}$ and growth slower than $\exp((\pi - \varepsilon)|\zeta|)$ as $\zeta \to \infty$.

Step 2: Behavior near the dominant singularity and along the slit The Lindelöf integral actually shows that the analytic extension $G(z)$ has well-defined limits as z approaches the slit $[1, \infty)$ from above or below, i.e., the limits $\lim_{\varepsilon \searrow 0} G(e^t + i\varepsilon)$ and $\lim_{\varepsilon \searrow 0} G(e^t - i\varepsilon)$ exist for all $t \in \mathbb{R}$. Moreover the imaginary part along the slit is given by a Bromwich integral,

$$\lim_{\varepsilon \searrow 0} \mathrm{Im}\, G(e^t + i\varepsilon) = \frac{1}{2i} \int_{1/2-i\infty}^{1/2+i\infty} e^{t\zeta}\, p(\zeta) \mathrm{d}\zeta \quad (t \geq 0). \tag{3.2}$$

The line of integration Re $\zeta = 1/2$ can be replaced by any other line Re $\zeta = x > 0$. As $t \searrow 0$, the imaginary part vanishes faster than any power of t, whereas the real part can be approximated to arbitrarily high order by a Taylor polynomial.

Step 3: Contour integrals We may now deform the contour of integration: in the z-plane, we replace the circle of radius 1 by a Hankel-type contour consisting of a circle of radius e^ε and a piece hugging the segment $[1, \varepsilon)$, see Fig. 2. In the t-plane, we replace the vertical segment joining 0 and $2\pi i$ by the three other sides of the rectangle

with corners $0, \varepsilon, \varepsilon + 2\pi i, 2\pi i$. This yields

$$
[z^m]G(z)^n = \frac{1}{2\pi i} \left(\int_0^\varepsilon e^{n\varphi(t)-mt} \, dt + \int_0^{2\pi} e^{n\varphi(\varepsilon+i\theta)-m(\varepsilon+i\theta)} \, id\theta \right.
$$
$$
\left. - \int_0^\varepsilon e^{n\varphi(t+2\pi i)-m(t+2\pi i)} \, dt \right). \tag{3.3}
$$

We focus on $N_n = o(n)$ and choose $\varepsilon = \eta_n$ as the solution of

$$
\mathrm{Re}\, \varphi'(\eta_n) = \frac{m}{n} = \mu + \frac{N_n}{n}. \tag{3.4}
$$

Notice $\eta_n \sim N_n/(n\sigma^2)$. Using the identities

$$
G(\bar{z}) = \overline{G(z)}, \quad \forall t \geq 0 : \; \varphi(t+2\pi i) = \overline{\varphi(t)}, \tag{3.5}
$$

Eq. (3.3) becomes

$$
[z^m]G(z)^n = H_n + V_n \tag{3.6}
$$

with

$$
\begin{aligned}
H_n &= \frac{1}{\pi} \int_0^{\eta_n} e^{n\mathrm{Re}\,\varphi(t)-mt} \, \sin\!\big(n\mathrm{Im}\,\varphi(t)\big) dt \\
V_n &= \frac{1}{\pi} \int_0^{\pi} e^{n\mathrm{Re}\,\varphi(\eta_n+i\theta)-m\eta_n} \, \cos\!\big(n\mathrm{Im}\,\varphi(\eta_n+i\theta)\big) d\theta.
\end{aligned} \tag{3.7}
$$

Standard arguments show that the dominant contribution to V_n come from small θ. Since $\mathrm{Im}\, G(e^t) \to 0$ faster than any power of t as $t \searrow 0$ and

$$
\mathrm{Im}\, G(e^t) = \mathrm{Im}\, e^{\varphi(t)} = e^{\mathrm{Re}\,\varphi(t)} \, \mathrm{Im}\, \varphi(t) \sim \mathrm{Im}\, \varphi(t), \tag{3.8}
$$

we may drop the trigonometric functions from Eq. (3.7) and find

$$
H_n \sim \frac{n}{\pi} \int_0^{\eta_n} e^{n\mathrm{Re}\,\varphi(t)-mt} \, \mathrm{Im}\, G(e^t) dt, \qquad V_n \sim \frac{1}{\pi} \int_0^{\pi} e^{n\mathrm{Re}\,\varphi(\eta_n+i\theta)-m\eta_n} \, d\theta. \tag{3.9}
$$

The vertical contribution is evaluated with the help of a Gaussian approximation around $\theta = 0$, which yields

$$
V_n \sim \frac{1}{\sqrt{2\pi n \mathrm{Re}\, \varphi''(\eta_n)}} \, e^{n\mathrm{Re}\,\varphi(\eta_n)-m\eta_n}. \tag{3.10}
$$

With Definition 2.8, we recognize in Eq. (3.10) the asymptotic expression from Theorem 2.9 and obtain

$$
V_n \sim \frac{1}{\sqrt{2\pi n\sigma^2}} \exp\left(-\frac{N_n^2}{2n\sigma^2} + \frac{N_n^3}{n^2} \sum_{j=0}^{r-1} \lambda_j \left(\frac{N_n}{n}\right)^j \right). \tag{3.11}
$$

The evaluation of H_n is more involved. As a preliminary step, we express $\operatorname{Im} G(e^t)$ through the Bromwich integral and find

$$H_n \sim \frac{n}{2\pi \mathrm{i}} \int_0^{\eta_n} \left(\int_{1/2-\mathrm{i}\infty}^{1/2+\mathrm{i}\infty} e^{\Phi_n(t,\zeta)} \, \mathrm{d}\zeta \right) \mathrm{d}t \qquad (3.12)$$

with Φ_n as in Eq. (2.18).

Step 4: Critical points of $\Phi_n(t, \zeta)$ In order to apply a Gaussian approximation to the bivariate integral (3.12), we look for a critical points (t_n, ζ_n) of Φ_n with $t_n \in (0, \eta_n)$ and $\zeta_n \in (0, \infty)$. The gradient $\nabla \Phi_n(t_n, \zeta_n)$ vanishes if and only if

$$\begin{cases} t_n = q'(\zeta_n), \\ \operatorname{Re}\left(\varphi'(t_n) - \varphi'(0)\right) = \dfrac{N_n - \zeta_n}{n}. \end{cases} \qquad (3.13)$$

Since $\operatorname{Re}\varphi'(t) = \mu + \sigma^2 t + O(t^2)$ as $t \searrow 0$, Eq. (3.13) implies

$$q'(\zeta_n) \sim \frac{N_n - \zeta_n}{n\sigma^2}. \qquad (3.14)$$

We recognize the equation for the critical points of f_n. Lemma 2.4 suggests the following: for $N_n \ll N_n^*$, there should be no critical point, for $N_n^* \ll N_n \ll n$, there should be two. Let us focus on the latter case and label the critical points as (t_n, ζ_n) and (t_n', ζ_n') with $t_n < t_n'$. In view of Lemmas 2.4 and 2.7, we expect $\zeta_n \approx x_n$ and $\zeta_n' \approx x_n'$, hence

$$t_n \sim q'(N_n), \quad \zeta_n = N_n + O(N_n^*) \qquad (3.15)$$

and $\zeta_n' < x_n^* < \zeta_n$. The Hessian of Φ_n is

$$\operatorname{Hess} \Phi_n(t, \zeta) = \begin{pmatrix} n\operatorname{Re}\varphi''(t) & 1 \\ 1 & -q''(\zeta) \end{pmatrix}. \qquad (3.16)$$

Using again Lemma 2.7 and $\zeta_n \approx x_n$, we expect

$$\det \operatorname{Hess} \Phi_n(t_n, \zeta_n) = -1 - \left(1 + o(1)\right)n\sigma^2 q''(\zeta_n) = -1 + O\left(\frac{N_n^*}{N_n}\right) < 0 \quad (3.17)$$

thus (t_n, ζ_n) is a saddle point. (More precisely, it is a saddle point of $\operatorname{Re} \Phi_n$, but the abuse of terminology is natural and not problematic in our context.) On the other hand, $\zeta_n' \approx x_n' < x_n^*$ with $1 + n\sigma^2 q'(x_n^*) = 0$ by definition of x_n^*, so we expect

$$\det \operatorname{Hess} \Phi_n(t_n', \zeta_n') = -1 - \left(1 + o(1)\right)n\sigma^2 q''(\zeta_n') > 0. \qquad (3.18)$$

Step 5: Gaussian approximation for H_n In order to evaluate the double integral in Eq. (3.12), we use a good change of variables and a Gaussian approximation. Let $\zeta(t)$ be the solution of $q'(\zeta) = t$, so that $\partial_\zeta \Phi_n(t, \zeta) = 0$ if and only if $\zeta = \zeta(t)$.

It is convenient to deform the contour and integrate along $\operatorname{Re} \zeta = \zeta(t)$ instead of $\operatorname{Re} \zeta = 1/2$. The integral becomes

$$H_n \sim \frac{n}{2\pi} \int_0^{\eta_n} \left(\int_{-\infty}^\infty e^{\Phi_n(t,\zeta(t)+is)} \, ds \right) dt. \qquad (3.19)$$

A straightforward computation shows that $F_n(t,s) = S_n(t, \zeta(t) + is)$, a function of two real variables t, s, has a critical point at $(t_n, 0)$ with positive definite Hessian

$$\operatorname{Hess} F_n(t_n, 0) = \begin{pmatrix} \beta_n & 0 \\ 0 & -\partial_\zeta^2 \Phi_n(t_n, \zeta_n) \end{pmatrix}, \quad \beta_n = \frac{\det \operatorname{Hess} \Phi_n(t_n, \zeta_n)}{\partial_\zeta^2 \Phi_n(t_n, \zeta_n)}, \qquad (3.20)$$

see Lemma B.1. $F_n : (0, \eta_n) \times \mathbb{R} \to \mathbb{C}$ has another critical point at $(t_n', 0)$, with negative determinant of the Hessian; later, we show that it does not contribute to the integral. The evaluation of H_n is concluded by replacing the double integral (3.19) by the integral of the Gaussian approximation around $(t_n, 0)$ which yields

$$H_n \sim \frac{n}{2\pi} \sqrt{\frac{(2\pi)^2}{\det \operatorname{Hess} F_n(t_n, 0)}} \, e^{F_n(t_n, 0)} = \frac{n}{\sqrt{|\det \operatorname{Hess} \Phi_n(t_n, \zeta_n)|}} \, e^{\Phi_n(t_n, \zeta_n)}. \qquad (3.21)$$

The argument leading to Eqs. (2.22) and (3.17) show

$$H_n \sim \frac{n}{\sqrt{1 - n\sigma^2 q''(x_{nr})}} \, e^{-f_{nr}(x_{nr})}. \qquad (3.22)$$

4 Analytic Continuation: Lindelöf and Bromwich Integrals

Here, we take care of steps 1 and 2, starting from Assumption 2.2. For concreteness' sake we write down the results for $b = 1/2$; they apply for general b with straightforward modifications. Define

$$\Lambda(w) = -\frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} p(\zeta) w^\zeta \frac{\pi}{\sin \pi \zeta} d\zeta \qquad (w \in \mathbb{C} \backslash (-\infty, 0]), \qquad (4.1)$$

the Lindelöf integral with symbol $p(\zeta)$.

Proposition 4.1 [11]

(a) $\Lambda(w)$ is analytic in the slit plane $\mathbb{C} \backslash (-\infty, -1]$.
(b) In the unit disk $|w| \le 1$,

$$\Lambda(w) = \sum_{k=1}^\infty p(k)(-w)^k = G(-w).$$

A detailed proof and many additional properties of $\Lambda(w)$ can be found in [8]. Proposition 4.1 shows right away that $G(z) = \Lambda(-z)$ has an analytic continuation from the unit disk to the open slit plane $\mathbb{C}\backslash[1, \infty)$. We use the same letter $G(z)$ for the analytic continuation, and set $\varphi(t) = \mathrm{Log}\, G(\mathrm{e}^t)$ with Log the principal branch of the logarithm, and $\mathrm{Im}\, t \in [0, 2\pi)$. We prove the following additional properties of $G(z)$ and $\varphi(t)$.

Theorem 4.2 (a) *The boundary value* $G(\mathrm{e}^t) = \lim_{\varepsilon \searrow 0} G(\mathrm{e}^t + \mathrm{i}\varepsilon)$ *exists for all* $t \in \mathbb{R}$ *and is a smooth function of* $t \in \mathbb{R}$.
(b) *The imaginary part* $\mathrm{Im}\, G(\mathrm{e}^t)$, $t \geq 0$ *is given by the Bromwich integral* (3.2).
(c) $\varphi(t)$ *is well-defined and smooth in a neighborhood of the origin; the derivatives* $\kappa_j = \varphi^{(j)}(0)$ *are real. As* $t \to 0$ *in the strip* $\mathrm{Im}\, t \in [0, 2\pi)$, *we have*

$$\varphi(t) = \mathrm{Log}\, G(\mathrm{e}^t) = \sum_{j=1}^{r} \kappa_j \frac{t^j}{j!} + O(t^{r+1}).$$

to arbitrarily high order r.

In (c) $z = \mathrm{e}^t$ is allowed to approach the slit $[1, \infty)$ as fast as we like; we may even take t real. Because the coefficients κ_j are real, we find in particular that $\mathrm{Im}\, G(\mathrm{e}^t)$ vanishes faster than any power of t as $t \to 0$, $t \in \mathbb{R}$.

