



# Kac–Ward Solution of the 2D Classical and 1D Quantum Ising Models

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**Abstract.** We give a rigorous derivation of the free energy of (i) the classical Ising model on the triangular lattice with translation-invariant coupling constants and (ii) the one-dimensional quantum Ising model. We use the method of Kac and Ward. The novel aspect is that the coupling constants may have negative signs. We describe the logarithmic singularity of the specific heat of the classical model and the validity of the Cimasoni–Duminil-Copin–Li formula for the critical temperature. We also discuss the quantum phase transition of the quantum model.

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## 1. Introduction

Onsager’s calculation in 1944 of the free energy of the Ising model on the square lattice was a remarkable achievement [21]. It helped to characterise the nature of the phase transition and yielded some critical exponents. Onsager’s method was algebraic in nature and was simplified by Kaufman [16]. The formula for the Ising free energy on the triangular lattice was first found by Houtappel [9] in 1950; he used a simplified version of Kaufman’s method with more elementary group theory. Further works on the triangular lattice (or its dual, the hexagonal lattice) include Wannier [28], and Husimi and Syozi [10, 11].

After the work of Onsager and Kaufman, people found two alternate approaches: combinatorial and fermionic. The former was proposed in 1952 by Kac and Ward [13]; it was later extended by Kasteleyn who noted the connection with dimer systems [15] (see also Temperley and Fisher [27]). Potts [23] and Stephenson [25] used the Kac–Ward method on the triangular lattice,

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for the free energy and for correlation functions. The fermionic method was proposed in 1964 by Schultz et al. [24].

In this article, we use the Kac–Ward approach. It consists of two parts. First is a remarkable identity that relates the partition function of the Ising model to (the square root of) the determinant of a suitable matrix; this holds for arbitrary planar graphs. Second, one uses the Fourier transform to block-diagonalise the matrix so as to obtain its determinant. The latter step involves a “mild” modification of the matrix to make it periodic; this mild step has been used over the years without mathematical justification. Only recently, careful analyses have been proposed by Kager et al. [14] (see [20] for a clear description) and by Aizenman and Warzel [1] (who elucidate the connection to the graph zeta function). These analyses are restricted to nonnegative coupling constants. Another line of research is the determination of the critical temperature for general two-periodic planar graphs by Li [18] and Cimasoni and Duminil-Copin [5]; this uses the results of Kenyon et al. [17] for dimer systems.

The main goal of this article is to extend the Kac–Ward method to the case of (translation-invariant) coupling constants of arbitrary signs. We work on the triangular lattice, which is the simplest case of frustrated systems with translation-invariant coupling constants. We start with the Cimasoni extension of the Kac–Ward formula to “faithful projections” of non-planar graphs [4] (see also Aizenman and Warzel [1] for a clear exposition). We use it for the torus  $\{1, \dots, L\}_{\text{per}} \times \{1, \dots, M\}_{\text{per}}$  with periodic boundary conditions. The main difficulties involve the non-planarity of the graph. We prove that these difficulties vanish in the limit  $L \rightarrow \infty$  for fixed  $M$ . Then, we can use the Fourier transform and we obtain the free energy formula for the infinite cylinder  $\mathbb{Z} \times \{1, \dots, M\}_{\text{per}}$ . The Onsager–Houtappel formula immediately follows by taking the limit  $M \rightarrow \infty$ .

As is well known, the exact form of the free energy allows to establish the occurrence of a phase transition characterised by the divergence of the specific heat (the second derivative of the free energy with respect to the temperature). We discuss cases where this phase transition occurs or fails to occur.

Our result for cylinders allows us to consider the one-dimensional quantum Ising model, whose free energy was first calculated in 1970 by Pfeuty [22]. We refer to [2, 3, 6, 8, 12, 19, 26] for recent studies. The quantum Ising model can be mapped to a 2D classical Ising model in the limit where the extra dimension becomes continuous. We also discuss the occurrence of a “quantum phase transition”.

