

## Chapter 1

# Reflection positivity and infrared bounds for quantum spin systems

Jakob E. Björnberg and Daniel Ueltschi

**Abstract.** The method of reflection positivity and infrared bounds allows to prove the occurrence of phase transitions in systems with continuous symmetries. We review the method in the context of quantum spin systems. The novel aspect is a proof for long-range interactions that involve the Euclidean distance between sites.

## 1 Introduction

### 1.1 The quest for understanding phase transitions

The description of phase transitions puzzled physicists at the dawn of Statistical Physics. The first mathematical description goes back to 1924 with Einstein's description of Bose–Einstein condensation, but its physical relevance was not recognised then. Peierls' argument for the Ising model was published in 1936 but did not settle the question either. Legend has it that in Amsterdam in 1937, during the conference celebrating the centenary of the birth of van der Waals, a spirited debate took place regarding the validity of the Gibbs formalism, and of the thermodynamic limit. This was supported by Born and Uhlenbeck but strongly contested by Sommerfeld. Kramers, who was chairman, apparently called for the question to be settled by a vote... and the outcome was positive, narrowly!

The setting has been clarified over the years and given precise mathematical meaning. The main challenge now is to prove the occurrence of phase transitions in specific systems.

The first results dealt with systems with discrete symmetries such as the Ising model. Regarding systems with continuous symmetries, Mermin and Wagner proved a negative result in 1966, namely that the Heisenberg model does not display spontaneous magnetisation in two dimensions, at any positive temperature [36]. It took another decade for the first positive result to appear, due to Fröhlich, Simon, and Spencer; they established in 1976 that the classical Heisenberg model undergoes a phase transition in dimensions three and higher [17]. Their work was inspired by ideas from

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quantum field theory, specifically by the Källén-Lehmann representation of two-point Green functions in relativistic quantum field theory, which suggested the right form of infrared bounds, and by reflection positivity, as formulated in the works of Jost [24], Osterwalder and Schrader [38], and Glaser [21]. (Furthermore, bounds in [12, 22] inspired the exponential infrared bounds proved in [17].)

The extension of these ideas to quantum spin systems was achieved in another groundbreaking article, by Dyson, Lieb, and Simon in 1978 [9]. The method was then further extended and streamlined in [14] and [15]. Further refinements include an extension to the ground states in two dimensions [37] and improved conditions that establish long-range order in the XY model in two dimensions [25–27].

It should be pointed out that the method does not apply to models where all coupling constants are positive [41]. An important problem, that remains open to this day, is to prove spontaneous magnetisation or long-range order in the Heisenberg ferromagnet.

Another extension of the method deals with "chessboard estimates", proposed by Fröhlich and Lieb [16] (they were partly motivated by [23]). Among many interesting works that use these ideas, let us mention the flux phase problem [30, 35]; spin reflection positivity applied to Hubbard models [31, 43, 44]; itinerant electron models [32–34]; high spin systems whose classical limit has long-range order [6, 7]; spin nematic phases [4, 42]; Néel order in spin-1 model with biquadratic interactions [29]; hard-core bosons [1, 26]; loop models associated with quantum spin systems [47] (motivated by [2, 45]) and other loop models associated with classical spin systems [39]. Finally, let us mention an alternate extension of [17] to quantum systems by Albert, Ferrari, Fröhlich, and Schlein [3].

There exist a few results about phase transitions in systems with continuous symmetry that were proved with different methods, see [8, 19]. But reflection positivity and infrared bound remains the most prolific method and the only one that has been applied to quantum systems.

A beautiful account of the method of reflection positivity in statistical mechanics has been written by Biskup [5]. It is restricted to classical systems, so the present survey can be seen as a complement, dealing with the quantum counterparts. Nevertheless, we have attempted to write a self-contained pedagogical account, that encompasses many of the results on phase transitions for quantum spin systems.

The handwritten notes of Tóth for his Prague lectures give a clear account of the method [46]. And an extensive overview, that retraces the origin of the key ideas, can be found in the handwritten notes of Fröhlich for his Vienna lectures [13].

## 1.2 Organisation of the survey

The setting for quantum spin systems is introduced in Section 2. We recall the existence of the infinite-volume limit of the free energy (Theorem 2.1), and discuss non-differentiability of the free energy and how it relates to long-range order. Subsequently:

- The existence of long-range order is stated in Section 3. For this we consider the case of positive temperature in dimensions 3 and higher as well as the ground state of two-dimensional systems.
- The case of nearest-neighbour interactions is covered in Theorem 3.2. As discussed after the theorem, long-range order has been proved for all  $d \geq 2$  and all  $S \in \frac{1}{2}\mathbb{N}$ , except for the case  $d = 2$  and  $S = \frac{1}{2}$  where it is restricted to models close to XY.
- The proof of long-range order in turn relies on an "infrared bound" on correlations which we state in Section 4. This involves transferring the bound from the Duhamel correlation function to the normal correlation function.
- We establish reflection positivity of our models in Section 5 and use it to prove the infrared bound on the Duhamel correlation function.
- The appendix contains correlation inequalities for quantum spin systems that we need in various places.

Almost all the material of this survey is standard. Our novel contribution is a proof for long-range interactions that decay with the Euclidean distance between sites, see Lemma 5.2. This was known in the classical case (Georgii explains that it fits the general setting of [14], see Example 17.32 in [20]) but not in the quantum case.

## 2 Setting and results

The domain of the system is a finite subset  $\Lambda \subset \mathbb{Z}^d$ . The state space is the Hilbert space  $\mathcal{H}_\Lambda = \otimes_{x \in \Lambda} \mathbb{C}^n$  where  $n = 2S + 1 = 2, 3, 4, \dots$ . Equivalently,  $\mathcal{H}_\Lambda$  can be defined as the linear span of the space of classical configurations,  $\mathcal{H}_\Lambda = \text{span}(\Sigma^\Lambda)$ , where  $\Sigma = \{-S, -S + 1, \dots, S\}$ . We consider the spin operators  $(S_x^{(i)})_{x \in \Lambda}^{i=1,2,3}$  that satisfy the relations:

$$\begin{aligned} [S_x^{(1)}, S_y^{(2)}] &= iS_x^{(3)}\delta_{x,y}, & [S_x^{(2)}, S_y^{(3)}] &= iS_x^{(1)}\delta_{x,y}, & [S_x^{(3)}, S_y^{(1)}] &= iS_x^{(2)}\delta_{x,y}, \\ (S_x^{(1)})^2 + (S_x^{(2)})^2 + (S_x^{(3)})^2 &= S(S+1)\text{Id}, \end{aligned} \tag{2.1}$$

for all  $x, y \in \Lambda$ . It can be shown that all these operators have eigenvalues  $\{-S, \dots, S\}$ . For  $S = \frac{1}{2}$ , these operators are given by (half) the Pauli matrices in the basis where

$\{S_x^{(3)}\}$  are diagonal, namely

$$\begin{aligned} S_x^{(1)} &= \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \text{Id}_{\Lambda \setminus \{x\}}, \\ S_x^{(2)} &= \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \text{Id}_{\Lambda \setminus \{x\}}, \\ S_x^{(3)} &= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \text{Id}_{\Lambda \setminus \{x\}}. \end{aligned} \quad (2.2)$$

These expressions are adapted to the tensor product structure  $\mathcal{H}_\Lambda = \otimes_{x \in \Lambda} \mathbb{C}^n$ . Using instead the interpretation  $\mathcal{H}_\Lambda = \text{span}(\Sigma^\Lambda)$ , we can define the spin operators as follows. Given  $\sigma_\Lambda \in \Sigma^\Lambda$ , let  $\sigma_\Lambda^{(x)}$  denote the configuration that is equal to  $\sigma_\Lambda$ , except at  $x$  where the value has been flipped; then

$$\begin{aligned} S_x^{(1)} |\sigma_\Lambda\rangle &= |\sigma_\Lambda^{(x)}\rangle, \\ S_x^{(2)} |\sigma_\Lambda\rangle &= i\sigma_x |\sigma_\Lambda^{(x)}\rangle, \\ S_x^{(3)} |\sigma_\Lambda\rangle &= \sigma_x |\sigma_\Lambda\rangle. \end{aligned} \quad (2.3)$$

The Hamiltonian of the system is the operator

$$H_{\Lambda,h} = - \sum_{i=1}^3 \sum_{x,y \in \Lambda} J_{x-y}^{(i)} S_x^{(i)} S_y^{(i)} - h \sum_{x \in \Lambda} S_x^{(3)} \quad (2.4)$$

and the partition function is

$$Z(\Lambda, \beta, h) = \text{Tr} e^{-\beta H_{\Lambda,h}}. \quad (2.5)$$

The functions  $J_{x-y}^{(i)}$  are called coupling parameters or coupling constants. We always assume that they are symmetric,  $J_x^{(i)} = J_{-x}^{(i)}$ , for all  $i \in \{1, 2, 3\}$  and  $x \in \mathbb{Z}^d$ .

We define the finite-volume free energy by

$$f_\Lambda(\beta, h) = - \frac{1}{\beta |\Lambda|} \log Z(\Lambda, \beta, h), \quad f_\Lambda(\infty, h) = \lim_{\beta \rightarrow \infty} f_\Lambda(\beta, h). \quad (2.6)$$

The limit  $\beta \rightarrow \infty$  exists and it corresponds to a trace in the eigenspace for  $H_{\Lambda,h}$  with lowest eigenvalue (the ground-state energy). As is well-known, we can take the limit of large volumes and we obtain a thermodynamic potential. This is a pillar of Statistical Mechanics. For this, we recall the notion of "van Hove sequences". We say that the sequence of finite domains  $(\Lambda_n)$  tends to  $\mathbb{Z}^d$  in the sense of van Hove, written  $\Lambda_n \uparrow \mathbb{Z}^d$ , if

- (i) it is increasing, i.e.  $\Lambda_n \subset \Lambda_{n+1}$ ;
- (ii) it invades  $\mathbb{Z}^d$ , i.e.  $\cup_n \Lambda_n = \mathbb{Z}^d$ ;
- (iii) its ratio boundary/volume tends to zero, i.e.  $\frac{|\{x \in \Lambda_n : \text{dist}(x, \Lambda_n^c) = 1\}|}{|\Lambda_n|} \rightarrow 0$ .

**Theorem 2.1.** *Assume that*

$$\sum_{x \in \mathbb{Z}^d} |J_x^{(i)}| < \infty$$

for all  $i \in \{1, 2, 3\}$ . Then there exists a function  $f(\beta, h)$  where  $0 \leq \beta \leq \infty$  and  $h \in \mathbb{R}$ , such that for any van Hove sequence  $(\Lambda_n)$ , we have

$$f(\beta, h) = \lim_{n \rightarrow \infty} f_{\Lambda_n}(\beta, h).$$

Convergence is locally uniform. Further, the function  $f(\beta, h)$  is concave and even in  $h$ .

We refer to [11, 40] for the proof of this theorem. For finite  $\beta$ , the function  $\beta f(\beta, h)$  is jointly concave in  $(\beta, \beta h)$  but here we only use the concavity in  $h$ . It is also well-known that  $f(\beta, h)$  is smooth when  $\beta$  is small — it is actually analytic.