Proof of Theorem 4.2 For $u \in \mathbb{C}$ in the closed strip $\mathrm{Im}\, u \in [-\pi, \pi]$, define

$$L(u) = -\frac{1}{2\pi \mathrm{i}} \int_{1/2-\mathrm{i}\infty}^{1/2+\mathrm{i}\infty} p(\zeta)\, \mathrm{e}^{\zeta u}\, \frac{\pi}{\sin \pi \zeta}\, \mathrm{d}\zeta. \tag{4.2}$$

When $\mathrm{Im}\, u$ is in the open strip $\mathrm{Im}\, u \in (-\pi, \pi)$, we have $w = \mathrm{e}^u \in \mathbb{C}\backslash(-\infty, 0]$ and $L(u) = \Lambda(\mathrm{e}^u)$. Along the vertical line $\mathrm{Re}\, \zeta = 1/2$, we have

$$\left| \frac{\exp(\zeta u)}{\sin(\pi\zeta)} \right| = 2\, \mathrm{e}^{\mathrm{Re}\, u/2}\, \frac{\exp(-s\,\mathrm{Im}\, u)}{\exp(\pi s) + \exp(-\pi s)} \leq 2\, \mathrm{e}^{\mathrm{Re}\, u/2} \quad \left(\zeta = \frac{1}{2} + \mathrm{i}s\right) \tag{4.3}$$

By Assumption 2.2(ii), since $\zeta^k p(\zeta)$ is integrable along $\mathrm{Re}\, \zeta = 1/2$. Eq. (4.3) then shows that the integral defining $L(u)$ is absolutely convergent, and it stays absolutely convergent if we replace the symbol $p(\zeta)$ by $\zeta^k p(\zeta)$. Standard arguments for parameter-dependent integrals then show that $L(u)$ is continuous on the closed strip, differentiable in the open strip, and we may exchange differentiation, limits, and integration, which shows that the restriction of L to the boundaries $\mathrm{Im}\, u = \pm\pi$ yield smooth functions.

When $z \to \mathrm{e}^t \in [1, \infty)$ along $\mathrm{Im}\, z > 0$, we have $w = -z \to -\mathrm{e}^t$ along $\mathrm{Im}\, w < 0$. Thus, we may write $w = \mathrm{e}^u$ with $\mathrm{Re}\, u \to t$ and $\mathrm{Im}\, u = \arg w \searrow -\pi$. Therefore,

$$\lim_{\varepsilon \searrow 0} G(\mathrm{e}^t + \mathrm{i}\varepsilon) = L(t - \mathrm{i}\pi) = \mathrm{i} \int_{-\infty}^{\infty} p\left(\tfrac{1}{2} + \mathrm{i}s\right) \mathrm{e}^{(1/2+\mathrm{i}s)t}\, \frac{\exp(\pi s)}{\exp(\pi s) + \exp(-\pi s)}\, \mathrm{d}s. \tag{4.4}$$

This proves the existence of the limit and, in view of the above mentioned properties of $L(u)$, the smoothness as a function of t. The complex conjugate is

$$-\mathrm{i} \int_{-\infty}^{\infty} p\left(\tfrac{1}{2} - \mathrm{i}s\right) \mathrm{e}^{(1/2 - \mathrm{i}s)t} \frac{\exp(\pi s)}{\exp(\pi s) + \exp(-\pi s)} \mathrm{d}s$$
$$= \mathrm{i} \int_{-\infty}^{\infty} p\left(\tfrac{1}{2} + \mathrm{i}s\right) \mathrm{e}^{(1/2 + \mathrm{i}s)t} \frac{\exp(-\pi s)}{\exp(\pi s) + \exp(-\pi s)} \mathrm{d}s. \tag{4.5}$$

Therefore,

$$\mathrm{Im}\, G(\mathrm{e}^t) = \frac{1}{2} \int_{-\infty}^{\infty} p\left(\tfrac{1}{2} + \mathrm{i}s\right) \mathrm{e}^{(1/2 + \mathrm{i}s)t} \, \mathrm{d}s = \frac{1}{2\mathrm{i}} \int_{1/2 - \mathrm{i}\infty}^{1/2 + \mathrm{i}\infty} p(\zeta) \, \mathrm{e}^{t\zeta} \, \mathrm{d}\zeta.$$

This proves (b). For (c), consider first real $t \in \mathbb{R}$. We have already checked (a) hence $G(\mathrm{e}^t)$ is in $C^\infty(\mathbb{R})$. It is real and strictly positive for $t \le 0$ (this follows from the series representation and $p(k) > 0$), and non-zero though possibly complex-valued for sufficiently small $t > 0$. Therefore, $\varphi(t) = \log G(\mathrm{e}^t)$ is well-defined and smooth in some interval $(-\infty, \delta)$, $\delta > 0$, and real-valued for $t \le 0$. In particular, the derivatives $\kappa_j = \varphi^{(j)}(0)$ exist and are real, and $\varphi(t)$ can be approximated to arbitrarily high order by Taylor polynomials. The extension to complex t, $\mathrm{Im}\, t \in [0, 2\pi)$, follows again from the smoothness of $L(u)$ in the closed strip $\mathrm{Im}\, u \in [-\pi, \pi]$.   □

Theorem 4.2(b) has an interesting consequence. Eq. (3.2) is, up to a factor π, the formula for the inverse Laplace transform, therefore

$$p(\lambda) = \pi \int_0^\infty \mathrm{e}^{-t\lambda} \, \mathrm{Im}\, G(\mathrm{e}^t) \mathrm{d}t \quad (\mathrm{Re}\, \lambda > 0). \tag{4.6}$$

In the special case of stretched exponential weights, we can draw on an extensive literature as $\exp(-\lambda^\alpha)$ is known to be the Laplace transform of a probability density, an α-*stable law* [16]. For $\alpha = 1/2$ [4]

$$\mathrm{Im}\, G(\mathrm{e}^t) = \frac{c\sqrt{\pi}}{2t^{3/2}} \mathrm{e}^{-1/(4t)} \quad (t \ge 0). \tag{4.7}$$

For general $\alpha \in (0, 1)$, we have instead [10, Theorem 2.4.6]

$$\mathrm{Im}\, G(\mathrm{e}^t) \sim \frac{c}{2} \sqrt{\frac{2\pi}{(1 - \alpha)\alpha^{-1/(1-\alpha)}}} \frac{\exp\left(-(1-\alpha)\left(\frac{\alpha}{t}\right)^{\alpha/(1-\alpha)}\right)}{t^{(2-\alpha)/(2-2\alpha)}} \tag{4.8}$$

as $t \searrow 0$. This is proven in [10] by applying a steepest descent approach to the Bromwich integral. For general weights, Eq. (4.8) is generalized as follows.

Assume that $t < \lim_{x \searrow a} |q''(x)|$. By Assumption 2.1, q'' is strictly increasing and negative on (a, ∞). By Eq. (2.6), we have $q''(x) \to 0$ as $x \to \infty$. Consequently, there exists a uniquely defined $\zeta(t)$ that solves $q'(\zeta(t)) = t$. We define

$$\psi(t) = t\zeta(t) - q(\zeta(t)) \tag{4.9}$$

and note the relations

$$\psi'(t) = \zeta(t), \quad \psi''(t) = \frac{1}{q''(\zeta(t))}, \tag{4.10}$$

so $\psi(t)$ is monotone increasing and strictly concave. Since $-\psi(-t)$ is the Legendre transform of the convex function $-q(x)$, it comes as no surprise that Assumption 2.1 on large x translates into information on small t.

Lemma 4.3 *The following holds:*
(a) $\lim_{t \searrow 0} t\psi'(t)/\log t = \infty$.
(b) $-\psi''(t) \geq c\frac{\psi'(t)}{t}$ *for some* $c > 0$ *and all sufficiently small* $t > 0$.
(c) $0 \leq \psi'''(t) \leq C\frac{|\psi''(t)|}{t}$ *for some* $C > 0$ *and all sufficiently small* $t > 0$.
The lemma has been proven in [14, Lemma 2.2].

Theorem 4.4 *As* $t \searrow 0$,

$$\operatorname{Im} G(e^t) \sim \frac{1}{2}\sqrt{2\pi|\psi''(t)|}\, e^{\psi(t)}.$$

Proof By Theorem 4.2(b), we may start from the Bromwich representation of $\operatorname{Im} G(e^t)$. The analyticity of $q(\zeta)$ allows us to replace the contour $\operatorname{Re}\zeta = 1/2$ by $\Gamma = \Gamma_1 \cup \Gamma_2$ where

$$\Gamma_1 = \{\zeta \in \mathbb{C} \mid |\zeta| = \zeta(t), \operatorname{Re}\zeta \geq 1/2\}, \quad \Gamma_2 = \{\zeta \in \mathbb{C} \mid |\zeta| > \zeta(t), \operatorname{Re}\zeta = 1/2\}. \tag{4.11}$$

To lighten notation, set $r = \zeta(t)$ and suppress the t- and r-dependence from the notation. Let $\theta_0 = \arcsin(1/2r)$ and notice $\theta_0 \nearrow \pi/2$ as $r \to \infty$ ($t \searrow 0$).

For small θ, we have

$$t\, r\, e^{i\theta} - q(r\, e^{i\theta}) = \psi(t) - \frac{1}{2}r^2 q''(r)(e^{i\theta} - 1)^2 + O\big(r^3 q'''(r)(e^{i\theta} - 1)^2\big)$$

$$= \psi(t) - \frac{1}{2}r^2|q''(r)|\big(\theta^2 + O(\theta^3)\big) \tag{4.12}$$

The estimate is uniform in $r = \zeta(t)$ by Assumption 2.3(iii). Let $\varepsilon(r) \searrow 0$ with $\varepsilon(r)^2 r^2 q''(r)/\log r \to \infty$ (this is possible by Assumption 2.1), then

$$\frac{1}{2i}\int_{-\varepsilon(r)}^{\varepsilon(r)} \exp\big(t r\, e^{i\theta} - q(r\, e^{i\theta})\big) i r\, e^{i\theta}\, d\theta \sim \frac{1}{2}\sqrt{2\pi|\psi''(t)|}\, e^{\psi(t)}. \tag{4.13}$$

As we veer away from $r = \zeta(t)$ along Γ_1, the real part of $\zeta t - q(\zeta)$ decreases. Indeed, for $\theta \in (0, \pi/2)$

$$\frac{d}{d\theta}\operatorname{Re}\big(t r\, e^{i\theta} - q(r\, e^{i\theta})\big) = \operatorname{Re}\big(i\zeta(t - q'(\zeta))\big)\Big|_{\zeta = r\, e^{i\theta}}$$

$$= -\operatorname{Im}\big(\zeta q'(r) - \zeta q'(\zeta)\big)\Big|_{\zeta = r\, e^{i\theta}} \leq 0. \tag{4.14}$$

At the very end, we have used Assumption 2.3(ii) It follows that

$$
\left| \frac{1}{2i} \int_{\varepsilon(r)}^{\theta_0} \exp\!\big(tr\,e^{i\theta} - q(r\,e^{i\theta})\big) ir\,e^{i\theta}\,d\theta \right|
$$
$$
\leq \frac{r\pi}{4} \exp\!\Big(\psi(t) - \frac{1}{2}r^2 |q''(r)|\varepsilon(r)(1 + o(1))\Big) = o\Big(\exp(\psi(t))\Big)
$$

Taking complex conjugates, we obtain a similar estimate for the integral from $-\theta_0$ to $-\varepsilon(r)$. Together with (4.13), we obtain

$$
\frac{1}{2i} \int_{\Gamma_1} e^{t\zeta - q(\zeta)}\,d\zeta \sim \frac{1}{2}\sqrt{2\pi\,|\psi''(t)|}\,e^{\psi(t)}. \tag{4.15}
$$

It remains to estimate the contribution from Γ_2. Because of the monotonicity (4.14), we have

$$
\mathrm{Re}\,(t\zeta_0 - q(\zeta_0)) \leq \psi(t) - \frac{1}{2}r^2 |q''(r)|\varepsilon(r)^2(1 + o(1)) \tag{4.16}
$$

Assumption 2.3(ii) ensures that

$$
\left| \int_{\Gamma_2} e^{t\zeta - q(\zeta)}\,d\zeta \right| \leq \exp\!\Big(\mathrm{Re}\,\big[t\zeta_0 - q(\zeta_0)\big] + O(\log r))\Big) = o\Big(\exp(\psi(t))\Big). \tag{4.17}
$$

$\square$

5 Critical Point and Gaussian Approximation

Here, we prove Lemmas 2.4–2.7 and Proposition 2.12 and we address steps 4 and 5 of the proof strategy.

5.1 Variational Analysis of $f_n(x)$: Critical Scales

Proof of Lemma 2.4 We treat the case $a = 0$. Under Assumption 2.1(i), q' is strictly convex and decreasing. Therefore

$$
f_n'(x) = q'(x) - \frac{N_n - x}{n\sigma^2}
$$
$$
\geq q'(x_n^*) + q''(x_n^*)(x - x_n^*) - \frac{N_n - x}{n\sigma^2} = \frac{N_n^* - N_n}{n\sigma^2}
\tag{5.1}
$$

with equality if and only if $x = x_n^*$. If $N_n < N_n^*$, we obtain $f_n'(x) > 0$ on $(0, \infty)$ and parts (a) and (b) of the lemma follow right away.

If $N_n > N_n^*$, then $f_n'(x_n^*) = (N_n^* - N_n)/(n\sigma^2) < 0$ and $\lim_{x \to \infty} f_n'(x) = \infty$, so by the intermediate value theorem f_n' has at least one zero in (x_n^*, ∞). On the other hand

$$
f_n''(x) = q''(x) + (n\sigma^2)^{-1} = q''(x) - q''(x_n^*) > 0 \text{ on } (x_n^*, \infty) \tag{5.2}
$$

so f_n' is strictly increasing and f_n' has exactly one zero x_n in (x_n^*, ∞), moreover $f_n(x_n) = \min_{(x_n^*, \infty)} f_n$. Since $q'(x_n) > 0$ by Assumption 2.1 and $q'(x_n) = (N_n - x_n)/(n\sigma^2)$, we must have $x_n < N_n$. This proves the first part of (c).