The paper is organised as follows: We state our main theorem about the free energy of the Ising model on triangular lattices in Sect. 2.1. We then discuss the possibility of a phase transition in the form of logarithmic singularity of the specific heat in Sect. 2.2. In Sect. 2.3, we consider the special case where two coupling constants are equal; we show that the Cimasoni–Duminil-Copin–Li formula (see Eq. (2.20)) may yield the correct critical temperature even when the couplings are not all positive. The derivation of the free energy is described

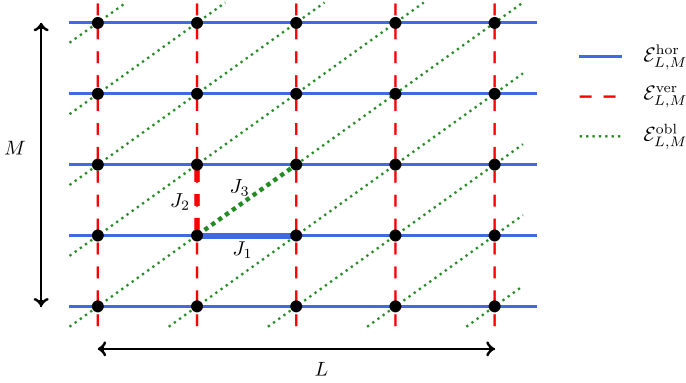


FIGURE 1. Our lattice is the torus  $\mathbb{T}_{L,M}$  with horizontal, vertical, and north-east edges

in Sect. 3. The quantum Ising model is discussed in Sect. 4; we describe the quantum phase transition at the end of the section.

## 2. The Classical Ising Model on the Triangular Lattice

### 2.1. The Free Energy

We view the triangular lattice as a square lattice with additional north-east edges. Let  $L, M \in \mathbb{N}$ . Let  $\mathbb{T}_L$  be the torus of  $L$  sites,  $\mathbb{T}_L \simeq \mathbb{Z} / L\mathbb{Z}$ , and let  $\mathbb{T}_{L,M}$  be the two-dimensional torus

$$\mathbb{T}_{L,M} = \mathbb{T}_L \times \mathbb{T}_M. \quad (2.1)$$

We let  $\mathcal{E}_{L,M} = \mathcal{E}_{L,M}^{\text{hor}} \cup \mathcal{E}_{L,M}^{\text{ver}} \cup \mathcal{E}_{L,M}^{\text{obl}}$  denote the set of edges of  $\mathbb{T}_{L,M}$  where

$$\mathcal{E}_{L,M}^{\text{hor}} = \{\{x, x + e_1\} : x \in \mathbb{T}_{L,M}\} \quad (\text{horizontal edges})$$

$$\mathcal{E}_{L,M}^{\text{ver}} = \{\{x, x + e_2\} : x \in \mathbb{T}_{L,M}\} \quad (\text{vertical edges})$$

$$\mathcal{E}_{L,M}^{\text{obl}} = \{\{x, x + e_1 + e_2\} : x \in \mathbb{T}_{L,M}\} \quad (\text{oblique north-east edges})$$

This is illustrated in Fig. 1. Let  $J_1, J_2, J_3 \in \mathbb{R}$  be three parameters; we define the coupling constants  $(J_e)_{e \in \mathcal{E}_{L,M}}$  to be

$$J_e = \begin{cases} J_1 & \text{if } e \in \mathcal{E}_{L,M}^{\text{hor}}, \\ J_2 & \text{if } e \in \mathcal{E}_{L,M}^{\text{ver}}, \\ J_3 & \text{if } e \in \mathcal{E}_{L,M}^{\text{obl}}. \end{cases} \quad (2.2)$$