In what follows we need to consider periodic boundary conditions. Given  $\ell \in \mathbb{N}$ , let  $\Lambda_\ell = \{0, 1, \dots, \ell - 1\}^d$ , and let the Hamiltonian  $H_{\Lambda_\ell, h}^{\text{per}}$  be given by Eq. (2.4), but with coupling parameters replaced by the following periodised ones:

$$J_{x, \text{per}}^{(i)} = \sum_{z \in \mathbb{Z}^d} J_{x+\ell z}^{(i)}. \quad (2.7)$$

We can define the periodised partition function  $Z^{\text{per}}(\Lambda_\ell, \beta, h)$  accordingly, and the free energy

$$f_{\Lambda_\ell}^{\text{per}}(\beta, h) = -\frac{1}{\beta \ell^d} \log Z^{\text{per}}(\Lambda_\ell, \beta, h). \quad (2.8)$$

As  $\ell \rightarrow \infty$ , these free energies converge to the free energies  $f(\beta, h)$  of Theorem 2.1.

We introduce the finite-volume equilibrium states

$$\langle \cdot \rangle_{\Lambda, \beta, h} = \frac{\text{Tr} [\cdot e^{-\beta H_{\Lambda, h}}]}{Z_{\Lambda, \beta, h}}, \quad \langle \cdot \rangle_{\Lambda, \infty, h} = \lim_{\beta \rightarrow \infty} \langle \cdot \rangle_{\Lambda, \beta, h}. \quad (2.9)$$

We also consider the states  $\langle \cdot \rangle_{\Lambda_\ell, \beta, h}^{\text{per}}$  with periodic boundary conditions, where we use  $H_{\Lambda_\ell, h}^{\text{per}}$  instead of  $H_{\Lambda_\ell, h}$ . In Section 3 we make several assumptions on the coupling constants and we show that the system then exhibits *long-range order* at low temperature, in the sense that there exists a lower bound  $c$  such that

$$\frac{1}{|\Lambda|^2} \sum_{x, y \in \Lambda} \langle S_x^{(3)} S_y^{(3)} \rangle_{\Lambda, \beta, 0} \geq c > 0, \quad (2.10)$$

where  $c$  is positive uniformly in the volume  $\Lambda$ . As for the domains  $\Lambda$ , the statement is relevant if it holds for all domains in a van Hove sequence.

In order to motivate the importance of this property, we show that it implies the occurrence of a first-order phase transition as  $h$  crosses 0. This also implies that

there exist many distinct Gibbs states at  $(\beta, 0)$ . Unfortunately, the following theorem involves conditions on the coupling constants that are *orthogonal* to the conditions under which long-range order is proved. We do not know of a proof of this theorem in the case where  $J_x^{(1)}$  and  $J_x^{(2)}$  have different signs, which is the situation of Section 3.

**Theorem 2.2.** *We assume that the system displays long-range order in the form of Eq. (2.10). We also assume that the coupling constants satisfy  $J_x^{(1)} = J_x^{(2)}$  for all  $x \in \mathbb{Z}^d$ . Then*

$$\frac{\partial}{\partial h} f(\beta, h) \Big|_{h=0-} > 0 > \frac{\partial}{\partial h} f(\beta, h) \Big|_{h=0+}.$$

Notice that we did not assume anything on  $J_x^{(3)}$ ; but these interactions are presumably larger than  $J_x^{(1)}, J_x^{(2)}$ , in order that long-range order in the 3rd direction of spin be possible.

*Proof.* Let us introduce the magnetisation operator

$$M_\Lambda = \sum_{x \in \Lambda} S_x^{(3)}. \quad (2.11)$$

Let  $|M_\Lambda|$  be the unique positive semi-definite square root of  $M_\Lambda^2$ . We have  $|M_\Lambda| \leq |\Lambda|S \text{Id}$ , so that  $M_\Lambda^2 \leq |\Lambda|S|M_\Lambda|$ . Since Gibbs states are positive linear functionals, we get

$$\left\langle \frac{M_\Lambda^2}{|\Lambda|^2} \right\rangle_{\Lambda, \beta, 0} \leq S \left\langle \frac{|M_\Lambda|}{|\Lambda|} \right\rangle_{\Lambda, \beta, 0}. \quad (2.12)$$

Long-range order implies that  $\frac{1}{|\Lambda|^2} \langle M_\Lambda^2 \rangle_{\Lambda, \beta, 0} \geq c$ , so the right side above is positive.

In order to get an inequality for the derivative of the free energy, let us introduce  $\tilde{f}_\Lambda(\beta, h)$  to be the free energy of the model with Hamiltonian

$$\tilde{H}_{\Lambda, h} = - \sum_{i=1}^3 \sum_{x, y \in \Lambda} J_{x-y}^{(i)} S_x^{(i)} S_y^{(i)} - h|M_\Lambda|. \quad (2.13)$$

We now check that  $\tilde{f}_\Lambda(\beta, h)$  converges as  $\Lambda \uparrow \mathbb{Z}^d$  to the free energy  $f(\beta, h)$  for  $h \geq 0$ . For this, notice that  $M_\Lambda$  (and  $|M_\Lambda|$ ) commute with  $H_{\Lambda, 0} = \tilde{H}_{\Lambda, 0}$  (here we use the assumption  $J_x^{(1)} = J_x^{(2)}$ ). For  $h \geq 0$  we have the inequalities (for the second one, observe that the spectrum of  $M_\Lambda$  is symmetric around 0)

$$\text{Tr} e^{-\beta H_{\Lambda, \beta, 0} + \beta h M_\Lambda} \leq \text{Tr} e^{-\beta H_{\Lambda, \beta, 0} + \beta h |M_\Lambda|} \leq 2 \text{Tr} e^{-\beta H_{\Lambda, \beta, 0} + \beta h M_\Lambda}. \quad (2.14)$$

Taking the logarithm and dividing by  $\beta|\Lambda|$ , and taking the relevant limits, we get that  $f$  and  $\tilde{f}$  are equal.

We now use the concavity in  $h$  of  $\tilde{f}_\Lambda$  and the fact that  $\sup_n (\liminf_m a_{m,n}) \leq \liminf_m (\sup_n a_{m,n})$  and we get

$$\begin{aligned} \left. \frac{\partial}{\partial h} f(\beta, h) \right|_{h=0+} &= \sup_{h>0} \frac{f(\beta, h) - f(\beta, 0)}{h} = \sup_{h>0} \liminf_{\Lambda \uparrow \mathbb{Z}^d} \frac{\tilde{f}_\Lambda(\beta, h) - \tilde{f}_\Lambda(\beta, 0)}{h} \\ &\leq \liminf_{\Lambda \uparrow \mathbb{Z}^d} \sup_{h>0} \frac{\tilde{f}_\Lambda(\beta, h) - \tilde{f}_\Lambda(\beta, 0)}{h} = \liminf_{\Lambda \uparrow \mathbb{Z}^d} \left. \frac{\partial}{\partial h} \tilde{f}_\Lambda(\beta, h) \right|_{h=0} \\ &= \liminf_{\Lambda \uparrow \mathbb{Z}^d} \left\langle -\frac{|M_\Lambda|}{|\Lambda|} \right\rangle_{\Lambda, \beta, 0}. \end{aligned} \quad (2.15)$$

The last expectation is with respect to the Gibbs state with Hamiltonian  $\tilde{H}_{\Lambda,0} = H_{\Lambda,0}$ . The right side is positive and  $\left. \frac{\partial}{\partial h} f(\beta, h) \right|_{h=0+}$  is indeed negative. Since  $f$  is even in  $h$  we get the other inequality as well.  $\blacksquare$

We now discuss spin rotations. They show that Hamiltonians with different couplings are related by a unitary transformation, which allows to make assumptions on the couplings without loss of generality. The following lemma applies to spin operators in  $\mathbb{C}^{2S+1}$ , and immediately extends to tensor products. Given  $\vec{a} \in \mathbb{R}^3$ , let

$$S^{(\vec{a})} = \vec{a} \cdot \vec{S} = a_1 S^{(1)} + a_2 S^{(2)} + a_3 S^{(3)}. \quad (2.16)$$

By linearity, the commutation relations (2.1) generalise as

$$[S^{(\vec{a})}, S^{(\vec{b})}] = iS^{(\vec{a} \times \vec{b})}. \quad (2.17)$$

Finally, let  $R_{\vec{a}} \vec{b}$  denote the vector  $\vec{b}$  rotated around  $\vec{a}$  by the angle  $\|\vec{a}\|$ .

**Lemma 2.3.**

$$e^{-iS^{(\vec{a})}} S^{(\vec{b})} e^{iS^{(\vec{a})}} = S^{(R_{\vec{a}} \vec{b})}.$$

*Proof.* We replace  $\vec{a}$  by  $s\vec{a}$ , and we check that both sides of the identity satisfy the same differential equation. We find

$$\frac{d}{ds} e^{-iS^{(s\vec{a})}} S^{(\vec{b})} e^{iS^{(s\vec{a})}} = -i[S^{(s\vec{a})}, e^{-iS^{(s\vec{a})}} S^{(\vec{b})} e^{iS^{(s\vec{a})}}], \quad (2.18)$$

and

$$\frac{d}{ds} S^{(R_{s\vec{a}} \vec{b})} = \left( \frac{d}{ds} R_{s\vec{a}} \vec{b} \right) \cdot \vec{S} = \left( \vec{a} \times R_{s\vec{a}} \vec{b} \right) \cdot \vec{S} = -i[S^{(s\vec{a})}, S^{(R_{s\vec{a}} \vec{b})}]. \quad (2.19)$$

We used (2.17) for the last identity.  $\blacksquare$

We obtain the following consequence. Here  $|z|$  denotes the sum of the coordinates of  $z \in \mathbb{Z}^d$ .

**Proposition 2.4.** *Let  $\rho(1), \rho(2), \rho(3)$  be a permutation of 1, 2, 3 and let  $\sigma_1, \sigma_2, \sigma_3 \in \{-1, +1\}$  satisfy  $\sigma_1\sigma_2\sigma_3 = +1$ . Let  $H_{\Lambda, h}$  be as in (2.4) and let*

$$\tilde{H}_{\Lambda, h} = - \sum_{i=1}^3 \sum_{x, y \in \Lambda} \tilde{J}_{x-y}^{(i)} S_x^{(i)} S_y^{(i)} - \sum_{x \in \Lambda} \tilde{h}_x S_x^{(\rho^{-1}(3))} \quad (2.20)$$

where  $\tilde{J}_x^{(i)} = (\sigma_{\rho(i)})^{|x|} J_x^{(\rho(i))}$  and  $\tilde{h}_x = (\sigma_{\rho^{-1}(3)})^{|x|} h$ . Then for any finite  $\Lambda \subset \mathbb{Z}^d$

$$\mathrm{Tr} e^{-\beta H_{\Lambda, h}} = \mathrm{Tr} e^{-\beta \tilde{H}_{\Lambda, h}}. \quad (2.21)$$

Using Proposition 2.4 one may transfer results on non-differentiability of the free energy for one set of coupling parameters to another.