If in addition to $N_n > N_n^*$, we have $\limsup_{n\to\infty} N_n/(n\sigma^2) < \lim_{x\searrow a} q'(x)$, then $\lim_{x\searrow a} f_n'(x) > 0$. We have already observed that $f_n'(x_n^*) < 0$. By the intermediate value theorem, f_n' has at least one zero x_n' in $(0, x_n^*)$. Since $f_n''(x) = q''(x) - q''(x_n^*) < 0$ on $(0, x_n^*)$, the zero is unique and corresponds to maximizer. This completes the proof of (c). The proof of (d) is similar to (c) and therefore omitted.   □

Proof of Lemma 2.5 We treat the case $a = 0$. Write $f_n(x) = I_n(x, N_n)$ with $I_n(x, y) = q(x) + [y - x]^2/[2n\sigma^2]$. For $y > N_n^*$, let $x_n(y) > x_n^* > x_n'(y)$ be the solutions of $\partial_x I_n(x, y) = 0$, with $x_n'(y)$ well-defined for $N_n \leq n\sigma^2 \sup q'$ only. Notice that $x \mapsto I_n(x, y)$ is increasing in $(0, x_n'(y))$, decreasing in $(x_n'(y), x_n(y))$, and increasing in $(x_n(y), \infty)$. We have

$$\frac{\mathrm{d}}{\mathrm{d}y}\big[I_n(x_n(y), y) - I(0, y)\big] = \frac{y - x_n(y)}{n\sigma^2} - \frac{y}{n\sigma^2} = -\frac{x_n(y)}{n\sigma^2} < 0. \tag{5.3}$$

As $y \searrow N_n^*$ at fixed n, a careful examination of the proof of Lemma 2.4 shows $x_n(y) \searrow x_n^*$ and $x_n'(y) \nearrow x_n^*(y)$, hence $I_n(x_n(y), y) \to I_n(x_n^*, N_n^*)$. But $x \mapsto I_n(x, N_n^*)$ is strictly increasing on $(0, \infty)$ because for $N_n = N_n^*$, $\partial_x I_n(\cdot, N_n) = f_n'(x) \geq 0$ by Eq. (5.1), hence $I_n(x_n^*, N_n^*) > I_n(0, N_n^*)$ and by continuity

$$\lim_{y\searrow N_n^*}\big[I_n(x_n(y), y) - I(0, y)\big] > 0. \tag{5.4}$$

Assumption 2.1 implies that $q(y) = o(y)$ as $y \to \infty$. It follows that

$$\lim_{y\to\infty}\big[I_n(y, y) - I(0, y)\big] = \lim_{y\to\infty}\left[q(y) - \frac{y^2}{2n\sigma^2}\right] = -\infty. \tag{5.5}$$

Equations (5.3)–(5.5) guarantee the existence and uniqueness of a solution $y = N_n^{**}$ to the equation $I_n(x_n(y), y) = I(0, y)$, and (a)–(c) follow with the observation $f_n(x_n) - f_n(0) = [I_n(x_n(y), y) - I_n(0, y)]|_{y=N_n}$.   □

Proof of Lemma 2.6 As noted after Assumption 2.1, we have $\lim_{x\to\infty} x^2 q''(x) = -\infty$, moreover from the definition (2.10) of x_n^* and the observation $x_n^* \to \infty$ we get

$$1 = \lim_{n\to\infty} n\sigma^2 |q''(x_n^*)| \gg \frac{n\sigma^2}{(x_n^*)^2} \tag{5.6}$$

hence $x_n^* \gg \sqrt{n}$. The inequality $x_n^* < N_n^*$ follows from the definition 2.10 of N_n^* and the positivity of q'. The inequality $N_n^* < N_n^{**}$ holds true by definition of N_n^{**}. By Assumption 2.1(v) there exists $C > 0$ such that $q(x) \leq Cx^\alpha$ for all sufficiently large x. Fix $C' > C$ and $N_n \geq (2C'n\sigma^2)^{1/(2-\alpha)}$. Then, for large n,

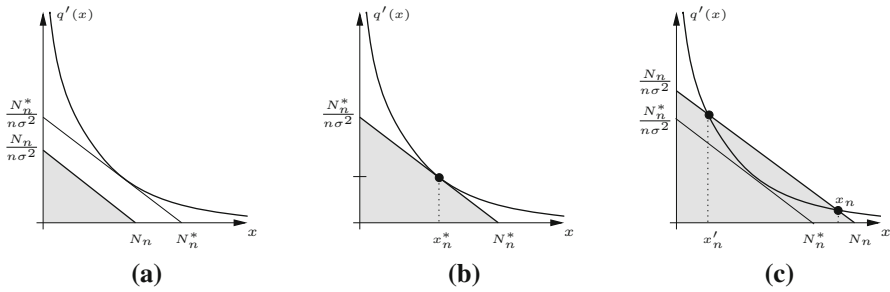$$\frac{q(N_n)}{N_n/(2n\sigma^2)} \leq 2n\sigma^2 C N_n^{\alpha-2} \leq \frac{C}{C'} < 1. \tag{5.7}$$

Fig. 3 Solutions to $q'(x) = (N_n - x)/(n\sigma^2)$ (= critical points of f_n) as in Lemma 2.7. **a** $N_n < N_n^*$: no solutions. **b** $N_n = N_n^*$: one solution. **c** $N_n > N_n^*$: two solutions

Write $C/C' = 1 - \varepsilon$. It follows that $f_n(N_n) = q(N_n) < (1 - \varepsilon)\frac{N_n}{n\sigma^2} \le (1 - \varepsilon)(1 + o(1))f_n(a)$, and a fortiori $\min f_n \le f_n(N_n) < f_n(a)$, which shows $N_n \ge N_n^{**}$. This proves $N_n^{**} = O(n^{1/(2-\alpha)})$ and completes the proof of the first part of the lemma.

Next suppose by contradiction that $N_n^*/x_n^* \to 1$. Then by Eq. (2.10) we must have $n\sigma^2 q'(x_n^*)/x_n^* \to 0$. Since $x_n^* \to \infty$, Assumption 2.1 yields $n\sigma^2 q''(x_n^*) \to 0$, contradicting the definition of x_n^*. Similarly, the assumption $N_n^*/x_n^* \to \infty$ leads to $n\sigma^2 q''(x_n^*) \to \infty$, contradicting again the definition of x_n^*. So N_n^*/x_n^* stays bounded away from 1 and from ∞. □

Proof of Lemma 2.7 Remember $q'(x_n^*) = [N_n^* - x_n^*]/[n\sigma^2]$ by definition of N_n^*, and $q'(x_n) = [N_n - x_n]/[n\sigma^2]$ by definition of x_n. Since q' is strictly decreasing we deduce

$$\frac{N_n - x_n}{n\sigma^2} = q'(x_n) < q'(x_n^*) = \frac{N_n^* - x_n^*}{n\sigma^2} < \frac{N_n^*}{2\sigma^2}, \tag{5.8}$$

so $N_n - x_n \le N_n^*$ (see Fig. 3). Since q' is strictly convex, we have $q'(x_n^*) > q'(x_n) + q''(x_n)(x_n^* - x_n)$ hence

$$q''(x_n) > \frac{q'(x_n) - q'(x_n^*)}{x_n - x_n^*} = \frac{(N_n - x_n) - (N_n^* - x_n^*)}{n\sigma^2(x_n - x_n^*)} = \frac{O(N_n^*)}{n\sigma^2(N_n + O(N_n^*))} \tag{5.9}$$

We also know that $q''(x_n) < 0$, so we obtain

$$f_n''(x_n) = q''(x_n) + \frac{1}{n\sigma^2} = \frac{1}{n\sigma^2}\left(1 + O\left(\frac{N_n^*}{N_n}\right)\right). \tag{5.10}$$

□

For the proof of Theorem 2.9 in the case $N_n \sim N_n^*$, we need the following.

Lemma 5.1 *Let $N_n = N_n^*$. We have*

$$f_n(x_n^*) - \frac{(N_n^*)^2}{2n\sigma^2} \ge \varepsilon\frac{(N_n^*)^2}{2n\sigma^2}. \tag{5.11}$$

for some $\varepsilon > 0$ and all sufficiently large n,

Proof Assume $a = 0$ and $q(0) = 0$. By Lemma 2.5, we already know $f_n(x_n^*) - f_n(0) > 0$, we prove $f_n(x_n^*) - f_n(0) > \varepsilon f_n(0)$. Since $f_n'(x_n^*) = 0$ for $N_n = N_n^*$,

$$f_n'(x) = \int_{x_n^*}^{x} f_n''(y)\mathrm{d}y = \frac{1}{n\sigma^2}\int_{x}^{x_n^*}\left(n\sigma^2|q''(y)| - 1\right)\mathrm{d}y. \tag{5.12}$$

For $y \geq x_n^*/2$, the integrand stays bounded away from zero, hence

$$f_n'(x) \geq \frac{c(x_n^* - x)}{n\sigma^2} \tag{5.13}$$

for all $x \geq x_n^*/2$ and some $c > 0$. Then,

$$f_n(x_n^*) - f_n(0) = \int_{0}^{x_n^*} f_n'(y)\mathrm{d}y \geq \frac{c x_n^{*2}}{8n\sigma^2} \tag{5.14}$$

and the statement follows from $N_n^* = O(x_n^*)$ (Lemma 2.6) and $f_n(0) = \frac{(N_n^*)^2}{2n\sigma^2}(1 + o(1))$. The proof for $a > 0$ is based on a similar estimate of $f_n(x_n^*) - f_n(2a)$ and therefore omitted. □

Proof of Proposition 2.12 Fix $r \in \mathbb{N}_0$. We have

$$f_{nr}'(x) = f_n'(x) + O\left(\left(\frac{N_n - x}{n\sigma^2}\right)^2\right). \tag{5.15}$$

Clearly, $f_{nr}'(N_n) = q'(N_n) > 0$. Fix $\delta \in (0, \varepsilon)$ and $x \in (x_n^*, (1 + \delta)x_n^*)$. Remembering (2.10) and the monotonicity of q', we obtain

$$\begin{aligned}
f_{nr}'(x) &\leq \frac{N_n^* - x_n^*}{n\sigma^2} - (1 + o(1))\frac{N_n - (1 + \delta)x_n^*}{n\sigma^2} \\
&= -\frac{1}{n\sigma^2}\left(N_n + o(N_n) - N_n^* - \delta x_n^*\right) \leq -\frac{1}{n\sigma^2}\left((1 + o(1))\varepsilon N_n^* - \delta x_n^*\right)
\end{aligned} \tag{5.16}$$

which is eventually negative because of $x_n^* \leq N_n^*$ and $\delta < \varepsilon$. It follows that f_{nr}' does indeed have a zero x_{nr} which lies between $(1 + \delta)x_n^*$ and N_n. On $(x_n^*, (1 + \delta)x_n^*)$ f_{nr} is strictly decreasing by (5.16), on $((1 + \delta)x_n^*, N_n)$ the second derivative satisfies

$$f_{nr}''(x) = \frac{1}{n\sigma^2}\left(1 - n\sigma^2|q''(x)| + O\left(\frac{N_n}{n}\right)\right) \tag{5.17}$$

which stays bounded away from 0, hence f_{nr} is strictly convex. It follows that x_{nr} is the unique zero of f_{nr}' and the maximizer of f_{nr} in (x_n^*, N_n). Moreover,

$$(1 + o(1))\frac{N_n - x_{nr}}{n\sigma^2} = q'(x_{nr}) \leq q'(x_n^*) = \frac{N_n^* - x_n^*}{n\sigma^2} \tag{5.18}$$

hence $x_{nr} = N_n - (1 + o(1))n\sigma^2 q'(x_{nr}) = N_n + O(N_n^*)$. The asymptotic expression for $f_{nr}(x_{nr})$ is easily checked.   □

5.2 Critical Points

We look for critical points (t_n, ζ_n) with $t_n \searrow 0$ and $\zeta_n \in (a, \infty)$ of

$$\Phi_n(t, \zeta) = -q(\zeta) + n\varphi(t) - mt + t\zeta \qquad (5.19)$$

(remember $m = \mu n + N_n$). Since our contour integrals involve integrals over $z = e^t$, it is convenient to work with functions of a single variable t: for $t < \lim_{x \to a} |q''(x)|$, let

$$\Psi_n(t) = \Phi_n(t, \zeta(t)) = n\varphi(t) - mt + \psi(t). \qquad (5.20)$$

where $\zeta(t)$ is the solution of $q'(\zeta(t)) = t$ as on p.14. Then, (t, ζ) is a critical point of $\Phi_n(t, \zeta)$ if and only if $\zeta = \zeta(t)$ and $\Psi_n'(t) = 0$. So instead of looking for bivariate critical points, we may look for zeros of $\Psi_n'(t)$ in $(0, \sup |q''(x)|)$. For later purpose, we note the relations

$$\Psi_n'(t) = n\varphi'(t) - m + \zeta(t) = -(N_n - \zeta(t)) + n\sigma^2 t(1 + O(t)) \qquad (5.21)$$

$$\Psi_n''(t) = n\varphi''(t) + \psi''(t) = n\sigma^2(1 + O(t)) + \frac{1}{q''(\zeta(t))}, \qquad (5.22)$$

with $\psi(t)$ defined in (4.9). The variable t and the analysis of Ψ_n are in some sense dual to the variable x (or ζ) and the variational problem $f_n(x) = \min$ analyzed in Sect. 5.1. The analysis becomes more involved, however, because we need to take into account correction terms from $\sum_{j \geq 3} \kappa_j t^j / j!$.

Lemma 5.2 (Inflection point of Ψ_n) *Let $\delta > 0$ such that $\inf_{(0,\delta)} \varphi''(t) > 0$. Then, for all sufficiently large n, Ψ_n has an inflection point $t_n^* \in (0, \delta)$. Any inflection point satisfies $t_n^* \sim q'(x_n^*)$, and Ψ_n is concave below the smallest inflection point and convex above the largest inflection point.*

If the inflection point is unique, then Ψ_n is concave on $(0, t_n^*)$ and (t_n^*, δ). In general, we do not know whether the inflection point is unique; however, the asymptotic behavior $t_n^* \sim q'(x_n^*)$ is uniquely determined, and all statements below hold for every inflection point t_n^*.