A spin configuration  $\sigma$  is an assignment of a classical spin  $\pm 1$  to each site of  $\mathbb{T}_{L,M}$ ,  $\sigma = (\sigma_x)_{x \in \mathbb{T}_{L,M}} \in \{-1, +1\}^{\mathbb{T}_{L,M}}$ . The Ising Hamiltonian is the function of spin configurations given by

$$H_{L,M}(\sigma) = - \sum_{e=\{x,y\} \in \mathcal{E}_{L,M}} J_e \sigma_x \sigma_y. \quad (2.3)$$

The partition function is

$$Z_{L,M}(J_1, J_2, J_3) = \sum_{\sigma} e^{-H_{L,M}(\sigma)} \quad (2.4)$$

and the finite-volume free energy density is

$$f_{L,M}(J_1, J_2, J_3) = -\frac{1}{LM} \log Z_{L,M}(J_1, J_2, J_3). \quad (2.5)$$

We consider two infinite-volume limits, to the infinite cylinder and to the plane. Namely, we define

$$\begin{aligned} f_M(J_1, J_2, J_3) &= \lim_{L \rightarrow \infty} f_{L,M}(J_1, J_2, J_3); \\ f(J_1, J_2, J_3) &= \lim_{L \rightarrow \infty} f_{L,L}(J_1, J_2, J_3). \end{aligned} \quad (2.6)$$

As is well known, we can consider arbitrary van Hove sequences of increasing domains, see, for example, [7], and we also get  $f(J_1, J_2, J_3)$ . The next theorem gives the free energy for the cylinder and for the two-dimensional lattice. The cylinder formula turns out to be convenient, and it is useful in the calculation of the 1D quantum Ising model.

**Theorem 2.1.** *For any  $J_1, J_2, J_3 \in \mathbb{R}$ , we have (with  $k_3 = k_1 + k_2$ ):*

(a) *On the cylinder  $\mathbb{Z} \times \mathbb{T}_M$ :*

$$\begin{aligned} f_M(J_1, J_2, J_3) &= -\log 2 - \frac{1}{4\pi M} \int_{-\pi}^{\pi} dk_1 \sum_{k_2 \in \tilde{\mathbb{T}}_M} \log \left[ \prod_{i=1}^3 \cosh(2J_i) + \prod_{i=1}^3 \sinh(2J_i) \right. \\ &\quad \left. - \sum_{i=1}^3 \sinh(2J_i) \cos k_i \right] \end{aligned}$$

where  $\tilde{\mathbb{T}}_M = \frac{2\pi}{M}\mathbb{T}_M + \frac{\pi}{M}$ .

(b) *On the square or triangular lattice:*

$$\begin{aligned} f(J_1, J_2, J_3) &= -\log 2 - \frac{1}{8\pi^2} \int_{[-\pi, \pi]^2} dk_1 dk_2 \log \left[ \prod_{i=1}^3 \cosh(2J_i) + \prod_{i=1}^3 \sinh(2J_i) \right. \\ &\quad \left. - \sum_{i=1}^3 \sinh(2J_i) \cos k_i \right]. \end{aligned}$$

Setting  $J_3 = 0$  and  $J_1 = J_2 = J$ , we get Onsager's formula for the isotropic Ising model on the square lattice, namely

$$\begin{aligned} f(J, J, 0) &= -\log 2 - \frac{1}{8\pi^2} \int_{[0, 2\pi]^2} dk_1 dk_2 \\ &\quad \log \left[ \cosh^2(2J) - \sinh(2J)(\cos k_1 + \cos k_2) \right]. \end{aligned} \quad (2.7)$$

The proof of part (a) of the theorem can be found at the end of Sect. 3. The next lemma establishes that  $f$  is equal to the limit  $M \rightarrow \infty$  of  $f_M$  so that (b) immediately follows from (a).