*Proof.* For any unitary matrix  $U$  acting  $\mathcal{H}_{\Lambda}$ , we have

$$\mathrm{Tr} e^{-\beta H_{\Lambda, h}} = \mathrm{Tr} [U^* e^{-\beta H_{\Lambda, h}} U] = \mathrm{Tr} [e^{-\beta U^* H_{\Lambda, h} U}]. \quad (2.22)$$

Consider  $U$  of the form

$$U = \bigotimes_{\substack{x \in \Lambda \\ |x| \text{ odd}}} W_x \bigotimes_{x \in \Lambda} V_x. \quad (2.23)$$

By combining rotations by an angle  $\frac{\pi}{2}$ , we can choose  $V_x$  such that  $V_x S_x^{(\rho^{-1}(3))} V_x^* = S_x^{(3)}$  while  $V_x S_x^{(\rho^{-1}(i))} V_x^* = \pm S_x^{(i)}$  for  $i = 1, 2$ , where the  $\pm$  does not depend on  $x$ . By combining rotations by an angle  $\pi$ , we can choose  $W_x$  such that  $W_x S_x^{(i)} W_x^* = \sigma_i S_x^{(i)}$ . Then  $U^* H_{\Lambda, h} U = \tilde{H}_{\Lambda, h}$ . ■

### 3 Long-range order

We state two results about long-range order. The first theorem holds for a larger class of coupling constants and for  $S$  large enough. The second theorem is restricted to nearest-neighbour interactions, but it has the advantage to apply to more values of  $S$  and more dimensions. To briefly summarise the consequences of those results, we will see that long-range order (in the form (2.10)) holds under the following conditions:

- for certain long-range interactions (specified below) if  $\beta \geq \beta_0$  for some  $\beta_0 < \infty$  provided  $d \geq 3$  and  $S$  is large enough, or
- for nearest-neighbour interactions if  $\beta \geq \beta_0$  for some  $\beta_0 < \infty$  provided  $d \geq 3$  and  $S \geq \frac{1}{2}$ , or
- for nearest-neighbour interactions in the ground-state  $\beta = \infty$  provided  $d \geq 2$  and either  $S \geq 1$ , or  $S \geq \frac{1}{2}$  and  $-J^{(2)}/J^{(1)} \leq 0.13$ .



The original results dealt with nearest-neighbour interactions [9, 17]. Then Fröhlich, Israel, Lieb, and Simon formulated a more abstract notion of reflection positive interactions [14]. Here we opt to list a series of examples that fit the setting; we provide explicit proofs in each case, see Lemma 5.2. Explicitly, we consider interactions of the following forms:

- Nearest neighbour:  $J_x^{(i)} = 0$  unless  $\|x\|_1 = 1$ , in which case it equals some constant  $J^{(i)}$ ;
- Yukawa:  $J_x^{(i)} = a^{(i)} e^{-b^{(i)}\|x\|_1}$  for constants  $a^{(i)} \in \mathbb{R}$ ,  $b^{(i)} > 0$ ;
- power-law:  $J_x^{(i)} = a^{(i)}\|x\|_1^{-s}$  with  $a^{(i)} \in \mathbb{R}$  and  $s > d$ ;
- interactions given by nearest-neighbour random walk:  $J_x^{(i)} = c^{(i)} \sum_{w:0 \rightarrow x} (\frac{\lambda^{(i)}}{2d})^{|w|}$  where  $c^{(i)} \in \mathbb{R}$  and  $\lambda^{(i)} \in [0, 1)$  and the sum is over nearest-neighbour walks in  $\mathbb{Z}^d$  from 0 to  $x$ ;
- the random walks above can be defined on a lattice with small mesh  $a$ ; as  $a \rightarrow 0$  we get long-range interactions that involve the Euclidean distance between sites such as  $J_x^{(i)} = a^{(i)}\|x\|_2^{-s}$  with  $a^{(i)} \in \mathbb{R}$  and  $s > d$ ;
- convex combinations of the above.

Let  $\Lambda_\ell^*$  denote the dual of  $\Lambda_\ell$  in Fourier theory, namely

$$\Lambda_\ell^* = \frac{2\pi}{\ell} \left\{ -\frac{\ell}{2} + 1, \dots, \frac{\ell}{2} \right\}^d \subset [-\pi, \pi]^d. \quad (3.1)$$

**Theorem 3.1.** *Assume that  $J^{(i)}(x)$  is one of the interactions above; we assume in addition that  $\ell$  is even and that*

$$J_x^{(3)} \geq J_x^{(1)} \geq -J_x^{(2)} \geq 0, \quad \text{for all } x \in \mathbb{Z}^d.$$

Then

$$\frac{1}{\ell^d} \sum_{x \in \Lambda_\ell} \langle S_0^{(3)} S_x^{(3)} \rangle_{\Lambda_\ell, \beta, 0}^{per} \geq \frac{1}{3} S(S+1) - \frac{1}{\ell^d} \sum_{k \in \Lambda_\ell^* \setminus \{0\}} \sqrt{\frac{e(k)}{2\varepsilon(k)}} - \frac{1}{2\beta\ell^d} \sum_{k \in \Lambda_\ell^* \setminus \{0\}} \frac{1}{\varepsilon(k)}. \quad (3.2)$$

Here we defined

$$\varepsilon(k) = \sum_{x \in \mathbb{Z}^d} J_{x, \text{per}}^{(3)} (1 - \cos kx) \quad (3.3)$$

while the function  $e(k)$  is defined in (4.26). Notice that  $\varepsilon(k)$  is bounded and that  $\varepsilon(k) \sim k^2$  around  $k = 0$ ; it is positive for  $k \neq 0$ . It is worth pointing out that  $e(k) \leq \text{const } S^2$  around  $k = 0$ . Therefore the right-hand-side of (3.2) is necessarily positive when  $d \geq 3$  and  $S, \beta$  are large enough.

We now assume that  $J^{(i)}$  are nearest-neighbour couplings, that is,

$$J_x^{(i)} = \begin{cases} J^{(i)} & \text{if } \|x\| = 1, \\ 0 & \text{otherwise.} \end{cases} \quad (3.4)$$

We further normalise them so that  $J^{(3)} = 1$ . In this case we derive sharper lower bounds for long-range order. Let us introduce the following two sums:

$$\begin{aligned} I_\ell^{(d)} &= \frac{1}{\ell^d} \sum_{k \in \Lambda_\ell^* \setminus \{0\}} \sqrt{\frac{\varepsilon(k + \pi)}{\varepsilon(k)}}, \\ \tilde{I}_\ell^{(d)} &= \frac{1}{\ell^d} \sum_{k \in \Lambda_\ell^* \setminus \{0\}} \sqrt{\frac{\varepsilon(k + \pi)}{\varepsilon(k)}} \left( \frac{1}{d} \sum_{i=1}^d \cos k_i \right)_+. \end{aligned} \quad (3.5)$$

Here,  $\varepsilon(k) = 2 \sum_{i=1}^d (1 - \cos k_i)$  and  $\varepsilon(k + \pi) = 2 \sum_{i=1}^d (1 + \cos k_i)$ , and  $(\cdot)_+$  denotes the positive part. Their infinite volume limits converge to the integrals

$$\begin{aligned} I^{(d)} &= \lim_{\ell \rightarrow \infty} I_\ell^{(d)} = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \sqrt{\frac{\varepsilon(k + \pi)}{\varepsilon(k)}} dk, \\ \tilde{I}^{(d)} &= \lim_{\ell \rightarrow \infty} \tilde{I}_\ell^{(d)} = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \sqrt{\frac{\varepsilon(k + \pi)}{\varepsilon(k)}} \left( \frac{1}{d} \sum_{i=1}^d \cos k_i \right)_+ dk. \end{aligned} \quad (3.6)$$

One can check that, as  $d \rightarrow \infty$ , these integrals satisfy  $I^{(d)} \rightarrow 1$  [9] and  $\tilde{I}^{(d)} \rightarrow 1$  [26]. We also introduce the expression

$$\alpha_\ell(\beta) = J^{(1)} \langle S_0^{(1)} S_{e_1}^{(1)} \rangle_{\Lambda_\ell, \beta, 0} + J^{(2)} \langle S_0^{(2)} S_{e_1}^{(2)} \rangle_{\Lambda_\ell, \beta, 0} \quad (3.7)$$

and  $\alpha(\beta) = \liminf_{\ell \rightarrow \infty} \alpha_\ell(\beta)$ . We also denote  $\alpha_\ell(\infty)$  the  $\beta \rightarrow \infty$  limit.

**Theorem 3.2.** *Assume that  $\ell$  is even and that the nearest-neighbour coupling constants satisfy*

$$J^{(3)} = 1 \geq J^{(1)} \geq -J^{(2)} \geq 0.$$

*Then we have the two lower bounds:*

$$\begin{aligned} \frac{1}{\ell^d} \sum_{x \in \Lambda_\ell} \langle S_0^{(3)} S_x^{(3)} \rangle_{\Lambda_\ell, \beta, 0}^{per} &\geq \\ &\left\{ \begin{aligned} &\frac{1}{3} S(S+1) - \frac{1}{2} (I_\ell^{(d)} + \frac{\sqrt{2}}{\ell^d}) \sqrt{\alpha_\ell(\beta)} - \frac{1}{2\beta\ell^d} \sum_{k \in \Lambda_\ell^* \setminus \{0\}} \frac{1}{\varepsilon(k)}, \\ &\sqrt{\alpha_\ell(\beta)} \left[ \frac{\sqrt{\alpha_\ell(\beta)}}{1 - J^{(2)}/J^{(1)}} - \frac{1}{2} \tilde{I}_\ell^{(d)} \right] - \frac{1}{2\beta\ell^d} \sum_{k \in \Lambda_\ell^* \setminus \{0\}} \frac{1}{\varepsilon(k)} \left( \frac{1}{d} \sum_{i=1}^d \cos k_i \right)_+. \end{aligned} \right. \end{aligned}$$

The theorem is proved at the end of Section 4.

We want to formulate sufficient conditions under which at least one of the lower bounds is positive, uniformly in  $\ell$ . The terms involving  $1/\beta$  converge as  $\ell \rightarrow \infty$  if  $d \geq 3$  and they can be made arbitrarily small by taking  $\beta$  sufficiently large. For  $d = 2$  the bounds are useful in the ground state, i.e. when the limit  $\beta \rightarrow \infty$  is taken before  $\ell \rightarrow \infty$ .

We get a uniform lower bound if either  $\frac{1}{3}S(S+1) > \frac{1}{2}I^{(d)}\sqrt{\alpha(\beta)}$ , or  $\frac{\sqrt{\alpha(\beta)}}{1-J^{(2)}/J^{(1)}} > \frac{1}{2}\tilde{I}^{(d)}$ . Irrespective of the value of  $\alpha(\beta)$ , at least one of the lower bound is positive if

$$\frac{\frac{1}{3}S(S+1)}{\frac{1}{2}I^{(d)}} > \frac{1}{2}\tilde{I}^{(d)}(1 - J^{(2)}/J^{(1)}) \iff 1 - J^{(2)}/J^{(1)} < \frac{\frac{4}{3}S(S+1)}{I^{(d)}\tilde{I}^{(d)}}. \quad (3.8)$$

Values of  $I^{(d)}$  and  $\tilde{I}^{(d)}$  can be found numerically; they are listed in Table 1 for  $d = 2, 3, 4$ . This allows to verify that the condition (3.8) holds for all values of  $J^{(1)}, J^{(2)}$  such that  $J^{(1)} \geq -J^{(2)} \geq 0$ , all dimensions  $d \geq 2$ , and all spin values  $S \in \frac{1}{2}\mathbb{N}$ , with the *one exception* of the case  $d = 2$  and  $S = \frac{1}{2}$ . In this case, (3.8) holds when  $-J^{(2)}/J^{(1)} \in [0, 0.109]$ .

Kubo and Kishi [27] improved the interval to  $[0, 0.13]$  and this is the current best result. To do this, they use the variational principle with the constant state  $\otimes_{x \in \Lambda_\ell} |\frac{1}{2}\rangle$  to get a bound on the ground state energy. Combined with the correlation inequalities stated in Lemma A.1, they get a lower bound for  $\alpha(\infty) = \lim_{\beta \rightarrow \infty} \alpha(\beta)$ , namely

$$\alpha(\infty) \geq \frac{1/4}{2 - J^{(2)}/J^{(1)}}. \quad (3.9)$$

(In [27] they consider the case  $J^{(1)} = J^{(3)} = 1$  but it is easily extended.) This implies that the second bound of Theorem 3.2 is positive in the interval  $[0, 0.13]$ .