Remark 4 It follows that t_n^* is of the order of N_n^*/n: from Lemma 5.2 and the definition of N_n^*, we have $t_n^* \sim \frac{N_n^* - x_n^*}{n\sigma^2}$, and then Lemma 2.6 yields

$$\delta \frac{N_n^*}{n} \leq t_n^* \leq C \frac{N_n^*}{n}. \qquad (5.23)$$

Proof of Lemma 5.2 As a preliminary observation, we note that any solution t_n^* of $\Psi_n'(t) = 0$ converges to zero: this is because $-\psi''(t_n^*) = n\varphi''(t) \geq n \inf_{(0,\delta)} \varphi'' \to \infty$.

As $t \searrow 0$ at fixed n, we have $\varphi''(t) \to \sigma^2$ and $\psi''(t) \to -\infty$ hence $\Psi_n''(t) \to -\infty$. On the other hand, we may choose ε_n in such a way that $\varepsilon_n \searrow 0$ and $\Psi_n''(\varepsilon_n) \to \infty$: indeed, $\psi''(q'(x_n^*)) = 1/q''(x_n^*) = -n\sigma^2$ by definition of x_n^*, so choosing $\varepsilon_n \gg q'(x_n^*)$ in such a way that $|\psi''(\varepsilon_n)| \ll |\psi''(q'(x_n^*))| = n\sigma^2$ we find $\Psi_n''(\varepsilon_n) = (1 + o(1))n\sigma^2 + o(n\sigma^2) \to \infty$. It follows from the intermediate value theorem that $\Psi_n''(t) = 0$ has a solution t_n^* in $(0, \varepsilon_n)$. It satisfies

$$\frac{q''(x_n^*)}{q''(\zeta(t_n^*))} = -\frac{\psi''(t_n^*)}{n\sigma^2} = \frac{\varphi''(t_n^*)}{\sigma^2} = 1 + O(t_n^*) \to 1. \tag{5.24}$$

Assumption 2.1(iv) and its consequence (2.5) imply that $\zeta(t_n^*)/x_n^* \to 1$ and $t_n^* = q'(\zeta(t_n^*)) \sim q'(x_n^*)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

$\Psi_n'(t)$ is positive for small t and decreasing on $(0, t_n^*)$ and increasing on (t_n^*, δ). Define $\zeta_n^* = \zeta(t_n^*)$ and notice $\zeta_n^* \sim x_n^*$ from the proof of the previous lemma.

For $N_n \ll n$, let $\eta_n \sim N_n/(n\sigma^2)$ be the solution of (3.4). Notice $\Psi_n'(\eta_n) = \zeta(\eta_n) \to \infty$.

Lemma 5.3 *Let $\delta > 0$ be as in Lemma 5.2, $\varepsilon > 0$, and t_n^* an inflection point of Ψ_n. Let $N_n \to \infty$ with $(1+\varepsilon)N_n^* \le N_n \ll n$. Then, for sufficiently large n, Ψ_n' has exactly two zeros in $(0, \delta)$, one zero $t_n \in (0, t_n^*)$ and another $t_n' \in (t_n^*, \eta_n)$. Set $\zeta_n := \zeta(t_n)$. We have*

$$0 \le N_n - \zeta_n = O(N_n^*),$$

$\limsup_{n\to\infty} t_n/t_n^* < 1$, $\liminf_{n\to\infty} \zeta_n/x_n^* > 1$, *and* $\liminf t_n'/\eta_n > 0$.

Remark 5 When $N_n \gg N_n^*$, we use Eq. (2.4), Lemmas 2.6 and 5.2 and find

$$\zeta_n \sim N_n, \quad t_n = q'(\zeta_n) \sim q'(N_n) \ll q'(x_n^*) \sim t_n^* = O\left(\frac{N_n^*}{n}\right) \tag{5.25}$$

hence $t_n = o(t_n^*)$. When $N_n = O(N_n^*)$, we have instead $c^{-1}t_n^* \le t_n \le c\, t_n^*$ for some $c > 0$ and all sufficiently large n: The upper bound is part of Lemma 5.3. For the lower bound, we note that $t_n = q'(\zeta_n) \ge q'(N_n)$ because q' is decreasing and $\zeta_n \le N_n$. Since N_n is of the order of N_n^*, Eq. (2.4) shows that $q'(N_n)$ is of the order of $q'(N_n^*)$ which in turn is of the order of $q'(x_n^*) \sim t_n^*$.

Proof of Lemma 5.3 We check first that $t_n^* < \eta_n$. With $C \ge 1$ as in Lemma 2.6, we have

$$t_n^* \sim q'(x_n^*) = \frac{N_n^* - x_n^*}{n\sigma^2} \le (1 - C^{-1})\frac{N_n^*}{n\sigma^2} \tag{5.26}$$

so

$$t_n^*/\eta_n \le (1 + o(1))(1 - C^{-1})N_n^*/N_n \le (1 + o(1))\frac{1 - C^{-1}}{1 + \varepsilon} \tag{5.27}$$

is bounded away from 1. As $t \to 0$ at fixed n, $\Psi'_n(t) \sim \zeta(t) \to \infty$, and as $n \to \infty$, $\Psi'_n(\eta_n) \to \infty$. Furthermore,

$$
\begin{aligned}
\Psi'_n(t_n^*) &= -(N_n - \zeta_n^*) + (1 + o(1))n\sigma^2 q'(\zeta_n^*) \\
&= -\big(N_n - (1 + o(1))x_n^*\big) + (1 + o(1))n\sigma^2 q'(x_n^*) \\
&= -\big(N_n - (1 + o(1))x_n^*\big) + (1 + o(1))\big(N_n^* - x_n^*\big) \\
&= -(N_n - N_n^*) + o(N_n^*) \to -\infty.
\end{aligned}
\tag{5.28}
$$

The intermediate value theorem proves the existence of a zero $t_n \in (0, t_n^*)$ and another zero $t'_n \in (t_n^*, \eta_n)$.

Suppose by contradiction that $t_n/t_n^* \to 1$. Then, the identities $t_n = q'(\zeta_n)$, $t_n^* = q'(\zeta_n^*)$ and Eq. (2.4) imply $\zeta_n \sim \zeta_n^* \sim x_n^*$ and an estimate analogous to (5.28) shows $\Psi'_n(t_n) \to -\infty$, in contradiction with $\Psi'_n(t_n) = 0$. It follows that t_n/t_n^* and ζ_n/x_n^* are bounded away from 1. In addition, $t_n \le t_n^* = O(N_n^*/n)$ and

$$
N_n - \zeta_n \sim n\sigma^2 t_n = O(N_n^*).
\tag{5.29}
$$

For the lower bound of t'_n, we use $t'_n \ge t_n^* \sim q'(x_n^*)$ and Assumption 2.1 to get $\zeta'_n \le (1 + o(1))x_n^*$ where $q'(\zeta'_n) = t'_n$. Since $t'_n = O(\eta_n) \to 0$, $\Psi'_n(t'_n) = 0$ together with (5.21) yields

$$
t'_n \sim \frac{N_n - \zeta'_n}{n\sigma^2} \sim \eta_n\left(1 - \frac{\zeta'_n}{N_n}\right) \ge \eta_n\left(1 - \frac{N_n^*}{N_n}\right)
\tag{5.30}
$$

and $N_n \ge (1 + \varepsilon)N_n^*$ implies $\liminf t'_n/\eta_n > 0$. Notice that, in view of (5.27), we have a fortiori $\liminf t'_n/t_n^* > 0$.

We have actually shown that for every inflection point t_n^*, $t/t_n^* \le 1$ stays bounded away from 1. In particular, in case of non-uniqueness of t_n^*, we may choose t_n^* as the *smallest* inflection point of Ψ_n. Then, Ψ'_n is strictly increasing on $(0, t_n^*)$; consequently, the zero t_n is unique. A similar argument shows that t'_n is unique. □

When $\liminf_{n \to \infty} N_n/n > 0$, the sequence η_n is either no longer defined or it does not converge to zero. The previous lemma is modified as follows.

Lemma 5.4 *Assume $N_n \to \infty$ with $\liminf N_n/n > 0$. Then, there exists $\delta_0 > 0$ such that for large n, Ψ_n has exactly one critical point t_n in $(0, \delta_0)$. The critical point lies in $(0, t_n^*)$, and it satisfies $0 \le N_n - \zeta(t_n) = O(N_n^*)$.*

Proof The existence and uniqueness of a critical point in $(0, t_n^*)$ as well as the properties of $\zeta(t_n)$ are proven as in the previous lemma. Fix $\delta_0 > 0$ such that $\varphi'(\delta_0) \le \mu + \frac{1}{2}\liminf(N_n/n) =: \mu + \varepsilon/2$. Then

$$
\Psi'_n(\delta_0) = -n\left(\mu + \frac{N_n}{n} - \varphi'(\delta_0)\right) + \zeta(\delta_0) \le -n\left(\frac{\varepsilon}{2} + o(1)\right) + \zeta(\delta_0) \to -\infty.
\tag{5.31}
$$

It follows that $\Psi'_n < 0$ on (t_n^*, δ_0). □

5.3 Hessians

Let $t_n \in (0, t_n^*)$ be the critical point of $\Psi_n(t)$, and $\zeta_n = \zeta(t_n)$. Thus, (t_n, ζ_n) is a critical point of $\Phi_n(t, \zeta)$. Lemma B.1 shows

$$\Psi_n''(t_n) = -\frac{\det \text{Hess}\, \Phi_n(t_n, \zeta_n)}{|q''(\zeta_n)|} = -\frac{1 - n\text{Re}\,\varphi''(t_n)|q''(\zeta_n)|}{|q''(\zeta_n)|}. \tag{5.32}$$

Lemma 5.5 *Assume* $N_n \to \infty$ *with* $\liminf_{n\to\infty}(N_n/N_n^*) > 1$ *and let* (t_n, ζ_n) *be the unique critical point of* Φ_n *in* $(0, t_n^*) \times (a, \infty)$.

(a) *If* $N_n \gg N_n^*$, *then* $\det \text{Hess}\, \Phi_n(t_n, \zeta_n) \to -1$.
(b) *If* $N_n = O(N_n^*)$, *then* $\det \text{Hess}\, \Phi_n(t_n, \zeta_n) = -(1 - n\sigma^2|q''(\zeta_n)|) + o(1)$ *and it stays bounded away from zero.*

Proof (a) If $N_n \gg N_n^*$, then by Lemmas 5.3 and 5.4, we have $\zeta_n = N_n + O(N_n^*) \sim N_n$; hence, in particular $\zeta_n \gg N_n^* \geq x_n^*$. Exploiting the monotonicity and the convexity of q', we have

$$0 \geq q''(\zeta_n) \geq \frac{q'(\zeta_n) - q'(x_n^*)}{\zeta_n - x_n^*} = \frac{O(N_n^*/(n\sigma^2))}{N_n(1 + o(1))} = -\frac{1}{n\sigma^2}O\left(\frac{N_n^*}{N_n}\right) \tag{5.33}$$

from which we get

$$\det \text{Hess}\, \Phi_n(t_n, \zeta_n) = -1 + n\sigma^2(1 + O(t_n))|q''(\zeta_n)| = -1 + O\left(\frac{N_n^*}{N_n}\right) \to -1. \tag{5.34}$$

(b) If $N_n = O(N_n^*)$: By Lemma 5.3, $\zeta_n \geq (1 + \varepsilon)x_n^*$ for some $\varepsilon > 0$. Consequently $|q''(\zeta_n)| \leq (1 - \delta)|q''(x_n^*)| = (1 - \delta)/(n\sigma^2)$ for some $\delta > 0$ and large n, from which we deduce that

$$1 - n\text{Re}\,\varphi''(t_n)|q''(\zeta_n)| \geq 1 - (1 + O(t_n))(1 - \delta) = 1 - \delta + O(t_n), \tag{5.35}$$

in particular the expression stays bounded away from zero. We also have

$$\det \text{Hess}\, \Phi_n(t_n, \zeta_n) = -(1 - n\sigma^2|q''(\zeta_n)|) + o(t_n)n\sigma^2|q''(\zeta_n)| \tag{5.36}$$

Since $\zeta_n > x_n^*$, the estimate (5.33) still holds true and

$$t_n n\sigma^2|q''(\zeta_n)| = t_n O\left(\frac{N_n^*}{N_n}\right) = O(t_n) \to 0. \tag{5.37}$$

$\square$

5.4 Gaussian Approximation

Next we address the Gaussian approximation for the evaluation of H_n. Because of Theorem 4.4, we need not deal with a bivariate integral and instead may use

$$\frac{1}{\pi} \int_0^{\varepsilon_n} e^{n\operatorname{Re}\varphi(t) - mt} \operatorname{Im} G(e^t) dt \sim \frac{1}{\sqrt{2\pi}} \int_0^{\varepsilon_n} \sqrt{|\psi''(t)|}\, e^{\Psi_n(t)}\, dt \qquad (5.38)$$

as $n \to \infty$ and $\varepsilon_n \searrow 0$. Remember that $\Psi_n(t_n) = \Phi_n(t_n, \zeta_n)$ and from Eq. (5.32) and Lemma 5.5,

$$\frac{\psi_n''(t_n)}{\Psi_n''(t_n)} = \frac{1}{1 - n\sigma^2 |q''(\zeta_n)|} \qquad (5.39)$$

with a denominator bounded away from zero. The following technical lemma helps estimate the prefactor $\sqrt{|\psi''(t)|}$. Set

$$R_n(t) := \Psi_n(t) + \log\sqrt{|\psi''(t)|}. \qquad (5.40)$$

Lemma 5.6 *Let $N_n \to \infty$ with $\liminf(N_n/N_n^*) > 1$ and $t_n \in (0, t_n^*)$ the zero of $\Psi_n'(t)$ from Lemma 5.3. Then, R_n' has at least one zero $s_n \in (0, t_n)$. Moreover, there exists a sequence $\delta_n \searrow 0$ such that every such zero lies in $((1 - \delta_n)t_n, t_n)$.*