**Lemma 2.2.** *As  $M \rightarrow \infty$ , the cylinder free energy density converges to the two-dimensional free energy density:*

$$f(J_1, J_2, J_3) = \lim_{M \rightarrow \infty} f_M(J_1, J_2, J_3).$$

*Proof.* We omit the dependence on coupling constants to alleviate the notation. Let  $J_0 = \max_{i=1,2,3} |J_i|$ . Writing  $L = kM + R$  with  $R \in \{0, M-1\}$ , we have

$$Z_{M,M}^k e^{-4J_0 kM - 6J_0 RM} \leq Z_{L,M} \leq Z_{M,M}^k e^{4J_0 kM + 6J_0 RM}. \quad (2.8)$$

Taking the logarithm and dividing by  $LM$ , we get

$$\frac{kM}{L} f_{M,M} + \frac{4J_0 k + 6J_0 R}{L} \geq f_{L,M} \geq \frac{kM}{L} f_{M,M} - \frac{4J_0 k + 6J_0 R}{L}. \quad (2.9)$$

We take the limit  $L \rightarrow \infty$ ; since  $kM/L \rightarrow 1$ ,  $k/L \rightarrow 1/M$ , and  $R/L \rightarrow 0$ , we obtain

$$f_{M,M} + \frac{4J_0}{M} \geq \lim_{L \rightarrow \infty} f_{L,M} \geq f_{M,M} - \frac{4J_0}{M}. \quad (2.10)$$

The lemma follows by taking the limit  $M \rightarrow \infty$ .  $\square$

## 2.2. Logarithmic Singularity of the Specific Heat

We explore the consequences of the formula of Theorem 2.1(b) regarding the possibility of phase transitions. More specifically, given fixed parameters  $J_1, J_2, J_3$ , we consider the function  $f: \mathbb{R}_+ \rightarrow \mathbb{R}$ :

$$f(\beta) = f(\beta J_1, \beta J_2, \beta J_3). \quad (2.11)$$

We are looking for values of  $\beta$  where  $f$  is not analytic. We show the well-known fact that the second derivative of  $f$  (which is related to the physical quantity called the specific heat) has a logarithmic singularity at a special value  $\beta_c$ , called the critical point. This is illustrated in Fig. 2, which displays the free energy  $f(\beta)$  and its first and second derivatives in the case of the homogenous triangular lattice ( $J_1 = J_2 = J_3 = 1$ ). By Theorem 2.1 (b), we have

$$f(\beta) = -\log 2 - \frac{1}{8\pi^2} \int_{[-\pi, \pi]^2} dk_1 dk_2 \log[g(\beta) + h(\beta; k_1, k_2)], \quad (2.12)$$

where (recalling that  $k_3 = k_1 + k_2$ )

$$\begin{aligned} g(\beta) &= \prod_{i=1}^3 \cosh(2\beta J_i) + \prod_{i=1}^3 \sinh(2\beta J_i) - \sum_{i=1}^3 \sinh(2\beta J_i), \\ h(\beta; k_1, k_2) &= \sum_{i=1}^3 \sinh(2\beta J_i) (1 - \cos k_i). \end{aligned} \quad (2.13)$$

It turns out that the term inside the logarithm is always positive.

**Lemma 2.3.** *For all  $J_1, J_2, J_3 \in \mathbb{R}$ , all  $\beta > 0$ , and all  $k_1, k_2 \in [-\pi, \pi]$ , we have*