$d$	$I^{(d)}$	$\tilde{I}^{(d)}$
2	1.393	0.6468
3	1.157	0.3499
4	1.094	0.2540

**Table 1.** Numerical values of the integrals  $I^{(d)}$  and  $\tilde{I}^{(d)}$  defined in (3.6).

## 4 Infrared bounds

This section explores estimates of the Fourier transform of correlations and their consequences. Such estimates are particularly relevant at small Fourier parameters; this corresponds to large wavelengths, i.e. the infrared spectrum for light, hence the name given by physicists.

We need to introduce the conventions about the Fourier transform used in this survey. Recall that  $\Lambda_\ell^* = \frac{2\pi}{\ell} \{-\frac{\ell}{2} + 1, \dots, \frac{\ell}{2}\}^d$ . The Fourier transform of a function

$f : \Lambda_\ell \rightarrow \mathbb{C}$  is

$$\widehat{f}(k) = \sum_{x \in \Lambda_\ell} e^{-ikx} f(x), \quad k \in \Lambda_\ell^*, \quad (4.1)$$

where we write  $kx$  for the usual inner product  $\sum_{i=1}^d k_i x_i$ . One can check that the inverse relation is then

$$f(x) = \frac{1}{\ell^d} \sum_{k \in \Lambda_\ell^*} e^{ikx} \widehat{f}(k). \quad (4.2)$$

Note that  $\varepsilon(k) = \widehat{J}^{(3)}(0) - \widehat{J}^{(3)}(k)$ .

The first infrared bound involves the Duhamel correlation function  $\eta(x)$ , defined by

$$\eta(x) = \frac{1}{\beta} \frac{1}{Z_{\text{per}}(\Lambda_\ell, \beta, h)} \int_0^\beta ds \text{Tr} S_0^{(3)} e^{-sH_{\Lambda, h}^{\text{per}}} S_x^{(3)} e^{-(\beta-s)H_{\Lambda, h}^{\text{per}}}. \quad (4.3)$$

The method of reflection positivity allows to establish the following infrared bound.

**Lemma 4.1.** *Let  $h = 0$  and  $\ell$  be even. Assume that the coupling constants  $J^{(*)}$  satisfy the assumptions of Theorem 3.1. Then*

$$\widehat{\eta}(k) \leq \frac{1}{2\beta\varepsilon(k)}, \quad \text{for all } k \in \Lambda_\ell^* \setminus \{0\}.$$

The proof of this lemma can be found at the end of Section 5.

#### 4.1 Falk–Bruch inequality

We cannot use directly the infrared bound on the Duhamel function because of a lack of suitable lower bound for  $\eta(0)$ . The way out is to derive another bound on the ordinary correlation function. This can be done using the Falk–Bruch inequality, that was proposed independently in [10] and [9].

Let  $\mathcal{H}$  be a separable Hilbert space,  $H$  a bounded hermitian operator such that  $\text{Tr} e^{-H} < \infty$ , and let  $\mathcal{B}$  denote the space of bounded operators on  $\mathcal{H}$ . We define the *Duhamel inner product* by

$$(A, B) = \frac{1}{Z} \int_0^1 ds \text{Tr} e^{-(1-s)H} A^* e^{-sH} B, \quad A, B \in \mathcal{B}, \quad (4.4)$$

with  $Z = \text{Tr} e^{-H}$ . We have

$$\begin{aligned} \frac{d}{ds} \text{Tr} e^{-(1-s)H} A^* e^{-sH} B &= \text{Tr} e^{-(1-s)H} [H, A^*] e^{-sH} B \\ &= \text{Tr} e^{-(1-s)H} A^* e^{-sH} [B, H], \end{aligned} \quad (4.5)$$

and we obtain the useful identity

$$([A, H], B) = (A, [B, H]). \quad (4.6)$$

Further,

$$(A, [B, H]) = \frac{1}{Z} \int_0^1 ds \frac{d}{ds} \text{Tr} e^{-(1-s)H} A^* e^{-sH} B = \langle [B, A^*] \rangle \quad (4.7)$$

where

$$\langle \cdot \rangle = \frac{1}{Z} \text{Tr} \cdot e^{-H}. \quad (4.8)$$

For a given  $A \in \mathcal{B}$ , let us introduce the function  $F(s) = \text{Tr} e^{-(1-s)H} A^* e^{-sH} A$ . We have

$$\frac{d^2}{ds^2} F(s) = \text{Tr} e^{-(1-s)H} [A, H]^* e^{-sH} [A, H] \geq 0 \quad (4.9)$$

(positivity can be shown by casting the right side in the form  $\text{Tr} B^* B$ ). The function  $F(s)$  is therefore convex. Then

$$\frac{1}{2} \langle A^* A + A A^* \rangle = \frac{1}{2Z} (F(0) + F(1)) \geq \frac{1}{Z} \int_0^1 F(s) ds = (A, A) \quad (4.10)$$

with equality if and only if  $[A, H] = 0$ . The Cauchy–Schwarz inequality of the Duhamel inner product (4.4) gives

$$|(A, [B, H])|^2 \leq (A, A) ([B, H], [B, H]). \quad (4.11)$$

Using Eq. (4.7) to write the Duhamel inner product of commutators as expectations in the state  $\langle \cdot \rangle$ , and the inequalities (4.10) and (4.11) as well as cyclicity of the trace, we get *Bogolubov's inequality*

$$|\langle [B, A^*] \rangle|^2 \leq \frac{1}{2} \langle A^* A + A A^* \rangle \langle [[B, H], B^*] \rangle. \quad (4.12)$$

Inequality (4.10) gives an upper bound for the Duhamel inner product, but we actually need a lower bound. For this, we consider the function

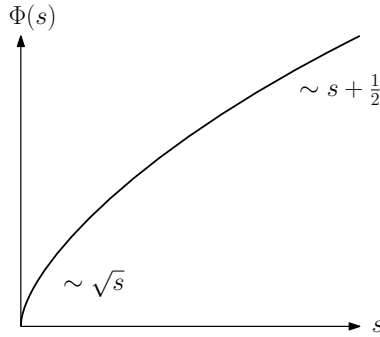
$$\Phi(s) = \sqrt{s} \coth \frac{1}{\sqrt{s}}. \quad (4.13)$$

This function is increasing, concave, and is depicted in Fig. 1. One can check that

$$\sqrt{s} \leq \Phi(s) \leq \sqrt{s} + s. \quad (4.14)$$

**Lemma 4.2** (Falk–Bruch inequality). *For all  $A \in \mathcal{B}$  such that the denominators differ from zero, we have*

$$\frac{2 \langle A^* A + A A^* \rangle}{\langle [A^*, [H, A]] \rangle} \leq \Phi \left( \frac{4(A, A)}{\langle [A^*, [H, A]] \rangle} \right).$$



**Figure 1.** The function  $\Phi$  of the Falk–Bruch inequality.

It is worth noting that the double commutator is nonnegative, as can be seen from Eq. (4.7). Indeed, taking  $A \mapsto [A^*, H]$  and  $B \mapsto A^*$ , we can express it using the Duhamel inner product as

$$\langle [A^*, [H, A]] \rangle = ([A^*, H], [A^*, H]) \geq 0. \quad (4.15)$$

*Proof of Lemma 4.2.* Recall the function  $F(s)$  defined before (4.9). The Falk–Bruch inequality can be written as

$$2 \frac{F(0) + F(1)}{F'(1) - F'(0)} \leq \Phi \left( \frac{4 \int_0^1 F(s) ds}{F'(1) - F'(0)} \right). \quad (4.16)$$

If  $\{\varphi_j\}$  is an orthonormal set of eigenvectors of  $H$  with eigenvalues  $\lambda_j$ , we can write

$$F(s) = \sum_{i,j} |(\varphi_i, A\varphi_j)|^2 e^{-\lambda_j} e^{(\lambda_j - \lambda_i)s} = \int_{-\infty}^{\infty} e^{st} d\mu(t), \quad (4.17)$$

where  $\mu$  is a positive measure. We have

$$\begin{aligned} F(0) + F(1) &= \int (e^t + 1) d\mu(t), \\ F'(1) - F'(0) &= \int t(e^t - 1) d\mu(t), \\ \int_0^1 F(s) ds &= \int \frac{e^t - 1}{t} d\mu(t). \end{aligned} \quad (4.18)$$

Let us consider the probability measure  $d\nu(t) = (\int t(e^t - 1)d\mu(t))^{-1}t(e^t - 1)d\mu(t)$ . We have

$$\begin{aligned} \frac{F(0) + F(1)}{F'(1) - F'(0)} &= \int \frac{1}{t} \coth \frac{t}{2} d\nu(t), \\ \frac{\int F(s)ds}{F'(1) - F'(0)} &= \int \frac{1}{t^2} d\nu(t). \end{aligned} \quad (4.19)$$

Since  $\Phi$  is concave we can use Jensen's inequality and we get (4.16):

$$\begin{aligned} \Phi\left(\frac{4 \int_0^1 F(s)ds}{F'(1) - F'(0)}\right) &= \Phi\left(\int \frac{4}{t^2} d\nu(t)\right) \geq \int \Phi\left(\frac{4}{t^2}\right) d\nu(t) \\ &= \int \frac{2}{t} \coth \frac{t}{2} d\nu(t) = 2 \frac{F(0) + F(1)}{F'(1) - F'(0)}. \end{aligned} \quad (4.20)$$

■

The Falk–Bruch inequality is saturated when the measure  $d\mu$  is a Dirac on a single value. This is the case if  $H$  is the Hamiltonian of the harmonic oscillator, and  $A$  is the creation or annihilation operator.

The following inequality follows from Lemma 4.2 and the upper bound in Eq. (4.14).

**Corollary 4.3.** *We have*

$$\frac{1}{2} \langle A^* A + A A^* \rangle \leq \frac{1}{2} \sqrt{(A, A) \langle [A^*, [H, A]] \rangle} + (A, A).$$

For our purpose we have  $H \sim \beta$  and  $(A, A) \sim \frac{1}{\beta}$  with  $\beta$  large, so that this inequality is quite optimal. We use it below since it is simpler.

## 4.2 Infrared bound for the usual correlation function

In the rest of this section  $\ell$  and  $\beta$  will be fixed, and we drop the subscripts on  $\langle \cdot \rangle_{\Lambda_\ell, \beta, 0}^{\text{per}}$ , writing simply  $\langle \cdot \rangle$ .