Proof As $t \searrow 0$ at fixed n, using Lemma 4.3, we have

$$R_n'(t) = \Psi_n'(t) + \frac{1}{2}\frac{\psi'''(t)}{\psi''(t)} = n\big(\operatorname{Re}\varphi'(t) - \mu\big) - N_n + \psi(t) + \frac{1}{2}\frac{\psi'''(t)}{\psi''(t)}$$
$$= n(\sigma^2 t + O(t^2)) - N_n + \psi'(t) + O(1/t)$$
$$= -N_n + o(1) + (1 + o(1))\psi'(t) \to \infty, \qquad (5.41)$$

hence, $t \mapsto R_n(t)$ is initially increasing. At $t = t_n$, we have

$$R_n'(t_n) = \Psi_n'(t_n) + \frac{1}{2}\frac{\psi'''(t_n)}{\psi''(t_n)} = \frac{1}{2}\frac{\psi'''(t_n)}{\psi''(t_n)} < 0. \qquad (5.42)$$

The intermediate value theorem guarantees the existence of a zero s_n of R_n'. Set

$$y_n := \psi'(s_n) + \frac{1}{2}\frac{\psi'''(s_n)}{\psi''(s_n)}. \qquad (5.43)$$

In view of Lemma 4.3, we have $y_n \sim \psi'(s_n)$ and by Eqs. (2.4) and (4.10), $q'(y_n) \sim s_n$. From $s_n \le t_n \le t_n^*$ and Lemma 2.7 we get that y_n is larger than x_n^* and actually bounded away from it. By the definition of s_n,

$$0 = n\big(\operatorname{Re}\varphi'(s_n) - \mu\big) - N_n + y_n = n\sigma^2 s_n(1 + O(s_n)) - [N_n - y_n] \qquad (5.44)$$

hence

$$\frac{N_n - y_n}{n\sigma^2} \sim s_n \sim q'(y_n). \tag{5.45}$$

Let $x_n \in (x_n^*, N_n)$ be the solution of $q'(x_n) = (N_n - x_n)/(n\sigma^2)$. Now

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(q'(y) - \frac{N_n - y}{n\sigma^2}\right) = q''(y) + \frac{1}{n\sigma^2} = \frac{1 - n\sigma^2|q''(y)|}{n\sigma^2} \tag{5.46}$$

and $n\sigma^2|q''(y)|$ stays bounded away from 1 when $y \geq (1+\delta)x_n^*$, we find that for some $c > 0$, we have

$$\left|q'(y_n) - \frac{N_n - y}{n\sigma^2}\right| \geq \frac{c|y_n - x_n|}{n\sigma^2} \tag{5.47}$$

hence $y_n - x_n = o(N_n - y_n) = o(N_n)$. If $N_n \gg N_n^*$, Lemma 2.7 says $x_n \sim N_n$ and we deduce $y_n - x_n = o(x_n)$. If $N_n = O(N_n^*)$, we use $x_n \geq x_n^*$ in conjunction with Lemma 2.6 and find $y_n - x_n = o(N_n^*) = o(x_n^*) = o(x_n)$.

Thus, we have checked that $y_n \sim x_n$, which in turn yields $s_n \sim q'(x_n)$. An entirely similar argument shows $t_n \sim q'(x_n)$, so we must have $s_n \sim t_n$. This holds for every zero in $(0, t_n)$, in particular the smallest one, and the lemma follows. □

Lemma 5.7 *Let $N_n \to \infty$ with $\liminf(N_n/N_n^*) > 1$ and $N_n = O(N_n^*)$. Then,*

$$\int_0^{t_n'} \sqrt{|\psi''(t)|}\, \mathrm{e}^{\Psi_n(t)}\, \mathrm{d}t \sim \sqrt{\frac{2\pi}{1 - n\sigma^2 q''(\zeta_n)}}\, \mathrm{e}^{\Psi_n(t_n)}.$$

Proof By Lemmas 5.2 and 5.3, Ψ_n is increasing on $(0, t_n)$ and decreasing on (t_n, t_n'). We use a Gaussian approximation to Ψ_n around t_n and adapt [14, Lemma 2.1]. First, we check that the window $(t_n - \varepsilon_n, t_n + \varepsilon_n)$ contributing most to the Gaussian integral fits amply into $(0, t_n')$, i.e.,

$$\varepsilon_n := \frac{1}{\sqrt{|\Psi_n''(t_n)|}} = o\big(\min(t_n, t_n' - t_n)\big). \tag{5.48}$$

By Lemma 5.3, we have $t_n \leq (1-\delta)t_n^* \leq (1-\delta)t_n'$ for some $\delta \in (0, 1)$, hence $t_n' - t_n \geq \delta t_n$ and

$$\min(t_n, t_n' - t_n) \geq \delta t_n. \tag{5.49}$$

Eq. (5.39) and Lemma 5.5 show that $\Psi_n''(t_n)$ is of the order of $\psi''(t_n)$, ε_n^2 of the order of $1/|\psi_n''(t_n)|$. By Lemma 4.3, $t_n^2|\psi_n''(t_n)| \to \infty$ hence $\varepsilon_n^2/t_n^2 \to 0$ and (5.48) follows. The same argument shows that for

$$c_n \to \infty \quad \text{with} \quad c_n\varepsilon_n = o(t_n) \tag{5.50}$$

we still get $c_n\varepsilon_n = o\big(\min(t_n, t_n' - t_n)\big)$. Second, we observe that the prefactor $\psi''(t)$ is essentially constant on the relevant window: Because of $\psi'''(u)/\psi''(u) = O(1/u)$

(Lemma [4.3]), we have

$$\frac{\psi''(t)}{\psi''(t_n)} = \exp\left(-\int_{t_n}^{t} \frac{\psi'''(u)}{\psi''(u)} du\right) = \exp\left(O\left(\log \frac{t}{t_n}\right)\right) \quad (5.51)$$

and

$$\sup\left\{\left|\left|\frac{\psi''(t)}{\psi''(t_n)} - 1\right|\right| \, |t - t_n| \le c_n \varepsilon_n\right\} = O(c_n \varepsilon_n) \to 0. \quad (5.52)$$

Third, we note that cubic corrections can be neglected:

$$\Psi_n(t) = \Psi_n(t_n) + \frac{1}{2}\left(1 + o(1)\right)\Psi_n''(t_n)(t - t_n)^2 \quad (5.53)$$

uniformly in $|t - t_n| \le c_n \varepsilon_n$. To this aim write, with the help of (5.52),

$$\Psi_n''(t) = n\left(\varphi''(t_n) + O(c_n \varepsilon_n)\right) + \psi_n''(t_n)(1 + O(c_n \varepsilon_n))$$
$$= \Psi_n''(t_n) + \psi_n''(t_n)\left(O\left(\frac{nc_n \varepsilon_n}{\psi_n''(t_n)}\right) + O(c_n \varepsilon_n)\right)$$

Now $n/\psi''(t_n) = nq''(\zeta_n) = O(1)$ by Lemma [5.5] and $\psi''(t_n) = O(\Psi_n''(t_n))$ by Eq. (5.39) and the same lemma, hence

$$\Psi_n''(t) = (1 + O(c_n \varepsilon_n))\Psi_n''(t_n) \quad (5.54)$$

in $|t - t_n| \le c_n \varepsilon_n$ and (5.53) follows. Eqs. (5.52), (5.53) and (5.39) yield

$$\int_{t_n - c_n \varepsilon_n}^{t_n + c_n \varepsilon_n} \sqrt{|\psi''(t)|}\, e^{\Psi_n(t)}\, dt \sim \sqrt{\frac{2\pi \psi''(t_n)}{\Psi_n''(t_n)}} \sim \sqrt{\frac{2\pi}{1 - n\sigma^2 q''(\zeta_n)}}\, e^{\Psi_n(t_n)}. \quad (5.55)$$

Our next task is to estimate the integral on $(0, t_n - c_n \varepsilon_n)$ and $(t_n + c_n \varepsilon_n, t_n')$, taking into account that the prefactor $\sqrt{|\psi''(t)|}$ goes to infinity. On both intervals, we have

$$\Psi_n(t) \le \Psi_n(t_n) - \frac{1}{2}(1 + O(c_n \varepsilon_n))c_n^2. \quad (5.56)$$

For $t \ge t_n + c_n \varepsilon_n$, we have $-\psi''(t) \le -\psi''(t_n)$ hence

$$\sqrt{|\psi''(t)|}\exp(\Psi_n(t)) \le \exp\left(\Psi_n(t_n) - \frac{1}{2}(1 + o(1))c_n^2 + \frac{1}{2}\log|\psi''(t_n)|\right). \quad (5.57)$$

By Lemma [5.6], we may choose $c_n \varepsilon_n \searrow 0$ in such a way that R_n is increasing on $(0, t_n(1 - c_n \varepsilon_n))$, which yields

$$\Psi_n(t) + \log\sqrt{|\psi''(t)|} \le \Psi_n(t_n - c_n \varepsilon_n) + \log\sqrt{|\psi''(t_n - c_n \varepsilon_n)|}. \quad (5.58)$$

This estimate, combined with Eqs. (5.52) and (5.56), shows that (5.57) holds true not only in $(t_n + c_n \varepsilon_n, t_n')$ but also in $(0, t_n - c_n \varepsilon_n)$. Next, we check that we can choose $c_n \to \infty$ so that not only $c_n \varepsilon_n = o(t_n)$ but in fact

$$- (1 + o(1))c_n^2 + \log |\psi''(t_n)| \to -\infty. \tag{5.59}$$

By Lemma 4.3 (iii), as $t \searrow 0$, we may estimate $|\psi''(t)|$ as follows: fix $t_0 > 0$ and take $t \in (0, t_0)$, then

$$\log(-\psi''(t)) = \log(-\psi''(t_0)) - \int_t^{t_0} \frac{\psi'''(u)}{\psi''(u)} \mathrm{d}u \le \log(-\psi''(t_0)) + \int_t^{t_0} \frac{C}{u} \mathrm{d}u$$

$$= \log(-\psi''(t_0)) + C \log \frac{t_0}{t} = O(|\log t|). \tag{5.60}$$

In particular $\log |\psi''(t_n)| = O(\log t_n)$. On the other hand,

$$\frac{c_n^2}{|\log t_n|} = \left(\frac{c_n \varepsilon_n}{t_n}\right)^2 \frac{\Psi_n''(t_n)}{\psi''(t_n)} \frac{t_n^2 |\psi''(t_n)|}{|\log t_n|} \tag{5.61}$$

The ratio $\Psi_n''(t_n)/\psi''(t_n)$ stays bounded away from 0 and by Lemma 4.3,

$$\frac{t^2 |\psi''(t)|}{|\log t|} \to \infty. \tag{5.62}$$

Thus we may find a function $\omega(t) \to 0$ such that $\omega(t)^2 t^2 |\psi''(t)|/|\log t|$ still goes to infinity as $t \searrow 0$, set $c_n \varepsilon_n = t_n \omega(t_n)$, and then Eq. (5.59) holds true. The bound (5.57) then shows

$$\int_0^{t_n - c_n \varepsilon_n} \sqrt{|\psi''(t)|}\, \mathrm{e}^{\Psi_n(t)}\, \mathrm{d}t + \int_{t_n + c_n \varepsilon_n}^{t_n'} \sqrt{|\psi''(t)|}\, \mathrm{e}^{\Psi_n(t)}\, \mathrm{d}t = o\left(\mathrm{e}^{\Psi_n(t_n)}\right). \tag{5.63}$$

Equations (5.55) and (5.63) complete the proof of the lemma. □

Lemma 5.8 *Let $N_n \to \infty$ with $N_n^* \ll N_n \ll n$. Then*

$$\int_0^{t_n'} \sqrt{|\psi''(t)|}\, \mathrm{e}^{\Psi_n(t)}\, \mathrm{d}t \sim \sqrt{2\pi}\ \mathrm{e}^{\Psi_n(t_n)}.$$

Proof The proof of Lemma 5.7 applies without any changes, the end result simplifies because $1 - n\sigma^2 q''(\zeta_n) \to 1$ by Lemma 5.5. □

Lemma 5.9 *Let $N_n \to \infty$ with $\liminf(N_n/n) > 0$. Let $\delta_0 > 0$ as in Lemma 5.4. Then*

$$\int_0^{\delta_0} \sqrt{|\psi''(t)|}\, \mathrm{e}^{\Psi_n(t)}\, \mathrm{d}t \sim \sqrt{2\pi}\ \mathrm{e}^{\Psi_n(t_n)}.$$

Proof By Lemma 5.4, $\Psi_n(t)$ has a unique critical point t_n in $(0, \delta_0)$ and we may use a Gaussian approximation on this interval—there is no need to restrict to an interval whose length goes to zero. Apart from this difference, the proof is identical to Lemmas 5.7 and 5.8. □

6 Evaluation of Contour Integrals: Proof of the Main Theorems

The proof of the main theorems starts from the decomposition

$$\mathbb{P}(S_n = n\mu + N_n) = H_n + V_n \tag{6.1}$$

where H_n and V_n are defined as in (3.7). The correctness of (6.1) is checked as in Sect. 3, building on the properties of $G(z)$ proven in Sect. 4. For the proof of Theorem 2.10, it is convenient to decompose H_n further as $H_n = H_n^1 + H_n^2$ where

$$H_n^1 = \frac{1}{\pi} \int_0^{t_n'} e^{nRe\,\varphi(t)-(\mu n+N_n)t}\,\sin(nIm\,\varphi(t))dt \tag{6.2}$$

and H_n^2 is a similar integral, but with integration from t_n' to η_n. In the proof of Theorem 2.11, we adopt slightly modified definitions and replace the sequence $\eta_n \sim N_n/(n\sigma^2)$ in the domain of integration by another sequence $\varepsilon_n \searrow 0$ or by some fixed small $\varepsilon > 0$.

6.1 Evaluation of V_n

Theorem 2.11 only needs upper bounds for V_n, provided in Lemmas 6.1 and 6.2. Theorems 2.9 and 2.10 requires the full asymptotic behavior of V_n proven in Lemma 6.3.