$$g(\beta) + h(\beta; k_1, k_2) \geq 0.$$

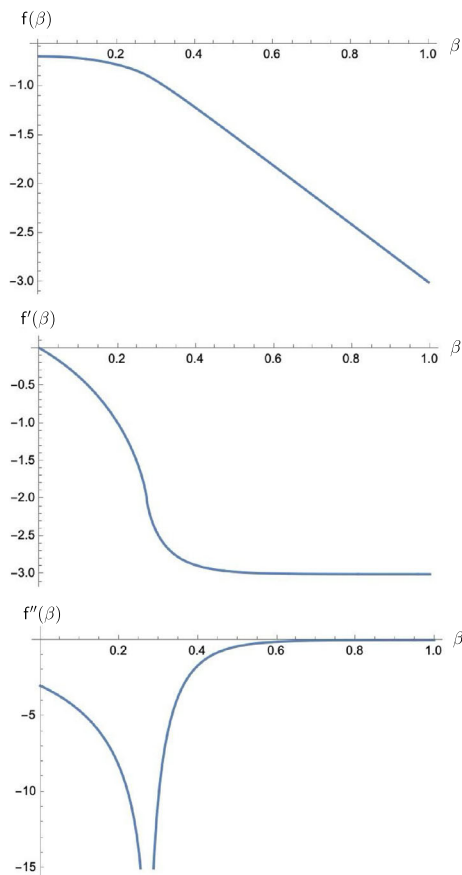


FIGURE 2. Plots of the free energy  $f(\beta)$  and its first and second derivatives for the translation-invariant triangular lattice ( $J_1 = J_2 = J_3 = 1$ ). The second derivative has a logarithmic singularity at  $\beta_c = \frac{1}{4} \log 3 = 0.274 \dots$

There should be a simple direct proof for this lemma but we could not find one. (In the case where  $J_1 = J_2$ , it follows from the proof of Theorem 2.6.) Instead we obtain it in Sect. 3 using suitable Kac–Ward identities, see Corollary 3.5(a). We now give a criterion for the free energy to be analytic in  $\beta$ .

**Lemma 2.4.** *Assume that  $g(\beta_0) + h(\beta_0; k_1, k_2) > 0$  for all  $k_1, k_2 \in [-\pi, \pi]$ . Then,  $f(\beta)$  is analytic in a complex neighbourhood of  $\beta_0$ .*

*Proof.* This is a standard complex analysis argument. There exists a complex neighbourhood  $\mathcal{N}$  of  $\beta_0$  such that  $\log[g(\beta) + h(\beta; k_1, k_2)]$  is analytic in  $\beta$  for each  $k_1, k_2$ . Then  $\int_\gamma \log[g(\beta) + h(\beta; k_1, k_2)] d\beta = 0$  for any contour  $\gamma$  in  $\mathcal{N}$ . By

Fubini's theorem,

$$\begin{aligned} & \int_{\gamma} d\beta \int_{[-\pi, \pi]^2} dk_1 dk_2 \log[g(\beta) + h(\beta; k_1, k_2)] \\ &= \int_{[-\pi, \pi]^2} dk_1 dk_2 \int_{\gamma} d\beta \log[g(\beta) + h(\beta; k_1, k_2)] = 0, \end{aligned} \quad (2.14)$$

so that  $f(\beta)$  is indeed analytic in  $\mathcal{N}$ .  $\square$

Next we establish a sufficient criterion for the logarithmic divergence of  $f''(\beta)$ . We assume here that the minimum of  $h(\beta_c; k_1, k_2)$  is at  $k_1 = k_2 = 0$  where this function is 0.

**Proposition 2.5.** *Assume that there exists  $\beta_c$  such that*

$$g(\beta_c) = 0, \quad g''(\beta_c) > 0.$$

*Further, we assume that there exists  $c > 0$  such that for all  $k_1, k_2 \in [-\pi, \pi]$ ,*

$$h(\beta_c; k_1, k_2) \geq c(k_1^2 + k_2^2).$$

*Then,  $f$  is continuously differentiable at  $\beta_c$ , but its second derivative diverges as  $\log |\beta - \beta_c|$  when  $\beta$  approaches  $\beta_c$ .*

It is not hard to verify that the second condition holds true when  $\sinh(2\beta_c J_i) + 2 \sinh(2\beta_c J_3) > 0$  for  $i = 1, 2$ .