We introduce Fourier transforms of spin operators. This allows to write the correlation functions in the form of Corollary 4.3. Accordingly, let

$$\widehat{S}_k^{(3)} = \sum_{x \in \Lambda_\ell} e^{-ikx} S_x^{(3)}, \quad k \in \Lambda_\ell^*. \quad (4.21)$$

One easily checks the inverse identity

$$S_x^{(3)} = \frac{1}{\ell^d} \sum_{k \in \Lambda_\ell^*} e^{ikx} \widehat{S}_k^{(3)}, \quad x \in \Lambda_\ell. \quad (4.22)$$

The Fourier transform of the usual correlation function is then equal to

$$\begin{aligned} \langle \widehat{S_0^{(3)} S_x^{(3)}} \rangle(k) &= \sum_{x \in \Lambda_\ell} e^{-ikx} \langle S_0^{(3)} S_x^{(3)} \rangle = \frac{1}{\ell^d} \sum_{x, y \in \Lambda_\ell} e^{-ik(x-y)} \langle S_x^{(3)} S_y^{(3)} \rangle \\ &= \frac{1}{\ell^d} \langle \widehat{S}_{-k}^{(3)} \widehat{S}_k^{(3)} \rangle. \end{aligned} \quad (4.23)$$

Notice that  $(\widehat{S}_k^{(3)})^* = \widehat{S}_{-k}^{(3)}$ , thus

$$\langle \widehat{S_0^{(3)} S_x^{(3)}} \rangle(k) \geq 0. \quad (4.24)$$

For the Duhamel correlation function we obtain

$$\widehat{\eta}(k) = (\widehat{S_0^{(3)}}, \widehat{S_x^{(3)}})(k) = \frac{1}{\ell^d} (\widehat{S}_k^{(3)}, \widehat{S}_k^{(3)}). \quad (4.25)$$

(There is no  $-k$  because the Duhamel inner product involves taking the adjoint.) Let

$$e(k) = \frac{1}{2} \sum_{x \in \mathbb{Z}^d} \left( (J_{x,\text{per}}^{(1)} - J_{x,\text{per}}^{(2)} \cos kx) \langle S_0^{(1)} S_x^{(1)} \rangle + (J_{x,\text{per}}^{(2)} - J_{x,\text{per}}^{(1)} \cos kx) \langle S_0^{(2)} S_x^{(2)} \rangle \right). \quad (4.26)$$

We will see in the proof of the next lemma that  $e(k) \geq 0$ , as it can be written as the expectation of double commutator in the form of Eq. (4.15).

**Lemma 4.4** (Infrared bound for the usual correlation function). *We have for all  $k \in \Lambda_\ell^* \setminus \{0\}$  that*

$$\langle \widehat{S_0^{(3)} S_x^{(3)}} \rangle(k) \leq \sqrt{\frac{e(k)}{2\varepsilon(k)}} + \frac{1}{2\beta\varepsilon(k)}.$$

*Proof.* We take  $A = \widehat{S}_k^{(3)}$  and  $H = \beta H_{\Lambda,0}^{\text{per}}$  in Corollary 4.3. We need to calculate the double commutator. First, we have

$$\begin{aligned} [H_{\Lambda,0}^{\text{per}}, \widehat{S}_k^{(3)}] &= \sum_{x \in \Lambda_\ell} [H_{\Lambda,0}^{\text{per}}, S_x^{(3)}] e^{-ikx} \\ &= - \sum_{i=1}^3 \sum_{x, y, z \in \Lambda_\ell} e^{-ikx} J_{y-z, \text{per}}^{(i)} [S_y^{(i)} S_z^{(i)}, S_x^{(3)}] \\ &= -2i \sum_{x, y \in \Lambda_\ell} e^{-ikx} \left( -J_{x-y, \text{per}}^{(1)} S_x^{(2)} S_y^{(1)} + J_{x-y, \text{per}}^{(2)} S_x^{(1)} S_y^{(2)} \right). \end{aligned} \quad (4.27)$$



We used the fact that operators at different sites commute, and also that  $J_x^{(i)} = J_{-x}^{(i)}$ . Next,

$$\begin{aligned}
 [\widehat{S}_{-k}^{(3)}, [H_{\Lambda,0}^{\text{per}}, \widehat{S}_k^{(3)}]] &= -2i \sum_{x,y \in \Lambda_\ell} e^{-ikx} \left[ e^{ikx} S_x^{(3)} + e^{iky} S_y^{(3)}, -J_{x-y,\text{per}}^{(1)} S_x^{(2)} S_y^{(2)} \right. \\
 &\quad \left. + J_{x-y,\text{per}}^{(2)} S_x^{(1)} S_y^{(2)} \right] \\
 &= 2 \sum_{x,y \in \Lambda_\ell} \left( (J_{x-y,\text{per}}^{(1)} - \cos(k(x-y))) J_{x-y,\text{per}}^{(2)} S_x^{(1)} S_y^{(1)} \right. \\
 &\quad \left. + (J_{x-y,\text{per}}^{(2)} - \cos(k(x-y))) J_{x-y,\text{per}}^{(1)} S_x^{(2)} S_y^{(2)} \right). \tag{4.28}
 \end{aligned}$$

Taking the expectation in the Gibbs state, we obtain

$$\langle [A^*, [H, A]] \rangle = \langle [\widehat{S}_{-k}^{(3)}, [\beta H_{\Lambda,0}^{\text{per}}, \widehat{S}_k^{(3)}]] \rangle = 4\beta \ell^d e(k). \tag{4.29}$$

We also see that  $e(k) \geq 0$  from Eq. (4.15). Lemma 4.4 follows from Corollary 4.3 and from the infrared bound on the Duhamel correlation function, Lemma 4.1. ■

We can now prove the occurrence of long-range order.

*Proof of Theorem 3.1.* We have the inequality (see Lemma A.1)

$$\langle S_0^{(3)} S_0^{(3)} \rangle \geq \frac{1}{3} \sum_{i=1}^3 \langle S_0^{(i)} S_0^{(i)} \rangle = \frac{1}{3} S(S+1). \tag{4.30}$$

This is where we use that  $J_x^{(3)} \geq J_x^{(1)} \geq -J_x^{(2)} \geq 0$ .

We now use the inverse Fourier transform on the two-point correlation function, namely

$$\frac{1}{3} S(S+1) \leq \langle S_0^{(3)} S_0^{(3)} \rangle = \frac{1}{\ell^d} \langle \widehat{S_0^{(3)} S_x^{(3)}} \rangle(0) + \frac{1}{\ell^d} \sum_{k \in \Lambda_\ell^* \setminus \{0\}} \langle \widehat{S_0^{(3)} S_x^{(3)}} \rangle(k). \tag{4.31}$$

Notice that the first term of the right side is equal to the long-range order parameter. Then

$$\frac{1}{\ell^d} \sum_{x \in \Lambda_\ell} \langle S_0^{(3)} S_x^{(3)} \rangle = \frac{1}{\ell^d} \langle \widehat{S_0^{(3)} S_x^{(3)}} \rangle(0) \geq \frac{1}{3} S(S+1) - \frac{1}{\ell^d} \sum_{k \in \Lambda_\ell^* \setminus \{0\}} \langle \widehat{S_0^{(3)} S_x^{(3)}} \rangle(k). \tag{4.32}$$

We can bound the last term with the help of Lemma 4.4, which gives Theorem 3.1. ■

*Proof of Theorem 3.2.* With nearest-neighbour interactions the function  $e(k)$  can be written as

$$e(k) = \alpha_\ell(\beta) \sum_{i=1}^d (1 + r \cos k_i), \tag{4.33}$$

where

$$r = \frac{-J^{(2)} \langle S_0^{(1)} S_{e_1}^{(1)} \rangle - J^{(1)} \langle S_0^{(2)} S_{e_1}^{(2)} \rangle}{J^{(1)} \langle S_0^{(1)} S_{e_1}^{(1)} \rangle + J^{(2)} \langle S_0^{(2)} S_{e_1}^{(2)} \rangle}. \quad (4.34)$$

Here  $e_1 = (1, 0, \dots, 0) \in \mathbb{Z}^d$  is the unit vector in the first direction. It follows from the fact that  $e(k) \geq 0$  for all  $k$ , that  $r \in [-1, 1]$ . Let

$$I_\ell^{(d)}(r) = \frac{1}{\ell^d} \sum_{k \in \Lambda_\ell^* \setminus \{0, \pi\}} \sqrt{\frac{\sum_{i=1}^d (1 + r \cos k_i)}{\sum_{i=1}^d (1 - \cos k_i)}} \quad (4.35)$$

where we omitted the term  $\frac{1}{\ell^d} \sqrt{1-r}$  for  $k = \pi = (\pi, \pi, \dots, \pi)$ . Adding it back and bounding it by  $\sqrt{2}/\ell^d$ , the lower bound is

$$\frac{1}{\ell^d} \sum_{x \in \Lambda_\ell} \langle S_0^{(3)} S_x^{(3)} \rangle \geq \frac{1}{3} S(S+1) - \frac{1}{2} \sqrt{\alpha_\ell(\beta)} (I_\ell^{(d)}(r) + \frac{\sqrt{2}}{\ell^d}) - \frac{1}{2\beta\ell^d} \sum_{k \in \Lambda_\ell^* \setminus \{0\}} \frac{1}{\varepsilon(k)}. \quad (4.36)$$

Observe that  $I_\ell^{(d)}(r)$  is concave with respect to  $r$  and that its derivative at  $r = 1$  is equal to

$$\left. \frac{d}{dr} I_\ell^{(d)}(r) \right|_{r=1} = \frac{1}{\ell^d} \sum_{k \in \Lambda_\ell^* \setminus \{0, \pi\}} \frac{\sum_{i=1}^d \cos k_i}{\sqrt{\sum_{i=1}^d (1 - \cos k_i) \sum_{i=1}^d (1 + \cos k_i)}}. \quad (4.37)$$

This is equal to zero, as can be seen with the change of variables  $k \mapsto k + (\pi, \dots, \pi)$ . Then  $I_\ell^{(d)}(r) \leq I_\ell^{(d)}(1) = I_\ell^{(d)}$ . Using this with the lower bound of Theorem 3.1, we obtain the first bound of Theorem 3.2.

For the second bound, we follow [25] and use the inverse Fourier transform. In what follows,  $x$  is the dummy variable summed over in the Fourier transform. We have

$$\begin{aligned} \langle S_0^{(3)} S_{e_1}^{(3)} \rangle &= \frac{1}{d\ell^d} \sum_{i=1}^d e^{ik_i} \langle \widehat{S_0^{(3)} S_x^{(3)}}(k) \rangle \\ &= \frac{1}{\ell^d} \langle \widehat{S_0^{(3)} S_x^{(3)}}(0) \rangle + \frac{1}{\ell^d} \sum_{k \in \Lambda_\ell^* \setminus \{0\}} \langle \widehat{S_0^{(3)} S_x^{(3)}}(k) \rangle \left( \frac{1}{d} \sum_{i=1}^d \cos k_i \right). \end{aligned} \quad (4.38)$$

We used lattice symmetries and the fact that  $\langle \widehat{S_0^{(3)} S_x^{(3)}} \rangle \geq 0$ , see Eq. (4.24). We have

$$\begin{aligned} \frac{1}{\ell^d} \langle \widehat{S_0^{(3)} S_x^{(3)}}(0) \rangle &\geq \langle S_0^{(3)} S_{e_1}^{(3)} \rangle - \frac{1}{\ell^d} \sum_{k \in \Lambda_\ell^* \setminus \{0\}} \langle \widehat{S_0^{(3)} S_x^{(3)}}(k) \rangle \left( \frac{1}{d} \sum_{i=1}^d \cos k_i \right)_+ \\ &\geq \langle S_0^{(3)} S_{e_1}^{(3)} \rangle - \frac{1}{\ell^d} \sum_{k \in \Lambda_\ell^* \setminus \{0\}} \left( \frac{1}{d} \sum_{i=1}^d \cos k_i \right)_+ \left[ \sqrt{\frac{e(k)}{2\varepsilon(k)}} + \frac{1}{2\beta\varepsilon(k)} \right]. \end{aligned} \quad (4.39)$$

Proceeding with  $e(k)$  as we did with the first lower bound, we get

$$\frac{1}{\ell^d} \langle \widehat{S_0^{(3)}} \widehat{S_x^{(3)}} \rangle(0) \geq \langle S_0^{(3)} S_{e_1}^{(3)} \rangle - \frac{1}{2} \sqrt{\alpha_\ell(\beta)} \tilde{I}_\ell^{(d)}(r) - \frac{1}{2\beta\ell^d} \sum_{k \in \Lambda_\ell^* \setminus \{0\}} \frac{1}{\varepsilon(k)}, \quad (4.40)$$

where

$$\tilde{I}_\ell^{(d)}(r) = \frac{1}{\ell^d} \sum_{k \in \Lambda_\ell^* \setminus \{0\}} \sqrt{\frac{\sum_{i=1}^d (1 + r \cos k_i)}{\sum_{i=1}^d (1 - \cos k_i)}} \left( \frac{1}{d} \sum_{i=1}^d \cos k_i \right)_+. \quad (4.41)$$