Lemma 6.1 *Suppose $N_n \to \infty$ and $N_n \geq \delta n$ for some $\delta > 0$ and all $n \in \mathbb{N}$. Then, for suitable $C = C_\delta > 0$, every sufficiently small $\eta > 0$, and all $n \in \mathbb{N}$*

$$\frac{1}{\pi} \int_0^\pi e^{nRe\,\varphi(\eta+i\theta)-(\mu n+N_n)\eta}\,d\theta \leq e^{-Cn\eta}.$$

Proof Let $\mathcal{S}_+$ be the half-strip $\{t \in \mathbb{C} \mid Re\,t \geq 0, Im\,t \in [0, \pi)\}$. By Theorem 4.2, we know that as $t \to 0$ in $\mathcal{S}_+$

$$Re\left(\varphi(t) - \mu t - \frac{N_n}{n}t\right) = -\frac{N_n}{n}Re\,t + \frac{1}{2}\sigma^2\left((Re\,t)^2 - (Im\,t)^2\right) + O(t^3). \tag{6.3}$$

Then, for sufficiently small $\varepsilon_1 > 0$ and all $t \in \mathcal{S}_+$ with $\max(|Re\,t|, |Im\,t|) \leq \varepsilon_1$, the right side of Eq. (6.3) is smaller than $-\delta Re\,t/2$, which shows

$$\frac{1}{\pi} \int_0^{\varepsilon_1} e^{nRe\,\varphi(\eta+i\theta)-(\mu n+N_n)\eta}\,d\theta \leq \frac{\varepsilon_1}{\pi} e^{-n\delta\eta/2} \tag{6.4}$$

for all $\eta \in (0, \varepsilon_1)$. On the other hand, on the unit circle, $G(z)$ is given by a power series with strictly positive coefficients, and therefore, $|G(z)|$ has a unique maximum at $z = 1$. It follows that for all $t = i\theta$ with $\theta \in [\varepsilon_1, 2\pi - \varepsilon_1]$, we know $\mathrm{Re}\,\varphi(t) < 0$. By continuity this extends to some thin vertical strip $\mathrm{Im}\,t \in [\varepsilon_1, 2\pi - \varepsilon_1]$, $\mathrm{Re}\,t \in [0, \varepsilon_2]$ so that $\mathrm{Re}\,(\varphi(t) - \mu t) \le -\mu \mathrm{Re}\,t$ and

$$\frac{1}{\pi} \int_{\varepsilon_1}^{\pi} e^{n\mathrm{Re}\,\varphi(\eta+i\theta)-(\mu n+N_n)\eta}\,d\theta \le \frac{\pi - \varepsilon_1}{\pi} e^{-n\mu\eta} \tag{6.5}$$

for all $\eta \in (0, \varepsilon_2)$. To conclude, we let $C_\delta := \min(\delta/2, \mu)$. $\qquad\square$

Lemma 6.2 *Assume $N_n = m - n\mu \to \infty$ and $N_n = o(n)$. Let $\varepsilon_n \searrow 0$ with $\varepsilon_n \le \eta_n = (1 + o(1))\frac{N_n}{n\sigma^2}$. Then, for suitable constant $C > 0$, as $n \to \infty$,*

$$\left| \frac{1}{2\pi} \int_0^{\pi} e^{n\mathrm{Re}\,\varphi(\varepsilon_n+i\theta)-m\varepsilon_n}\,d\theta \right| \le e^{-(1+o(1))N_n\varepsilon_n/2}.$$

Proof The proof is analogous to Lemma 6.1. We start from the estimate (6.3). In a sufficiently small ε_1-neighborhood of the corner 0 of the half-strip $\mathcal{S}_+$, we have

$$\mathrm{Re}\left(\varphi(t) - \frac{m}{n}t\right) \le -\left(\frac{N_n}{n} - \frac{1}{2}\sigma^2 \mathrm{Re}\,t\right)\mathrm{Re}\,t. \tag{6.6}$$

Notice that ε_1 can be chosen n-independent: we only need $-\sigma^2(\mathrm{Im}\,t)^2 + O((\mathrm{Im}\,t)^3) \le 0$ for $|\mathrm{Im}\,t| \le \varepsilon_1$. When $\mathrm{Re}\,t = \varepsilon_n$ with $\varepsilon_n \le \frac{N_n}{n\sigma^2}$, the upper bound in (6.6) is in turn bounded by $-(1 + o(1))N_n\varepsilon_n/2$.

When $\mathrm{Re}\,t = \varepsilon_n$ is small but $\mathrm{Im}\,t$ is bounded away from 0 and 2π, we estimate

$$\mathrm{Re}\left(\varphi(t) - \frac{m}{n}t\right) \le -\frac{m}{n}\mathrm{Re}\,t = -\left(\mu + \frac{N_n}{n}\right)t \le -\frac{N_n}{n}t. \tag{6.7}$$

and we conclude as in Lemma 6.1. $\qquad\square$

The proof of the next lemma is closely related to the treatment of moderate deviations for random variables with generating functions analytic beyond $z = 1$ given by Ibragimov and Linnik [10].

Lemma 6.3 *Let $N_n \to \infty$ along $\sqrt{n} \ll N_n = O(n^{1-\gamma})$ for some $\gamma > 0$. Define $\eta_n \sim N_n/(n\sigma^2)$ by (3.4) and V_n as in (3.7). Then Eq. (3.11) holds true.*

Proof Since $\mathrm{Re}\,\varphi'(t) = \mu + \sigma^2 t + O(t^2)$ is strictly increasing for small $t > 0$ and $N_n/n \to 0$, we may fix $\delta > 0$ small enough so that for large $n \ge n_\delta$, the Eq. (3.4) has indeed a unique solution $\eta_n \in (0, \delta)$, which satisfies $\eta_n \sim N_n/(n\sigma^2)$. Arguments analogous to the proof of Eq. (6.5) show

$$\frac{1}{\pi}\left| \int_\delta^{\pi} e^{n(\varphi(\eta_n+i\theta))-(\mu n+N_n)(\eta_n+i\theta)}\cos(n\mathrm{Im}\,\varphi(\eta_n + i\theta))d\theta \right| \le e^{-n\mu\eta_n}. \tag{6.8}$$

We also have, uniformly in $s \in (0, \delta_n)$,

$$\varphi(\eta_n + \mathrm{i}s) = \varphi(\eta_n) + \varphi'(\eta_n)\mathrm{i}s - \frac{1}{2}\varphi''(\eta_n)s^2 + O(s^3)$$

$$= \varphi(\eta_n) + \mathrm{i}\left(\mu + \frac{N_n}{n}\right)s - \operatorname{Im}\varphi'(\eta_n)s - \frac{1}{2}\varphi''(\eta_n)s^2 + O(s^3) \quad (6.9)$$

and therefore

$$\operatorname{Re}\varphi(\eta_n + \mathrm{i}s) - \left(\mu + \frac{N_n}{n}\right)\eta_n = \operatorname{Re}\varphi(\eta_n) - \frac{1}{2}\varphi''(\eta_n)s^2(1 + O(s)) - \operatorname{Im}\varphi'(\eta_n)s. \tag{6.10}$$

Since $\varphi''(\eta_n) \to \sigma^2$, standard arguments show

$$\frac{1}{\pi}\int_0^\delta e^{n\operatorname{Re}\varphi(\eta_n) - \frac{n}{2}\operatorname{Re}\varphi''(\eta_n)s^2(1+O(s))} \, \mathrm{d}s \sim \frac{\exp(n\operatorname{Re}\varphi(\eta_n))}{\sqrt{2\pi n\sigma^2}}, \tag{6.11}$$

moreover the contribution to the interval from $s \geq \delta_n := (\log n)/\sqrt{n}$ is negligible and Eq. (6.11) holds true with δ replaced by δ_n. By Theorem 4.2 and the relation $\operatorname{Im} G(e^t) = |G(e^t)|\operatorname{Im}\varphi(t)$, the imaginary part of $\varphi'(\eta_n)$ vanish faster than any power of $\eta_n = O(n^{-\gamma})$, hence $n\operatorname{Im}\varphi'(\eta_n)$ can be neglected; the same argument works for $n\operatorname{Im}\varphi''(\eta_n)$. For the cosine, we look separately at $(0, \delta_n)$ and (δ_n, δ). On $(0, \delta_n)$, again by Theorem 4.2, $\sup_{s\in(0,\delta_n)}|\operatorname{Im}\varphi(\eta_n + \mathrm{i}s)|$ vanishes faster than any power of $\max(\eta_n, \delta_n)$; hence, it can be neglected. On (δ_n, δ), we simply bound the cosine by 1. As the contribution from (δ_n, δ) to the integral (6.11) is negligible, combining with (6.8), we find in the end

$$V_n = (1 + o(1))\frac{\exp(n\operatorname{Re}\varphi(\eta_n) - \mu n - N_n)}{\sqrt{2\pi n\sigma^2}} + O\left(e^{-n\mu\eta_n}\right). \tag{6.12}$$

By Theorem 4.2 and Definition 2.8, we have

$$n\operatorname{Re}\varphi(\eta_n) - \mu n - N_n = \left(1 + O\left(\frac{N_n}{n}\right)\right)\frac{N_n^2}{2n\sigma^2} \tag{6.13}$$

with correction terms expressed in terms of the Cramér series. In particular, the exponent goes to $-\infty$ as $-N_n^2/n$, i.e., slower than the term $-n\mu\eta_n = -N_n\mu$ in the second term. Therefore, the second term in Eq. (6.12) is negligible compared to the first and Eqs. (3.10) and (3.11) hold true.   □

6.2 Evaluation of H_n

Here we focus on the regime $\liminf N_n/N_n^* > 1$; the case $\limsup N_n/N_n^* \leq 1$ is treated in the proof of Theorem 2.9. In order to apply the Gaussian approximation from Sect. 5.4, we need to drop the sine and replace $\operatorname{Im}\varphi(t)$ with $G(e^t)$.

Lemma 6.4 *Let $\varepsilon_n \searrow 0$ faster than some power of n, i.e., $n\varepsilon_n^p \to 0$ for some $p > 0$. Then,*

$$\sin\bigl(n\,\mathrm{Im}\,\varphi(t)\bigr) \sim n\,\mathrm{Im}\,G(\,\mathrm{e}^t\,)$$

uniformly for $t \in [0, \varepsilon_n]$.

Proof We have

$$\mathrm{Im}\,G(\,\mathrm{e}^t\,) = \mathrm{Im}\,\mathrm{e}^{\varphi(t)} = \mathrm{e}^{\mathrm{Re}\,\varphi(t)}\,\mathrm{Im}\,\varphi(t) = |G(\,\mathrm{e}^t\,)|\,\mathrm{Im}\,\varphi(t). \qquad (6.14)$$

We know that

$$\sup_{t\in[0,\varepsilon_n]} |\mathrm{Re}\,G(\,\mathrm{e}^t\,) - 1| = O(\varepsilon_n) \to 0, \qquad (6.15)$$

and the imaginary part vanishes faster than any power, in particular

$$\sup_{t\in[0,\varepsilon_n]} |\mathrm{Im}\,G(\,\mathrm{e}^t\,)| = O(\varepsilon_n^p) \to 0. \qquad (6.16)$$

It follows that a similar bound applies to $\mathrm{Im}\,\varphi(t)$. As a consequence,

$$\sup_{t\in[0,\varepsilon_n]}\left|\frac{n\,\mathrm{Im}\,\varphi(t) - \sin(n\,\mathrm{Im}\,\varphi(t))}{\sin(n\,\mathrm{Im}\,\varphi(t))}\right| = O\left(n^2 \sup_{t\in[0,\varepsilon_n]} |\mathrm{Im}\,\varphi(t)|^2\right)$$
$$= O\left(n^2\varepsilon_n^{2p}\right) \to 0, \qquad (6.17)$$

Thus $\sin(n\,\mathrm{Im}\,\varphi(t)) \sim n\,\mathrm{Im}\,\varphi(t)$, uniformly in $[0, \varepsilon_n]$. Eq. (6.14) in turn shows $\mathrm{Im}\,\varphi(t) \sim \mathrm{Im}\,G(\,\mathrm{e}^t\,)$ uniformly in $[0, \varepsilon_n]$. $\qquad\square$

The dominant contribution to the Gaussian integral in Lemmas 5.7–5.9 comes from windows of width $o(t_n)$ around t_n; by Remark 5 and Lemma 2.6, $t_n = O(t_n^*) = O(N_n^*/n) = O(n^{-[1-\alpha]/[2-\alpha]})$. This latter bound vanishes like some negative power of n, hence Lemma 6.4 is applicable on the interval contributing most to the Gaussian integrals. Outside, we use the inequality $|\sin(n\,\mathrm{Im}\,\varphi(t))| \le n\,\mathrm{Im}\,\varphi(t) = n(1 + o(1))\,\mathrm{Im}\,G(\,\mathrm{e}^t\,)$, and we find: For $N_n \to \infty$ with $\liminf_{n\to\infty} N_n/N_n^* > 1$, we have

$$H_n^1 \sim \frac{n}{\sqrt{1 - n\sigma^2 q''(\zeta_n)}}\,\mathrm{e}^{\Psi_n(t_n)}. \qquad (6.18)$$

For $N_n \to \infty$ with $N_n \gg N_n^*$,

$$H_n^1 \sim n\,\mathrm{e}^{\Psi_n(t_n)}. \qquad (6.19)$$

H_n^2 is estimated in the proof of Theorem 2.10 and is not needed in the proof of Theorem 2.11. Finally, for $\liminf N_n/n > 0$ and $\delta_0 > 0$ small enough as in Lemma 5.4,

$$\frac{1}{\pi} \int_0^{\delta_0} \mathrm{e}^{n\,\mathrm{Re}\,\varphi(t) - mt}\,\sin\bigl(n\,\mathrm{Im}\,\varphi(t)\bigr)\mathrm{d}t \sim n\,\mathrm{e}^{\Psi_n(t_n)}. \qquad (6.20)$$

Remember that $\Psi_n(t_n) = \Phi_n(t_n, \zeta_n) = -f_{nr}(x_{nr}) + o(1)$ and $1 - n\sigma^2|q''(\zeta_n)| \sim 1 - n\sigma^2|q''(x_{nr})|$ by the definition of Ψ_n and Eq. (2.22), so the right-hand sides in Eqs. (6.18)–(6.20) correspond to the relevant contribution in Theorems 2.10 and 2.11.