*Proof.* We already know that  $f(\beta)$  is concave and therefore continuous. For  $\beta \neq \beta_c$ , we have

$$f'(\beta) = -\frac{1}{8\pi^2} \int dk_1 dk_2 \frac{g'(\beta) + \frac{\partial}{\partial \beta} h(\beta; k_1, k_2)}{g(\beta) + h(\beta; k_1, k_2)}. \quad (2.15)$$

There exists a constant  $C$  such that

$$\left| \frac{g'(\beta) + \frac{\partial}{\partial \beta} h(\beta; k_1, k_2)}{g(\beta) + h(\beta; k_1, k_2)} \right| \leq C \frac{|\beta - \beta_c| + k_1^2 + k_2^2}{(\beta - \beta_c)^2 + c(k_1^2 + k_2^2)}. \quad (2.16)$$

As  $a \rightarrow 0+$ , we note that

$$\int_0^1 \frac{r dr}{a^2 + r^2} = \frac{1}{2} \log(a^2 + 1) - \log a \sim |\log a|. \quad (2.17)$$

Writing the integral (2.15) with polar coordinates around 0, and using (2.16) and (2.17), we easily check that  $f'$  is continuous at  $\beta_c$ . For the second derivative, we write

$$\begin{aligned} f''(\beta) &= -\frac{g''(\beta)}{8\pi^2} \int \frac{dk_1 dk_2}{g(\beta) + h(\beta; k_1, k_2)} - \frac{1}{8\pi^2} \int dk_1 dk_2 \frac{\frac{\partial^2}{\partial^2 \beta} h(\beta; k_1, k_2)}{g(\beta) + h(\beta; k_1, k_2)} \\ &\quad + \frac{1}{8\pi^2} \int dk_1 dk_2 \left( \frac{g'(\beta) + \frac{\partial}{\partial \beta} h(\beta; k_1, k_2)}{g(\beta) + h(\beta; k_1, k_2)} \right)^2. \end{aligned} \quad (2.18)$$

For the first term, we use the bounds  $g(\beta) < g''(\beta_c)(\beta - \beta_c)^2$  and  $h(\beta; k_1, k_2) < \text{const}(k_1^2 + k_2^2)$ ; using polar coordinates and (2.17), this term diverges as  $\log |\beta - \beta_c|$  when  $\beta \rightarrow \beta_c$ . The second term is easily seen to be bounded uniformly in

$\beta \rightarrow \beta_c$  using the second condition of the proposition and  $|\frac{\partial^2}{\partial^2\beta}h(\beta; k_1, k_2)| < \text{const}(k_1^2 + k_2^2)$ . For the third term, we use (2.16), and  $|\frac{\partial}{\partial\beta}h(\beta; k_1, k_2)| < \text{const}(k_1^2 + k_2^2)$ . Using polar coordinates, and neglecting constants, we get an upper bound of the form

$$\begin{aligned} & \int_0^1 \left( \frac{|\beta - \beta_c| + r^2}{(\beta - \beta_c)^2 + r^2} \right)^2 r dr \\ & \leq 4(\beta - \beta_c)^2 \int_0^{|\beta - \beta_c|^{1/2}} \frac{r dr}{((\beta - \beta_c)^2 + r^2)^2} + \int_{|\beta - \beta_c|^{1/2}}^1 \left( \frac{|\beta - \beta_c| + r^2}{(\beta - \beta_c)^2 + r^2} \right)^2 r dr. \end{aligned} \quad (2.19)$$

The first integral is easily seen to behave as  $|\beta - \beta_c|^{-1}$ , and it is controlled by the prefactor. The integrand of the second integral is a decreasing function of  $r$ ; we get an upper bound by replacing  $r$  with  $|\beta - \beta_c|^{1/2}$  which shows that it is bounded.

We have now verified that the only divergent term in (2.18) is the first one, and the divergence is logarithmic indeed.  $\square$

### 2.3. Case $J_1 = J_2$

We consider the special case where two coupling constants are identical. By using symmetries (spin flips along alternate rows or columns), we can assume without loss of generality that  $J_1 = J_2 \geq 0$ . Further, by rescaling  $\beta$ , we can take  $J_1 = J_2 = 1$ .