One easily checks that the derivative of  $\tilde{I}_\ell^{(d)}(r)$  is positive, so it is smaller than  $\tilde{I}_\ell^{(d)}(1) = \tilde{I}_\ell^{(d)}$ . Finally, we have using Lemma A.1 that

$$\langle S_0^{(3)} S_{e_1}^{(3)} \rangle \geq \frac{\alpha_\ell(\beta)}{1 - J^{(2)}/J^{(1)}}. \quad (4.42)$$

The second lower bound of Theorem 3.2 follows.  $\blacksquare$

## 5 Reflection positivity

Let  $\mathcal{H}$  be a separable Hilbert space, and let  $\mathcal{B}_{\text{left}}$ , resp.  $\mathcal{B}_{\text{right}}$ , denote the space of bounded operators on  $\mathcal{H} \otimes \mathcal{H}$  that are of the form  $a \otimes \mathbb{1}$ , resp.  $\mathbb{1} \otimes a$ , for some  $a \in \mathcal{B}(\mathcal{H})$ . Let  $\mathcal{R}$  denote the automorphism of  $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$  such that

$$\begin{aligned} \mathcal{R}(a \otimes \mathbb{1}) &= \mathbb{1} \otimes a, \\ \mathcal{R}(\mathbb{1} \otimes a) &= a \otimes \mathbb{1}. \end{aligned} \quad (5.1)$$

Let us fix an orthonormal basis  $\{e_i\}$  on  $\mathcal{H}$ , and define the complex conjugate  $\bar{a}$  of a bounded operator  $a$  by

$$\langle e_i, \bar{a} e_j \rangle = \overline{\langle e_i, a e_j \rangle}. \quad (5.2)$$

In matrix notation, that means taking the complex conjugate of its elements, without transposing as for hermitian adjoints. The reason to use the complex conjugate is that for all  $a, b \in \mathcal{H}$ , we have

$$\overline{\overline{a} b} = \bar{a} \bar{b}. \quad (5.3)$$

Here is the key inequality that is closely related to reflection positivity. Let  $I$  be an index set and  $\mu$  a finite measure on  $I$ . We assume that  $A, C_i \in \mathcal{B}_{\text{left}}$  and  $B, D_i \in \mathcal{B}_{\text{right}}$  for all  $i \in I$ .

**Lemma 5.1.** *We have*

$$\left| \text{Tr} e^{A+B+\int C_i D_i d\mu(i)} \right|^2 \leq \text{Tr} e^{A+\mathcal{R}A+\int C_i \mathcal{R}C_i d\mu(i)} \cdot \text{Tr} e^{\mathcal{R}B+B+\int \mathcal{R}D_i D_i d\mu(i)}.$$

*Proof.* We use the Duhamel formula in the following form. If  $A, B$  are bounded operators, then

$$e^{A+B} = \sum_{n \geq 0} \int_{0 < t_1 < \dots < t_n < 1} dt_1 \dots dt_n e^{t_1 A} B e^{(t_2 - t_1) A} B \dots B e^{(1 - t_n) A}. \quad (5.4)$$

In what follows, we use the shorthands

$$\int d\mathbf{i} \equiv \int d\mu(i_1) \dots \int d\mu(i_n) \quad \text{and} \quad \int d\mathbf{t} \equiv \int_{0 < t_1 < \dots < t_n < 1} dt_1 \dots dt_n. \quad (5.5)$$

We also write  $A = a \otimes \mathbb{1}$ ,  $B = \mathbb{1} \otimes b$ ,  $C_i = c_i \otimes \mathbb{1}$ , and  $D_i = \mathbb{1} \otimes d_i$ . Then

$$\begin{aligned} & \left| \text{Tr}_{\mathcal{H} \otimes \mathcal{H}} e^{A+B+\int C_i D_i d\mu(i)} \right|^2 \\ &= \left| \sum_{n \geq 0} \int d\mathbf{i} \int d\mathbf{t} \text{Tr}_{\mathcal{H} \otimes \mathcal{H}} e^{t_1(A+B)} C_{i_1} D_{i_1} \dots C_{i_n} D_{i_n} e^{(1-t_n)(A+B)} \right|^2 \\ &= \left| \sum_{n \geq 0} \int d\mathbf{i} \int d\mathbf{t} \text{Tr}_{\mathcal{H}} e^{t_1 a} c_{i_1} \dots c_{i_n} e^{(1-t_n)a} \text{Tr}_{\mathcal{H}} e^{t_1 b} d_{i_1} \dots d_{i_n} e^{(1-t_n)b} \right|^2 \\ &\leq \sum_{n \geq 0} \int d\mathbf{i} \int d\mathbf{t} \text{Tr}_{\mathcal{H}} e^{t_1 a} c_{i_1} \dots c_{i_n} e^{(1-t_n)a} \text{Tr}_{\mathcal{H}} e^{t_1 \bar{a}} \bar{c}_{i_1} \dots \bar{c}_{i_n} e^{(1-t_n)\bar{a}} \\ &\quad \cdot \text{Tr}_{\mathcal{H}} e^{t_1 \bar{b}} \bar{d}_{i_1} \dots \bar{d}_{i_n} e^{(1-t_n)\bar{b}} \text{Tr}_{\mathcal{H}} e^{t_1 b} d_{i_1} \dots d_{i_n} e^{(1-t_n)b} \\ &= \text{Tr}_{\mathcal{H} \otimes \mathcal{H}} e^{A+\mathcal{R}\bar{A}+\int C_i \mathcal{R}\bar{C}_i d\mu(i)} \cdot \text{Tr}_{\mathcal{H} \otimes \mathcal{H}} e^{\mathcal{R}\bar{B}+B+\int \mathcal{R}\bar{D}_i D_i d\mu(i)}. \end{aligned} \quad (5.6)$$

We used the ordinary Cauchy–Schwarz inequality for functions, here with argument  $(n, \mathbf{i}, \mathbf{t})$ . The complex conjugate was written with the help of (5.3).  $\blacksquare$

We now derive the infrared bound for the Duhamel correlation function, Lemma 4.1. In the rest of this Section, we fix an even integer  $\ell$  and consider periodic couplings (2.7). Recall that  $\Lambda_\ell = \{0, 1, \dots, \ell - 1\}^d$ . Let  $\Delta$  denote the discrete Laplacian from the coupling constant  $J_{\text{per}}^{(3)}$ , that acts on a field  $v = (v_x) \in \mathbb{R}^{\Lambda}$  as

$$(\Delta v)_x = \sum_{y \in \Lambda_\ell} J_{x-y, \text{per}}^{(3)} (v_y - v_x). \quad (5.7)$$

Notice the following identity, which is a discrete version of  $\int f(-\Delta g) = \int \nabla f \nabla g$  for functions:

$$(u, -\Delta v) = \frac{1}{2} \sum_{x, y} J_{x-y}^{(1)} (u_x - u_y)(v_x - v_y). \quad (5.8)$$

In the left side,  $(\cdot, \cdot)$  stands for the usual inner product on  $\mathbb{R}^{\Lambda_\ell}$ , i.e.  $(u, v) = \sum_{x \in \Lambda_\ell} u_x v_x$ . We introduce the following partition function that depends on a field  $v$ :

$$Z(v) = \text{Tr} e^{-\beta H(v)}, \quad (5.9)$$

with Hamiltonian given by

$$H(v) = H_{\Lambda_\ell, 0}^{\text{per}} - \sum_{x \in \Lambda_\ell} h_x S_x^{(3)}, \quad (5.10)$$

where the local magnetic field are obtained from  $v$  by

$$h_x = (\Delta v)_x. \quad (5.11)$$

Let

$$\tilde{Z}(v) = e^{\frac{1}{4}\beta(v, \Delta v)} Z(v). \quad (5.12)$$

We show that  $\tilde{Z}(v)$  is maximised by the field  $v \equiv 0$ , which is the key to proving Lemma 4.1.

Let  $\mathcal{R}$  denote a reflection across a plane cutting through edges. Namely, given a direction  $i = 1, \dots, d$  and a half integer  $\epsilon \in \{\frac{1}{2}, \frac{3}{2}, \dots, \frac{\ell-1}{2}\}$ , let  $\mathcal{R}$  be the bijection  $\Lambda_\ell \rightarrow \Lambda_\ell$  such that

$$\mathcal{R}x = x + 2(\epsilon - x_i)e_i. \quad (5.13)$$

Let

$$\Lambda_{\text{left}} = \{x \in \Lambda_\ell : \epsilon - \frac{\ell}{2} < x_i < \epsilon\}, \quad \Lambda_{\text{right}} = \{x \in \Lambda_\ell : \epsilon < x_i < \epsilon + \frac{\ell}{2}\}. \quad (5.14)$$

Given a field  $v_1 \in \mathbb{R}^{\Lambda_{\text{left}}}$ , let  $(\mathcal{R}v_1)_x = (v_1)_{\mathcal{R}x} \in \mathbb{R}^{\Lambda_{\text{right}}}$ .

**Lemma 5.2.** *Let the couplings  $J^{(i)}$  satisfy the assumptions of Theorem 3.1. Then, for any  $v_1 \in \mathbb{R}^{\Lambda_{\text{left}}}$  and  $v_2 \in \mathbb{R}^{\Lambda_{\text{right}}}$ , we have*

$$Z(v_1, v_2)^2 \leq Z(v_1, \mathcal{R}v_1) Z(\mathcal{R}v_2, v_2).$$

We first prove the lemma in the case of nearest-neighbour couplings; we then consider long-range interactions that depend on the  $\ell_1$  distance; finally, we consider the Euclidean distance.