6.3 Critical Scale: Proof of Theorem 2.10

Let $N_n \to \infty$ with $\liminf N_n/N_n^* > 1$ and $N_n = O(n^{1/(2-\alpha)})$. Let $\eta_n \sim N_n/(n\sigma^2)$ be the solution of (3.4), define H_n and V_n as in (3.7), and H_n^1 and H_n^2 as in (6.2) Thus we have

$$\mathbb{P}(S_n = n\mu + N_n) = V_n + H_n^1 + H_n^2. \qquad (6.21)$$

By Lemma 6.3, the asymptotics of V_n is given in terms of the Cramér series as in Eq. (3.11). H_n^1 is evaluated with Lemmas 5.7 and 5.8 as the right-hand side of Eq. (3.22). The proof is complete once we show $H_n^2 = o(V_n)$.

On (t_n', η_n), the function $\Psi_n(t)$ is increasing by Lemma 5.3,

$$\sup_{t \in (t_n', \eta_n)} \Psi_n(t) \le \Psi_n(\eta_n) = \Big(n\operatorname{Re}\varphi(\eta_n) - (n\mu + N_n)\eta_n\Big) + \psi(\eta_n). \qquad (6.22)$$

The term in big parentheses can be reexpressed with the Cramér series and is exactly equal to the exponent in the evaluation of V_n [see Eqs. (3.10) and (3.11)], while $\psi(\eta_n) \to -\infty$. The prefactor satisfies

$$\sup_{t \in (t_n', \eta_n)} \sqrt{|\psi''(t)|} \le \sqrt{|\psi''(t_n')|} = \exp\Big(O(\log t_n')\Big) = \exp\Big(O(\log n)\Big). \qquad (6.23)$$

(remember (5.4) and $t_n' \ge t_n^*$, with t_n^* of the order of $N_n^*/n \gg 1/\sqrt{n}$). On the other hand, $\eta_n \searrow 0$ faster than some power of n, so by Lemma 4.3, we have for some constant C

$$|\psi(\eta_n)| \gg |\log \eta_n| \ge C \log n \gg \log \sqrt{|\psi''(t_n')|}, \qquad (6.24)$$

whence

$$\psi(\eta_n) + \log n + \sup_{t \in (t_n', \eta_n)} \log \sqrt{|\psi''(t)|} \to -\infty. \qquad (6.25)$$

In view of (6.22), (3.10) and (3.11), we obtain

$$H_n^2 = \frac{n}{\pi} \int_{t_n'}^{\eta_n} e^{n\operatorname{Re}\varphi(t) - (n\mu + N_n)t} \operatorname{Im} G(e^t) dt = o(V_n). \qquad (6.26)$$

which concludes the proof. $\square$

6.4 Big-Jump: Proof of Theorem 2.11

Let $N_n \to \infty$ with $N_n \gg n^{1/(2-\alpha)}$. Notice that, by Lemma 2.6, we then have $N_n \gg N_n^{**}$. We distinguish two cases.

Case 1 $\liminf_{n\to\infty} N_n/n > 0$. Fix $\delta_0 > 0$ as in Lemma 5.4 and define H_n and V_n as in (3.7) but with δ_0 instead of η_n. The asymptotic behavior of H_n is given by Eq. (6.20). The proof is complete once we check $V_n = o(H_n)$.

V_n is exponentially small in n by Lemma 6.1. Remembering Eqs. (2.22) and (2.16), we get

$$\Psi_n(t_n) = - f_{nr}(x_{nr}) + o(1) = -q(N_n) + \frac{1}{2}(1 + o(1))n\sigma^2 q'(N_n)^2 + o(1)$$
$$\geq - q(N_n) + o(1) \geq -C N_n^\alpha. \tag{6.27}$$

It follows that $H_n \sim n \exp(\Psi_n(t_n))$ goes to zero slower than $\exp(-cn^\alpha)$ for some $c > 0$, hence $V_n = o(H_n)$ and

$$\mathbb{P}(S_n = n\mu + N_n) \sim H_n \sim n\, e^{-f_{nr}(x_{nr})}. \tag{6.28}$$

If $n^{1/(2-\alpha)} \ll N_n \ll n$, set $\varepsilon_n = t_n'$. By Remark 5, the critical point $t_n' \in (t_n^*, \eta_n)$ is bounded from below by some constant times $\eta_n \sim N_n/(n\sigma^2)$. We define H_n and V_n as in (3.7) but with ε_n instead of η_n. H_n is evaluated as in (6.19), which yields $H_n \sim n \exp(\Psi_n(t_n))$; Eq. (6.27) stays valid. V_n is estimated by Lemma 6.2, which yields $V_n = O(\exp(-N_n t_n'/2)))$. Now $N_n t_n' \to \infty$ much faster than N_n^α. Indeed, t_n' is bounded from below by some constant times N_n/n, hence

$$\frac{N_n^\alpha}{N_n t_n'} = O(n N_n^{\alpha-2}) \to 0. \tag{6.29}$$

It follows that $V_n = o(H_n)$ and Eq. (6.28) stays true. $\qquad\qquad\qquad\qquad\square$

6.5 Small-Steps: Proof of Theorem 2.9

The proof of Theorem 2.9 requires two more technical lemmas, proven at the end of this section. Remember the function $R_n(t)$ from (5.40).

Lemma 6.5 *If* $\limsup_{n\to\infty} N_n/N_n^* < 1$, *then* $R_n' > 0$ *on* $(0, \eta_n)$ *for all sufficiently large* n.

Lemma 6.6 *Let* $N_n \to \infty$ *with* $N_n \sim N_n^*$. *Set* $f_n^*(x) := q(x) + \frac{(N_n^*-x)^2}{2n\sigma^2}$. *Suppose that there are infinitely many* $n \in \mathbb{N}$ *for which the equation* $R_n'(t) = 0$ *has a solution* $s_n \in (0, \eta_n)$. *Then,* $s_n \sim t_n^*$ *and*

$$R_n(s_n) = -f_n^*(x_n^*) + o\Big(\frac{(N_n^*)^2)}{n}\Big) + O(\log n).$$

The zero s_n need not be unique—in case of non-uniqueness the lemma applies to every choice of s_n.

Proof of Theorem 2.9 Let $N_n \to \infty$ along $\sqrt{n} \ll N_n \leq (1 + o(1))N_n^*$. Let $\eta_n \sim N_n/(n\sigma^2)$ be the solution of (3.4) and define V_n and H_n as in (3.7). By Lemma 6.3,

the asymptotic behavior of V_n is given by Eqs. (3.10) and 3.11, so it remains to verify that $H_n = o(V_n)$. We estimate

$$H_n \leq \frac{n\eta_n}{\pi} \sup_{t \in (0,\eta_n)} \sqrt{|\psi''(t)|}\, e^{\Psi_n(t)} = \frac{n\eta_n}{\pi} \sup_{t \in (0,\eta_n)} e^{R_n(t)}. \tag{6.30}$$

Just as in Lemma 5.6, one checks that $R_n(t)$ is increasing for small t. We distinguish two cases.

Case 1 $\liminf_{n \to \infty} N_n/N_n^* < 1$. Then Lemma 6.5 shows, for large n,

$$\sup_{t \in (0,\eta_n)} R_n(t) \leq R_n(\eta_n) = \Big(n\mathrm{Re}\,\varphi(\eta_n) - (n\mu + N_n)\eta_n\Big) + \psi(\eta_n) + \log \sqrt{|\psi''(\eta_n)|} \tag{6.31}$$

and we deduce from (3.10)

$$\begin{aligned}
\frac{H_n}{V_n} &\leq \frac{n\eta_n}{\pi} \sqrt{2\pi n\sigma^2} \exp\Big(\psi(\eta_n) + \log \sqrt{|\psi''(\eta_n)|}\Big) \\
&= \exp\Big(\psi(\eta_n) + \log \sqrt{|\psi''(\eta_n)|} + O(\log n)\Big)
\end{aligned} \tag{6.32}$$

which goes to zero by an estimate analogous to (6.24).

Case 2 $N_n \sim N_n^*$. If R_n' reaches its maximum at $t = \eta_n$, the estimate (6.32) still applies. If along some subsequence (n_j), R_n' reaches its maximum at some interior point $s_n \in (0, \eta_n)$, then we must have $R_n'(s_n) = 0$ and by Lemmas 6.6 and 5.1

$$\sup_{t \in (0,\eta_n)} R_n(t) \leq -(1 + \varepsilon + o(1))\frac{N_n^{*2}}{2n\sigma^2} + O(\log n). \tag{6.33}$$

for some $\varepsilon > 0$ and all large n. Since $V_n = \exp(-(1 + o(1))\frac{N_n^{*2}}{2n\sigma^2} + O(\log n))$ we get

$$\frac{H_n}{V_n} \leq \exp\Big(-(\varepsilon + o(1))\frac{N_n^{*2}}{2n\sigma^2} + O(\log n)\Big). \tag{6.34}$$

By Assumption 2.1 and Lemma 2.6,

$$\frac{1}{n\sigma^2} = |q''(x_n^*)| \gg \frac{\log x_n^*}{x_n^{*2}} \geq \frac{\log n + O(1)}{x_n^{*2}} \tag{6.35}$$

so $\log n = o(N_n^{*2}/n)$ and the right-hand of (6.34) goes to zero.

We have checked in both cases that $H_n = o(V_n)$, which concludes the proof. $\square$

Proof of Lemma 6.5 Suppose that the equation $R_n'(s_n) = 0$ has a solution $s_n \in (0, \eta_n)$ for infinitely many n. Thus, (s_n) may be defined only along some subsequence (n_j), which we suppress from the notation. Define $y_n \sim \psi'(s_n)$ as in (5.43). Proceeding as

in Lemma 5.6, we find that $q'(y_n) \sim (N_n - y_n)/(n\sigma^2)$. By the convexity of q' and the definition of N_n^*, we have

$$(1 + o(1))\frac{N_n - y_n}{n\sigma^2} = q'(y_n) \geq \frac{N_n^* - y_n}{n\sigma^2} = \frac{N_n^* - N_n}{n\sigma^2} + \frac{N_n - y_n}{n\sigma^2} \qquad (6.36)$$

hence $N_n^* - N_n \leq o(N_n - y_n) = o(N_n)$ and $\limsup N_n^*/N_n \leq 1$ i.e. $\liminf N_n/N_n^* > 1$.

So if $\limsup_{n\to\infty} N_n/N_n^* < 1$, we must have $R_n' \neq 0$ on $(0, \eta_n)$ except possibly for finitely many n. From the proof of Lemma 5.6 we know that $\lim_{t\searrow 0} R_n'(t) = \infty$ for all $n \in \mathbb{N}$, and Lemma 6.5 follows. □

Proof of Lemma 6.6 For $t \in (0, \eta_n)$ with $\eta_n \sim N_n/(n\sigma^2)$ we have

$$\begin{aligned}
R_n'(t) &= n(\operatorname{Re}\varphi'(t) - \mu) - N_n + \psi'(t) + \frac{1}{2}\frac{\psi'''(t)}{\psi''(t)} \\
&= n\sigma^2 t - (N_n^* - \zeta(t)) + o(nt) + (N_n^* - N_n) + O\left(\frac{1}{t}\right) \\
&= n\sigma^2 t - (N_n^* - \zeta(t)) + o(N_n^*) + o(\zeta(t)). \qquad (6.37)
\end{aligned}$$

In terms of the dual variable $\zeta = \zeta(t) = \psi'(t)$, the equation $R_n'(t) = 0$ reads

$$n\sigma^2 q'(\zeta) - (N_n^* - \zeta) = o(N_n^*) + o(\zeta). \qquad (6.38)$$

At $\zeta = x_n^*$, the left-hand side of (6.38) vanishes by definition of N_n^*. It follows that

$$n\sigma^2 q'(\zeta) - (N_n^* - \zeta) = \int_{x_n^*}^{\zeta} (1 + n\sigma^2 q''(x))dx. \qquad (6.39)$$

Fix $\gamma > 0$. On $((1+\gamma)x_n^*, \infty)$, we have $1 - n\sigma^2|q''(x)| \geq 1 - n\sigma^2|q''((1+\gamma)x_n^*)| =: c_\gamma > 0$ and for all $x \geq x_n^*$, the integrand is non-negative. As a consequence,

$$n\sigma^2 q'(\zeta) - (N_n^* - \zeta) \geq c_\gamma(\zeta - (1+\gamma)x_n^*) \qquad (6.40)$$

for all $\zeta \geq (1+\gamma)x_n^*$. Suppose that Eq. (6.38) has a solution ζ_n' with $\zeta_n' \geq (1+2\gamma)x_n^*$ along some subsequence. Then $N_n^* = O(x_n^*) = o(\zeta_n')$ and

$$c_\gamma \gamma \zeta_n' \leq c_\gamma(\zeta_n' - (1+\gamma)x_n^*) = o(\zeta_n'), \qquad (6.41)$$

which is a contradiction (remember $\zeta_n' \geq x_n^* \to \infty$). It follows that for every $\gamma > 0$, there are at most finitely many n for which $\zeta_n' > (1+2\gamma)x_n^*$, hence $\limsup \zeta_n'/x_n^* \leq 1$.

The case $\psi'(\eta_n) \leq \zeta_n' \leq (1-2\gamma)x_n^*$ is treated in an analogous fashion, based on two observations: first, $n\sigma^2 q''(x) + 1 \leq -c_\gamma < 0$ for all $x \in (a, (1-\gamma)x_n^*)$ and some $c_\gamma > 0$. Second, since η_n is of the order of N_n^*/n i.e., of the order of $t_n^* \sim q'(x_n^*)$, the estimate (2.4) shows that ζ_n' is bounded from below by some constant times N_n^*, i.e.,

we still have $N_n^* = O(\zeta_n')$ and $\zeta_n' \to \infty$. We find $\liminf \zeta_n'/x_n^* \geq 1$, hence altogether $\lim \zeta_n'/x_n^* = 1$. This applies in particular to $\zeta_n' := \psi'(s_n)$ i.e. $s_n = q'(\zeta_n')$. Eq. 2.4 and $t_n^* \sim q'(x_n^*)$ (Lemma 5.2) yield $\lim s_n/t_n^* = 1$.