**Theorem 2.6.** *Let  $J_1 = J_2 = 1$ .*

- (a) *If  $J_3 > -1$ , there is a unique  $\beta_c$  such that  $f(\beta)$  is analytic in  $\mathbb{R}_+ \setminus \{\beta_c\}$  and  $f''(\beta)$  has a logarithmic divergence at  $\beta_c$ .*
- (b) *If  $J_3 \leq -1$ ,  $f(\beta)$  is analytic in  $\mathbb{R}_+$ .*

The theorem is illustrated with the phase diagram of Fig. 3.

It helps to bring in the Cimasoni–Duminil-Copin–Li formula for the critical density that was established for two-periodic planar lattices with nonnegative coupling constants [5, 18]. Its general formulation involves sums over even graphs in the periodised cell that generates the lattice (see [5, Theorem 1.1]). In the present situation, this equation is

$$a(\beta) = 0 \quad (2.20)$$

where the function  $a(\beta)$  is defined as

$$\begin{aligned} a(\beta) = & 1 + \tanh^2 \beta \tanh(\beta J_3) - 2 \tanh \beta - \tanh(\beta J_3) - \tanh^2 \beta \\ & - 2 \tanh \beta \tanh(\beta J_3). \end{aligned} \quad (2.21)$$

In order to make the connection to the free energy (2.12), we remark that

$$g(\beta) = \cosh^2(2\beta) \cosh(2\beta J_3) a^2(\beta). \quad (2.22)$$

Then,  $g(\beta)$  vanishes precisely when  $a(\beta)$  does; indeed,  $h(\beta; k_1, k_2)$  is nonnegative when the coupling constants are nonnegative, and its minimum is 0.



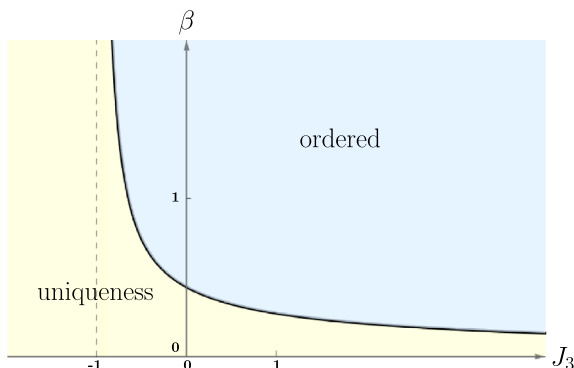


FIGURE 3. Phase diagram with  $J_1 = J_2 = 1$ . The free energy is proved to lack analyticity at the line that separates the “ordered” and “uniqueness” phases. The separation line is the inverse critical temperature  $\beta_c = \beta_c(J_3)$ ; it is solution of the equation  $\tanh \beta_c = j^{-1}(J_3)$  with  $j$  defined in Eq. (2.26). For  $J_3 \geq 0$ , the article [5] proves the existence of a unique infinite-volume Gibbs state for  $\beta < \beta_c$ , and of several distinct Gibbs states for  $\beta > \beta_c$ . For  $J_3 < 0$ , uniqueness is only proved for small  $\beta$ , and the existence of multiple Gibbs states is only proved for large  $\beta$  (using the Pirogov–Sinai theory, see, for example, [7])

Proposition 2.5 applies, establishing the singularity of the second derivative of the free energy.

We check in the proof below that the Cimasoni–Duminil-Copin–Li formula (2.20) holds whenever  $J_1 = J_2 \geq 0$ , and  $J_3 \in \mathbb{R}$  is allowed to be negative. One can also check that it *does not hold* if  $J_1 = J_2$  change signs; indeed, the free energy is the same due to symmetries (spin flips on a sublattice), but Eq. (2.20) is different and has different solutions.