*Proof of Lemma 5.2 for nearest-neighbour couplings.* We cast  $Z(v_1, v_2)$  in the form of Lemma 5.1. Using (5.8), we get

$$\begin{aligned} \tilde{Z}(v) &= \text{Tr} \exp \beta \left\{ \frac{1}{4} \sum_{x,y} v_x J_{x-y}^{(3)} (v_y - v_x) + \sum_{i=1}^3 \sum_{x,y} J_{x-y}^{(i)} S_x^{(i)} S_y^{(i)} \right. \\ &\quad \left. + \sum_{x,y} J_{x-y}^{(3)} S_x^{(1)} (v_y - v_x) \right\} \\ &= \text{Tr} \exp \beta \left\{ \sum_{i=1}^2 \sum_{x,y} J_{x-y}^{(i)} S_x^{(i)} S_y^{(i)} + \sum_{x,y} J_{x-y}^{(3)} \left( S_x^{(3)} + \frac{v_x}{2} \right) \left( S_y^{(3)} + \frac{v_y}{2} \right) \right. \\ &\quad \left. - \widehat{J}^{(3)}(0) \sum_x \left( S_x^{(3)} v_x + \frac{v_x^2}{4} \right) \right\}. \quad (5.15) \end{aligned}$$

This formula holds for general couplings and we will use it for long-range couplings too (with  $J_{x,\text{per}}^{(i)}$ ). We now assume that  $J_x^{(i)} = 0$  except when  $\|x\|_1 = 1$ . Then the above expression has the form of Lemma 5.1 by choosing

$$\begin{aligned} A &= \beta \sum_{x,y \in \Lambda_{\text{left}}} \left[ \sum_{i=1}^2 J_{x-y}^{(i)} S_x^{(i)} S_y^{(i)} + J_{x-y}^{(3)} \left( S_x^{(3)} + \frac{v_x}{2} \right) \left( S_y^{(3)} + \frac{v_y}{2} \right) \right] \\ &\quad - \widehat{J}^{(3)}(0) \sum_{x \in \Lambda_{\text{left}}} \left( S_x^{(3)} v_x + \frac{v_x^2}{4} \right) \\ B &= \beta \sum_{x,y \in \Lambda_{\text{right}}} \left[ \sum_{i=2}^3 J_{x-y}^{(i)} S_x^{(i)} S_y^{(i)} + J_{x-y}^{(3)} \left( S_x^{(3)} + \frac{v_x}{2} \right) \left( S_y^{(3)} + \frac{v_y}{2} \right) \right] \\ &\quad - \widehat{J}^{(3)}(0) \sum_{x \in \Lambda_{\text{right}}} \left( S_x^{(1)} v_x + \frac{v_x^2}{4} \right) \\ \int C_i D_i d\mu(i) &= \beta \sum_{\substack{x \in \Lambda_{\text{left}} \\ y \in \Lambda_{\text{right}} \\ \|x-y\|=1}} \left[ J_{x-y}^{(1)} S_x^{(1)} S_y^{(1)} - J_{x-y}^{(2)} (i S_x^{(2)}) (i S_y^{(2)}) \right. \\ &\quad \left. + J_{x-y}^{(3)} \left( S_x^{(3)} + \frac{v_x}{2} \right) \left( S_y^{(3)} + \frac{v_y}{2} \right) \right]. \quad (5.16) \end{aligned}$$

In the usual basis where all  $S_x^{(3)}$  are diagonal, we have  $\overline{S_x^{(1)}} = S_x^{(1)}$ ,  $\overline{i S_x^{(2)}} = i S_x^{(2)}$ ,  $\overline{S_x^{(3)}} = S_x^{(3)}$ . Then  $\overline{A} = A$  and  $\overline{B} = B$ . We have multiplied  $S_x^{(2)}$  by  $i$ , so taking the complex conjugate gives the operator back. Then  $\overline{C_i} = C_i$  and  $\overline{D_i} = D_i$ . In order to use Lemma 5.1 the measure  $\mu$  needs to be positive, which is guaranteed by  $J^{(1)}, J^{(3)} \geq 0$  and  $J^{(2)} \leq 0$ .

An important observation is that if certain interactions can be cast in the form above, then this can also be done with convex combinations of these interactions. We use this property below. ■

*Proof of Lemma 5.2 for long-range couplings,  $\ell_1$  case.* We now consider the case of Yukawa interaction,  $J_{x-y}^{(i)} = a^{(i)} e^{-b^{(i)}\|x-y\|_1}$ . We see from (5.15) that it suffices to consider a fixed  $i \in \{1, 2, 3\}$ . To simplify the notation we write  $J_x = J_{x,\text{per}}^{(i)}$  and we also consider only  $a^{(i)} = b^{(i)} = 1$ . It suffices to consider the cross-term

$$\sum_{\substack{x \in \Lambda_{\text{left}} \\ y \in \Lambda_{\text{right}}}} J_{x-y,\text{per}} A_x A_y \quad (5.17)$$

where  $A_x \in \{S_x^{(1)}, iS_x^{(2)}, S_x^{(3)} + \frac{v_x}{2}\}$ . It is also more convenient to work with the off-set box  $\Lambda_\ell = \{-\frac{\ell}{2} + \frac{1}{2}, \dots, \frac{\ell}{2} + \frac{1}{2}\}^d$  and reflections in the line  $x_1 = 0$ ; clearly this is equivalent. The following simple argument can be found in the article of Quitsmann and Taggi [39, Appendix A]. First note that the interaction factorises over the coordinates:

$$J_{x-y,\text{per}} = \prod_{j=1}^d J_{x_j-y_j,\text{per}}, \quad x_j, y_j \in \mathbb{Z}. \quad (5.18)$$

For  $x \in \Lambda_{\text{right}}$  and  $y \in \Lambda_{\text{left}}$  we have  $x_1 > 0 > y_1$  while the remaining  $x_j, y_j$  have arbitrary signs. We use different expressions for the periodised interaction in the cases  $j = 1$  and  $j \geq 2$ . First

$$J_{x_1-y_1,\text{per}} = \sum_{z \in \mathbb{Z}} e^{-|x_1-y_1-z\ell|} = \frac{1}{e^\ell - 1} e^{|x_1|} e^{|y_1|} + \frac{e^\ell}{e^\ell - 1} e^{-|x_1|} e^{-|y_1|}. \quad (5.19)$$

For  $j \geq 2$  we use the Fourier transform  $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(z) e^{-2\pi i z} d\xi$  of the function  $f: \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(z) = e^{|z\ell|}$ , or more precisely  $f(z - \frac{y-x}{\ell})$ , and the Poisson summation formula:

$$\sum_{z \in \mathbb{Z}} e^{-|x_j-y_j-z\ell|} = \sum_{n \in \mathbb{Z}} e^{-2\pi i(y-x)/\ell} \hat{f}(n) = \sum_{n \in \mathbb{Z}} e^{2\pi i x/\ell} e^{-2\pi i y/\ell} \frac{2\ell}{\ell^2 + 4\pi^2 n^2}. \quad (5.20)$$

Using these expressions we get

$$\sum_{\substack{x \in \Lambda_{\text{right}} \\ y \in \Lambda_{\text{left}}}} J_{x-y,\text{per}} A_x A_y = \frac{1}{e^\ell - 1} \sum_{i \in I} \mu(i) C_i \overline{\mathcal{R}C_i} + \frac{e^\ell}{e^\ell - 1} \sum_{i \in I} \mu(i) D_i \overline{\mathcal{R}D_i} \quad (5.21)$$

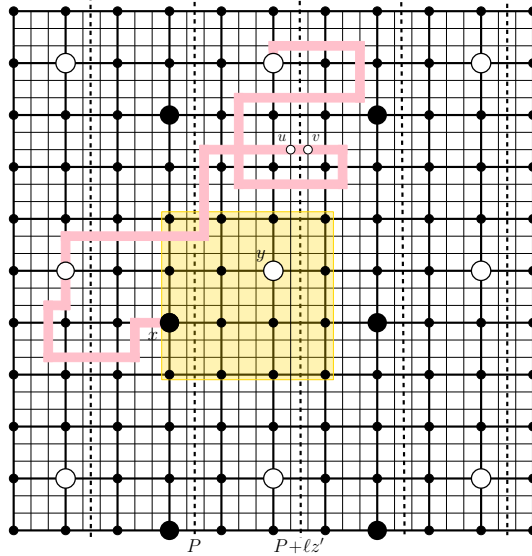
where  $i = (n_2, \dots, n_d) \in \mathbb{Z}^{d-1} =: I$  and  $\mu(i) = \prod_{j=2}^d \frac{2\ell}{\ell^2 + 4\pi^2 n_j^2} > 0$ ,

$$C_i = \sum_{x \in \Lambda_{\text{right}}} e^{|x_1|} \prod_{j=2}^d e^{2\pi i x/\ell} A_x \quad \text{and} \quad D_i = \sum_{x \in \Lambda_{\text{right}}} e^{-|x_1|} \prod_{j=2}^d e^{2\pi i x/\ell} A_x. \quad (5.22)$$

This is of the required form since  $\overline{A_x} = A_x$  for all  $x$ . The case of power-law potential  $J_x = a\|x\|_1^{-s}$  is a consequence of the Yukawa-case, since

$$\int_0^\infty t^{s-1} e^{-t\|x\|_1} dt = \frac{C}{\|x\|_1^s}. \quad (5.23)$$

■



**Figure 2.** Illustration for interactions given by random walks on  $a\mathbb{Z}^d$ . Here  $\ell = 4$  and  $a = \frac{1}{3}$ . The coloured box represents the original sites. The black (resp. white) circles are the site  $x$  and its periodised counterparts (resp.  $y$ ). Walks from  $x$  to  $y$  (or their counterparts) must cross an odd number of dotted lines; the middle crossing is  $(u, v)$ .

*Proof of Lemma 5.2 for long-range couplings,  $\ell_2$  case.* We now consider interactions given by nearest-neighbour random walks,

$$J_x = \sum_{w:0 \rightarrow x} \left( \frac{\lambda}{2d} \right)^{|w|}, \quad (5.24)$$

where  $\lambda \in (0, 1)$ . As before, it suffices to consider cross-terms, where  $x \in \Lambda_{\text{left}}$  and  $y \in \Lambda_{\text{right}}$ . By translating the walk  $w$  we can write

$$J_{x-y, \text{per}} = \sum_{z \in \mathbb{Z}^d} \sum_{w: x \rightarrow y + \ell z} \left( \frac{\lambda}{2d} \right)^{|w|}. \quad (5.25)$$

Such a walk  $w$  crosses the translates  $P + \ell z'$  ( $z' \in \mathbb{Z}^d$ ) of the plane of reflection  $P$  an odd number  $2m + 1$  times, for some  $m \geq 0$ . Let  $z' \in \mathbb{Z}^d$  be such that the  $(m + 1)$ th crossing is across  $P + \ell z'$ . By translating the walk by  $\ell z'$ , we obtain

$$J_{x-y, \text{per}} = \sum_{m \geq 0} \sum_{z, z' \in \mathbb{Z}^d} \sum_{\substack{w: x - \ell z' \rightarrow y + \ell(z - z') \\ 2m+1 \text{ crossings} \\ \text{crossing } (m+1) \text{ in } P}} \left( \frac{\lambda}{2d} \right)^{|w|}. \quad (5.26)$$



Finally we decompose the walk into two parts  $w_1$  and  $w_2$  where  $w_1$  is the part of the walk before the  $(m + 1)$ th crossing and  $w_2$  is the part of the walk after this crossing. Let  $u$  be the last vertex of  $w_1$  and  $v$  the first vertex of  $w_2$ . Thus  $u, v$  are neighbours on either side of  $P$ , which we abbreviate with  $(u, v) \in P$ . Reversing also  $w_1$  and making a change of variables we arrive at

$$J_{x-y, \text{per}} = \sum_{m \geq 0} \sum_{(u, v) \in P} \left( \sum_{z \in \mathbb{Z}^d} \sum_{\substack{w: u \rightarrow \ell z \\ m \text{ crossings}}} \left( \frac{\lambda}{2d} \right)^{|w|+1/2} \right) \left( \sum_{z' \in \mathbb{Z}^d} \sum_{\substack{w: v \rightarrow \ell z' \\ m \text{ crossings}}} \left( \frac{\lambda}{2d} \right)^{|w|+1/2} \right). \quad (5.27)$$

Then we can indeed write

$$\sum_{\substack{x \in \Lambda_{\text{left}} \\ y \in \Lambda_{\text{right}}}} J_{x-y, \text{per}} A_x A_y = \int C_i D_i d\mu(i) \quad (5.28)$$

where  $i = ((u, v), m)$  and  $C_i \in \mathcal{B}_{\text{left}}$  is given by

$$C_i = \sum_{z \in \mathbb{Z}^d} \sum_{\substack{w: u \rightarrow x + \ell z \\ m \text{ crossings}}} \left( \frac{\lambda}{2d} \right)^{|w|+1/2} A_x, \quad (5.29)$$

and similarly for  $D_i$ .