By Remark 5 and Lemma 2.6, we have $s_n \sim t_n^* \geq \text{const } N_n^*/n \gg 1/\sqrt{n}$ hence $\log s_n = O(\log n)$ and by (5.4)

$$\log |\psi''(s_n)| = O(\log s_n) = O(\log n). \tag{6.42}$$

Furthermore for $t \in (0, \eta_n)$

$$\begin{aligned}
\Psi_n(t) &= \frac{1}{2}n\big(\sigma^2 t^2 + o(\eta_n^2)\big) - N_n t + t\zeta(t) - q(\zeta(t) \\
&= \left\{ -q(\zeta(t)) + \frac{1}{2}n\sigma^2 t^2 - (N_n^* - \zeta(t))t \right\} + O\big((N_n^* - N_n)\eta_n\big) + o(n\eta_n^2)
\end{aligned} \tag{6.43}$$

The two remainders are $o((N_n^*)^2/n)$ by our choice of N_n. Write $g_n(t)$ for the term in curly braces. At $t = q'(x_n^*) = [N_n^* - x_n^*]^2/[n\sigma^2]$, we have $\zeta(t) = x_n^*$, $g_n(t) = -f_n^*(x_n^*)$ and $g_n'(t) = 0$. Moreover, for $t \in (0, \eta_n)$

$$g_n''(t) = n\sigma^2 + \zeta'(t) = n\sigma^2 - \frac{1}{|q''(\zeta(t))|} = O(n\sigma^2). \tag{6.44}$$

Here, we have used that $t \leq \eta_n$ implies that $\zeta(t)$ is bounded from below by a constant times N_n^* or equivalently, x_n^* and therefore, $|q''(\zeta(t))|$ is bounded from below by some constant times $1/(n\sigma^2)$. We deduce

$$\left|g_n(s_n) + f_n^*(x_n^*)\right| \leq \frac{1}{2}(t_n - q'(x_n^*))^2 \sup_{0,\eta_n} |g_n''| = o(t_n^2 n\sigma^2) = o(n\eta_n^2) = o\left(\frac{(N_n^*)^2}{n}\right). \tag{6.45}$$

The estimate on $R_n(s_n)$ follows from Eqs. (6.42), (6.43) and (6.45). □

Appendix A. Proofs of Lemmas 2.14 and 2.15

Proof of Lemma 2.14 The proof of Assumption 2.1 is straightforward and left to the reader. The function $q(\zeta) = \zeta^\alpha$ is analytic in $\text{Re } \zeta > 0$ and $p(\zeta) = \exp(-\zeta^\alpha)$ satisfies, for all $k \in \mathbb{N}$,

$$|\zeta^k p(\zeta)| = |\zeta|^k e^{-|\zeta|^\alpha \cos(\alpha \arg(\zeta))} \leq |\zeta|^k e^{-|\zeta|^\alpha \cos(\alpha \pi/2)}. \tag{A.1}$$

Since $\alpha \in (0, 1)$, we have $\cos \frac{\alpha\pi}{2} > 0$ and Eq. (A.1) shows that $|\zeta^k p(\zeta)|$ is integrable along $\mathrm{Re}\,\zeta = 1/2$ and that $p(\zeta)$ grows slower than any exponential $\exp(\varepsilon|\zeta|)$. This proves Assumption 2.2.

The equation $q''(x_n^*) = -1/(n\sigma^2)$ can be solved explicitly. N_n^* and N_n^{**} are best determined with the scaling relation (2.24). They have already been determined in [12], we omit the proof. For the insensitivity scale, we notice that

$$n\sigma^2 q'(N_n)^2 = n\sigma^2\alpha^2 N_n^{2\alpha-2} \tag{A.2}$$

which goes to zero if and only if $N_n \gg n^{-1/(2-2\alpha)}$.   □

Proof of Lemma 2.15 The function $q(x) = -\log c + (\log x)^\beta$ is clearly smooth on $(1, \infty)$. Then $q'(x) = \beta(\log x)^{\beta-1}/x$ and as $x \to \infty$,

$$q''(x) \sim -\frac{\beta(\log x)^{\beta-1}}{x^2}, \quad q'''(x) \sim \frac{2\beta(\log x)^{\beta-1}}{x^3}. \tag{A.3}$$

Assumption 2.1 is easily checked. For Assumption 2.2 we note $q(\zeta) = c + (\log \zeta)^\beta$ is analytic in $\mathrm{Re}\,\zeta > 1$. Fix $b > 1$ and write $\zeta = r\exp(i\theta)$. As $|\zeta| \to \infty$ along $\mathrm{Re}\,\zeta = b$ i.e. $\zeta = b + iy$, the argument θ goes to $\pm\pi/2$ and we have

$$\mathrm{Re}\,(\log|\zeta| + i\theta)^\beta = \mathrm{Re}\,\left(\log|y| + \frac{1}{2}\log\left(1 + \frac{b^2}{y^2}\right) + i\theta\right)^\beta \tag{A.4}$$

$$= (\log|y|)^\beta + o(1), \tag{A.5}$$

conditions (i) and (ii) in Assumption 2.2 are easily checked. Assumption 2.3(iii) follows from a computation similar to (A.3). Set $y_0 = \sqrt{r^2 - b^2}$. We have for $\zeta = b + iy$, $y \geq y_r$, uniformly in r,

$$\mathrm{Re}\,q(\zeta) - \mathrm{Re}\,q(z_r) = (\log y)^\beta - (\log y_0)^\beta + o(1)$$

$$\geq \beta(\log y_0)^{\beta-1}\log\frac{y}{y_0} + o(1) \tag{A.6}$$

hence

$$\int_{y_r}^\infty e^{-\mathrm{Re}\,q(b+iy)}\,\mathrm{d}y \leq e^{-\mathrm{Re}\,q(b+iy_0)+o(1)}\int_1^\infty e^{-(\log y_0)^{\beta-1}\log s}\,y_0 \mathrm{d}s$$

$$\sim e^{-\mathrm{Re}\,q(b+iy_0)}\frac{y_0}{(\log y_0)^{\beta-1}} = e^{-\mathrm{Re}\,q(b+iy_0)+O(\log r)}, \tag{A.7}$$

which proves Assumption 2.3(i). Next let $\zeta \in \mathbb{C}$ with $\mathrm{Re}\,\zeta > 1$, write $\zeta = r\exp(i\theta)$, then

$$\zeta q'(\zeta) = \beta(\log \zeta)^{\beta-1} = \beta(\log r + i\theta)^{\beta-1} = q'(r)\left(1 + \frac{i\theta}{\log r}\right)^{\beta-1} \tag{A.8}$$

and for large r and $\theta \in (0, \pi/2)$,

$$\frac{\operatorname{Im} \zeta q'(\zeta)}{\operatorname{Im} \zeta q'(r)} \sim \frac{\beta - 1}{r \log r} \frac{\theta}{\sin \theta} \to 0 \tag{A.9}$$

and so Assumption 2.3(ii) holds.

We now turn to the asymptotic behavior of the sequences x_n^*, N_n^* and N_n^{**}. Since $q''(x_n) = -\frac{1}{n\sigma^2}$, it is clear that $x_n \to \infty$ as $n \to \infty$. The equation is

$$\frac{1}{n\sigma^2} \sim \frac{\beta (\log x_n^*)^{\beta-1}}{(x_n^*)^2}. \tag{A.10}$$

Consequently,

$$(x_n^*)^2 \sim \beta n \sigma^2 \left(\tfrac{1}{2} \log(x_n^*)^2 \right)^{\beta-1}$$
$$\sim \beta n \sigma^2 \left(\tfrac{1}{2} \log n \right)^{\beta-1} \left(1 + \frac{\log \beta \sigma^2 + (\beta-1) \log \log x_n^*}{\log n} \right)^{\beta-1}. \tag{A.11}$$

The last bracket is asymptotically equal to 1 and we get the expression for x_n^*. Next, we have from (2.10)

$$N_n^* = x_n^* + n\sigma^2 \frac{\beta (\log x_n^*)^{\beta-1}}{x_n^*} \sim 2x_n^*. \tag{A.12}$$

The last asymptotics follows from (A.10).

We now turn to N_n^{**}. It is asymptotically given by the solution of the equations

$$\frac{N_n^2}{2n\sigma^2} = q(x_n) + \frac{(N_n - x_n)^2}{2n\sigma^2}, \tag{A.13}$$

$$q'(x_n) = \frac{N_n - x_n}{n\sigma^2}. \tag{A.14}$$

Equation (A.14) is equivalent to

$$x_n^2 - N_n x_n + \beta n \sigma^2 (\log x_n)^{\beta-1} = 0. \tag{A.15}$$

The relevant solution is

$$x_n = \tfrac{1}{2} \left(N_n + \sqrt{N_n^2 - 4\beta n \sigma^2 (\log x_n)^{\beta-1}} \right) = N_n \left(1 - \frac{\beta n \sigma^2}{N_n^2} (\log x_n)^{\beta-1} (1 + o(1)) \right). \tag{A.16}$$

It follows that $N_n - x_n \sim \frac{\beta n \sigma^2}{N_n} (\log x_n)^{\beta-1}$. We insert this in (A.13); using $\log x_n \sim \log N_n$, we get

$$\frac{N_n^4}{2n\sigma^2} - N_n^2 (\log N_n)^{\beta} - \tfrac{1}{2} \beta^2 n \sigma^2 (\log N_n)^{2\beta-2} = o(1). \tag{A.17}$$

The relevant solution is

$$
\begin{aligned}
N_n^2 &\sim n\sigma^2 \Big[(\log N_n)^\beta + \sqrt{(\log N_n)^{2\beta} + \beta^2 (\log N_n)^{2\beta-2}} \Big] \\
&\sim 2n\sigma^2 (\tfrac{1}{2}\log N_n^2)^\beta \sim 2^{1-\beta} n\sigma^2 \big(\log n + \log 2\sigma^2 + \beta \log\log N_n\big)^\beta .
\end{aligned}
\tag{A.18}
$$

Only the term $\log n$ matters in the last bracket and the result follows.

The last part of the lemma on insensitivity sequence is shown in [3, Section 8.3], the proof is therefore omitted.   □

Appendix B. Bivariate Hessian

As explained in Step 4 of the proof outline, the Hessian at (t_n, ζ_n) has determinant $-1 + o(1)$ and is a saddle point of $\Phi_n(t, \zeta)$, considered as a function of two real variables $t, \zeta > 0$. In order to get rid of off-diagonal elements in the Hessian and to give all eigenvalues the same sign, we take complex ζ and reparametrize, as sketched in Step 5.

Lemma B.1 *Let $\zeta(t)$ be the unique solution of $(\partial_\zeta \Phi_n)(t, \zeta) = t - q'(\zeta) = 0$. Set*

$$
F_n : (0, \infty) \times \mathbb{R} \to \mathbb{C}, \quad F_n(t, s) = \Phi_n(t, \zeta(t) + \mathrm{i}s).
$$

Then $(\nabla F_n)(t_n, 0) = 0$,

$$
\mathrm{Hess}\, F_n(t, 0) = \begin{pmatrix} \beta(t) & 0 \\ 0 & q''(\zeta(t)) \end{pmatrix}, \quad \beta(t) = -\frac{\det(\mathrm{Hess}\, \Phi_n)(t, \zeta(t))}{q''(\zeta(t))}.
$$

Note that $\zeta_n = \zeta(t_n)$, so as $n \to \infty$

$$
\det \mathrm{Hess}\, F_n(t_n, 0) = \beta(t_n)q''(\zeta_n) = -\det(\mathrm{Hess}\, \Phi_n)(t_n, \zeta_n) = 1 + o(1). \tag{B.1}
$$

Proof We have

$$
\begin{aligned}
\partial_t F_n(t, s) &= (\partial_t \Phi_n)(t, \zeta(t) + \mathrm{i}s) + (\partial_\zeta \Phi_n)(t, \zeta(t) + \mathrm{i}s)\zeta'(t), \\
\partial_s F_n(t, s) &= \mathrm{i}\partial_\zeta \Phi_n(t, \zeta(t) + \mathrm{i}s).
\end{aligned}
\tag{B.2}
$$

At $t = t_n$, $s = 0$, we have $\zeta(t) = \zeta_n$ and $(\nabla F_n)(t_n, 0) = \nabla \Phi_n(t_n, \zeta_n) = 0$. For the Hessian, we compute

$$
\begin{aligned}
\partial_s^2 F_n(t, 0) &= -\partial_\zeta^2 \Phi_n(t, \zeta(t)) = q''(\zeta(t)), \\
\partial_t \partial_s F_n(t, 0) &= \mathrm{i}(\partial_t \partial_\zeta \Phi_n)(t, \zeta(t)) + \mathrm{i}\partial_\zeta^2 \Phi_n(t, \zeta(t)) \\
\partial_t^2 F_n(t, 0) &= \frac{\mathrm{d}^2}{\mathrm{d}t^2} \Phi_n(t, \zeta(t)) = \beta(t).
\end{aligned}
\tag{B.3}
$$

By definition of $\zeta(t)$,

$$0 = \frac{d}{dt}\partial_\zeta \Phi_n(t, \zeta(t)) = (\partial_t \partial_\zeta \Phi_n)(t, \zeta(t)) + \partial_\zeta^2 \Phi_n(t, \zeta(t))\zeta'(t). \tag{B.4}$$

It follows that $\partial_t \partial_s F_n(t, 0) = 0$, and

$$\beta(t) = \frac{d^2}{dt^2}\Phi_n(t, \zeta(t)) = \frac{d}{dt}(\partial_t \Phi_n)(t, \zeta(t))$$
$$= (\partial_t^2 \Phi_n)(t, \zeta(t)) + (\partial_t \partial_\zeta \Phi_n)(t, \zeta(t))\zeta'(t). \tag{B.5}$$

We solve for $\zeta'(t)$ in Eq. (B.4), insert into Eq. (B.5), and obtain the formula for $\beta(t)$.
□

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