Next, observe that the proof also applies when the walks take place on  $a\mathbb{Z}^d$  where the inverse mesh  $a^{-1}$  is an odd integer. The middle crossing  $(u, v)$  may be an edge of  $a\mathbb{Z}^d$  rather than  $\mathbb{Z}^d$  but this is fine. We choose  $\lambda = 1 - ca^2$  with  $c > 0$ . As  $a \rightarrow 0$  the number of steps diverges which allows to invoke a local limit theorem (see e.g. [28]). Denoting  $\mathbb{P}_a[x]$  the probability that the walk on  $a\mathbb{Z}^d$ , starting at 0, reaches  $x$ , we have

$$\begin{aligned} \mathbb{P}_a[x] &= \sum_{n \geq 0} ca^2 (1 - ca^2)^n \left( \frac{2\pi n}{d} \right)^{-d/2} e^{-\frac{d}{2na^2} \|x\|_2^2} (1 + o(1)) \\ &= c \int_0^\infty ds e^{-cs} \left( \frac{2\pi s}{da^2} \right)^{-d/2} e^{-\frac{d}{2s} \|x\|_2^2} (1 + o(1)) \end{aligned} \quad (5.30)$$

We made the change of variables  $s = na^2$  and we replaced the sum by a Riemann integral. Consequently, the coupling constants  $J_x = \lim_{a \rightarrow 0} \frac{1}{c} \left( \frac{2\pi s}{da^2} \right)^{d/2} \mathbb{P}_a[x]$  are reflection positive. Explicitly, we have

$$J_x = \int_0^\infty ds s^{-d/2} e^{-cs} e^{-\frac{d}{2s} \|x\|_2^2}. \quad (5.31)$$

With the change of variables  $s \mapsto ds/2c$ , we get the interaction (up to a positive factor)

$$\int_0^\infty ds s^{-d/2} e^{-ds/2} e^{-\frac{c}{s} \|x\|_2^2}. \quad (5.32)$$

Integrating over  $c$  in the form  $\int_0^\infty dc c^{u-1}$ , we get

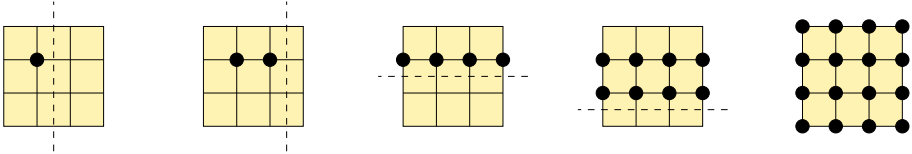
$$\frac{1}{\|x\|_2^{2u}} \int_0^\infty ds s^{u-d/2} e^{-ds/2}. \tag{5.33}$$

It is natural to suppose that  $2u > d$  so that the interaction is summable; this helps to make the integral over  $s$  convergent. The latter is a constant that does not involve  $x$ , so we indeed obtain that the following interaction is reflection positive:

$$J_x = \frac{1}{\|x\|_2^{2u}}. \tag{5.34}$$

■

**Corollary 5.3.** *For all  $v \in \mathbb{R}^{\Lambda_\ell}$ , we have  $\tilde{Z}(v) \leq \tilde{Z}(0)$ .*



**Figure 3.** Starting with a maximiser, reflections yield further maximisers where more and more values are identical.

*Proof.* Without loss of generality we can assume that  $v_0 = 0$ . We observe that  $\tilde{Z}(\lambda v) \rightarrow 0$  as  $|\lambda| \rightarrow \infty$ , so that  $\tilde{Z}(v)$  is maximised for finite  $v$ . Indeed, in the expression (5.12) we have  $e^{\frac{1}{4}\beta\lambda^2(v,\Delta v)} \sim e^{-c\lambda^2}$  and  $Z(\lambda v) \leq e^{C|\lambda|}$ .

Then let  $(v_1, v_2)$  be a maximiser with  $v_0 = 0$ . Using Lemma 5.2 with a plane crossing the edge  $(0, e_1)$ , we have that  $(v_1, \mathcal{R}v_1)$  is also a maximiser, with  $v_0 = v_{e_1} = 0$ . Using a plane crossing the edge  $(e_1, 2e_1)$ , we get a maximiser with more zeros. Iterating, we get a maximiser with a whole line of zeros. We then consider reflection planes in another direction to get a maximiser with a plane of zeros. We then consider reflection planes in further directions. See Fig. 3 for an illustration. ■

*Proof of Lemma 4.1.* From Corollary 5.3 and Eq. (5.12), we have the "Gaussian domination" bound

$$\frac{Z(sv)}{Z(0)} \leq e^{-\frac{1}{4}s^2\beta(v,\Delta v)}. \tag{5.35}$$

The derivative of  $Z(sv)$  with respect to  $s$  is equal to 0 at  $s = 0$  because of symmetries (for instance, a rotation around the 3rd spin axis by angle  $\pi$ , that takes  $S_x^{(i)}$  to  $-S_x^{(i)}$ ,  $i = 1, 2$ , and leaves  $S_x^{(3)}$  invariant). The second derivative can be calculated e.g. using

the Duhamel formula (5.4) and translation-invariance. Recalling the Duhamel correlation function  $\eta$  from (4.3), we get

$$\frac{1}{Z(0)} \frac{d^2}{ds^2} Z(sv) \Big|_{s=0} = \beta^2 \sum_{x,y \in \Lambda} h_x h_y \eta(x-y), \quad (5.36)$$

where we recall that  $h_x = (\Delta v)_x$ . We now choose the field  $v$  to be

$$v_x = \cos(kx), \quad k \in \Lambda_\ell^*. \quad (5.37)$$

Observe that  $\Delta v_x = -\varepsilon(k)v_x$ . The order  $s^2$  of the inequality (5.35) gives

$$\frac{1}{2} \beta^2 \varepsilon(k)^2 \sum_{x,y \in \Lambda_\ell} \cos(kx) \cos(ky) \eta(x-y) \leq \frac{1}{4} \beta \varepsilon(k) \sum_{x \in \Lambda_\ell} \cos(kx)^2. \quad (5.38)$$

Since  $\eta(x)$  and  $\widehat{\eta}(k)$  are both real, the left-hand-side satisfies

$$\begin{aligned} \sum_{x,y \in \Lambda_\ell} \cos(kx) \cos(ky) \eta(x-y) &= \sum_{x \in \Lambda_\ell} \cos(kx) \sum_{y \in \Lambda_\ell} e^{iky} \eta(x-y) \\ &= \sum_{x \in \Lambda_\ell} \cos(kx) \sum_{z \in \Lambda_\ell} e^{ik(x-z)} \eta(z) \\ &= \sum_{x \in \Lambda_\ell} \cos(kx) e^{ikx} \widehat{\eta}(k) \\ &= \widehat{\eta}(k) \sum_{x \in \Lambda_\ell} \cos(kx)^2. \end{aligned} \quad (5.39)$$

Inserting this in Eq. (5.38) we obtain Lemma 4.1. ■

## Appendix A Correlation inequalities for quantum systems

We needed inequalities on correlation functions in different spin directions. Such inequalities go back at least to [27]. The present lemma appeared in this form in [18]. It also holds with periodic boundary conditions.

**Lemma A.1.** *Assume that, for all  $x, y \in \Lambda$ , the coupling constants satisfy*

$$|J_{x-y}^{(2)}| \leq J_{x-y}^{(1)}.$$

*Then we have that*

$$\left| \langle S_0^{(2)} S_x^{(2)} \rangle_{\Lambda, \beta, h} \right| \leq \langle S_0^{(1)} S_x^{(1)} \rangle_{\Lambda, \beta, h},$$

*for all  $x \in \Lambda$ .*

*Proof.* Let  $|a\rangle$ ,  $a \in \{-S, \dots, S\}$  denote basis elements of  $\mathbb{C}^{2S+1}$ . Let the operators  $S^{(\pm)}$  be defined by

$$\begin{aligned} S^{(+)}|a\rangle &= \sqrt{S(S+1) - a(a+1)} |a+1\rangle, \\ S^{(-)}|a\rangle &= \sqrt{S(S+1) - (a-1)a} |a-1\rangle, \end{aligned} \quad (\text{A.1})$$

with the understanding that  $S^{(+)}|S\rangle = S^{(-)}| -S\rangle = 0$ . Then let  $S^{(1)} = \frac{1}{2}(S^{(+)} + S^{(-)})$ ,  $S^{(2)} = \frac{1}{2i}(S^{(+)} - S^{(-)})$ , and  $S^{(3)}|a\rangle = a|a\rangle$ . It is well-known that these operators satisfy the spin commutation relations. Further, the matrix elements of  $S^{(1)}$ ,  $S^{(\pm)}$  are all nonnegative, and the matrix elements of  $S^{(2)}$  are all less than or equal to those of  $S^{(1)}$  in absolute values. Using the Trotter formula and multiple resolutions of the identity, we have

$$\begin{aligned} |\text{Tr } S_0^{(2)} S_x^{(2)} e^{-\beta H_{\Lambda,0}}| &\leq \lim_{N \rightarrow \infty} \sum_{\sigma_0, \dots, \sigma_N \in \{-S, \dots, S\}^\Lambda} \left| \langle \sigma_0 | S_0^{(2)} S_x^{(2)} | \sigma_1 \rangle \right. \\ &\langle \sigma_1 | e^{\frac{\beta}{N} \sum J_{y-z}^{(3)} S_y^{(3)} S_z^{(3)}} | \sigma_1 \rangle \langle \sigma_1 | \left( 1 + \frac{\beta}{N} \sum_{y,z \in \Lambda} (J_{y-z}^{(1)} S_y^{(1)} S_z^{(1)} + J_{y-z}^{(2)} S_y^{(2)} S_z^{(2)}) \right) | \sigma_2 \rangle \\ &\dots \langle \sigma_N | e^{\frac{\beta}{N} \sum J_{y-z}^{(3)} S_y^{(3)} S_z^{(3)}} | \sigma_N \rangle \langle \sigma_N | \left( 1 + \frac{\beta}{N} \sum_{y,z \in \Lambda} (J_{y-z}^{(1)} S_y^{(1)} S_z^{(1)} + J_{y-z}^{(2)} S_y^{(2)} S_z^{(2)}) \right) | \sigma_0 \rangle \Big|. \end{aligned} \quad (\text{A.2})$$

Observe that the matrix elements of all operators are nonnegative, except for  $S_0^{(2)} S_x^{(2)}$ . Indeed, this follows from

$$\begin{aligned} &J_{y-z}^{(1)} S_y^{(1)} S_z^{(1)} + J_{y-z}^{(2)} S_y^{(2)} S_z^{(2)} \\ &= \frac{1}{4} (J_{y-z}^{(1)} - J_{y-z}^{(2)}) (S_y^{(+)} S_z^{(+)} + S_y^{(-)} S_z^{(-)}) + \frac{1}{4} (J_{y-z}^{(1)} + J_{y-z}^{(2)}) (S_y^{(+)} S_z^{(-)} + S_y^{(-)} S_z^{(+)}). \end{aligned} \quad (\text{A.3})$$

We get an upper bound for the right side of (A.2) by replacing  $|\langle \sigma_0 | S_0^{(2)} S_x^{(2)} | \sigma_1 \rangle|$  with  $\langle \sigma_0 | S_0^{(1)} S_x^{(1)} | \sigma_1 \rangle$ . We have obtained

$$|\text{Tr } S_0^{(2)} S_x^{(2)} e^{-\beta H_{\Lambda,0}}| \leq \text{Tr } S_0^{(1)} S_x^{(1)} e^{-\beta H_{\Lambda,0}}, \quad (\text{A.4})$$

which proves the claim. We actually set  $\hbar = 0$  in order to shorten the equations, but adding terms involving the  $S_x^{(3)}$  operators to the Hamiltonian is straightforward. ■

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