In addition to the non-analyticity of the free energy, Cimasoni and Duminil-Copin prove that the phase transition involves a change of the number of infinite-volume Gibbs states: There is just one for  $\beta \leq \beta_c$  and more than one for  $\beta > \beta_c$ . The proof relies on the GKS and FKG correlation inequalities, which hold for nonnegative coupling constants only. It would be interesting to extend this to the case of coupling constants with arbitrary signs.

*Proof of Theorem 2.6.* When  $J_3 \geq 0$ , the theorem is a special case of [5], so we assume now that  $J_3 \leq 0$  (even though the proof applies to positive  $J_3$  with minor changes). We check that there exists a unique  $\beta_c$  that satisfies the conditions of Proposition 2.5.

We first check that  $g(\beta) + h(\beta; k_1, k_2)$  can reach 0 only when  $k_1 = k_2 = 0$ . Let  $\alpha = -\frac{\sinh(2\beta J_3)}{\sinh(2\beta)}$ . Using trigonometric identities, we have

$$h(\beta; k_1, k_2) = 2 \sinh(2\beta) + 2 \sinh(2\beta J_3) + 2 \sinh(2\beta) \left[ \alpha \cos^2\left(\frac{k_1+k_2}{2}\right) - \cos\left(\frac{k_1+k_2}{2}\right) \cos\left(\frac{k_1-k_2}{2}\right) \right]. \quad (2.23)$$

We can minimise separately on the variables  $\frac{k_1+k_2}{2}$  and  $\frac{k_1-k_2}{2}$ . There exists a minimiser satisfying  $\cos(\frac{k_1+k_2}{2}) \geq 0$  and  $\cos(\frac{k_1-k_2}{2}) = 1$ . The minimum is then easily found, namely

$$\min_{k_1, k_2} h(\beta; k_1, k_2) = \begin{cases} 0 & \text{if } \alpha \leq \frac{1}{2}, \\ 2 \sinh(2\beta)(1 - \alpha - \frac{1}{4\alpha}) & \text{if } \alpha \geq \frac{1}{2}. \end{cases} \quad (2.24)$$

The first case corresponds to  $k_1 = k_2 = 0$ . Suppose that  $\alpha \geq \frac{1}{2}$  and that  $g(\beta) + \min h(\beta; k_1, k_2) = 0$ . This is equivalent to

$$(1 + \sinh^2(2\beta)) \sqrt{1 + \alpha^2 \sinh^2(2\beta)} - (1 + \sinh^2(2\beta)) \alpha \sinh(2\beta) - \frac{\sinh^2(2\beta)}{4\alpha} = 0. \quad (2.25)$$

The solution is  $\alpha = \frac{\sinh(2\beta)}{2\sqrt{1+\sinh^2(2\beta)}} < \frac{1}{2}$ ; this contradicts the assumption that  $\alpha \geq \frac{1}{2}$ . This proves that when  $J_1 = J_2$  and with arbitrary  $J_3 \in \mathbb{R}$ , the condition for  $\beta_c$  is  $g(\beta_c) = 0$ , which is equivalent to the Cimasoni–Duminil-Copin–Li equation  $a(\beta_c) = 0$ .

Instead of looking for  $\beta_c$  as function of  $J_3$ , it is more convenient to look for  $J_3$  as function of  $t = \tanh \beta$ . The equation is then

$$J_3 = \frac{\operatorname{artanh} \frac{1-2t-t^2}{1+2t-t^2}}{\operatorname{artanh} t} \equiv j(t). \quad (2.26)$$

The derivative of the function  $j(t)$  is

$$j'(t) = -\frac{(1+t^2)\operatorname{artanh} t + 2t \operatorname{artanh} \frac{1-2t-t^2}{1+2t-t^2}}{2t(1-t^2)\operatorname{artanh}^2 t}. \quad (2.27)$$

It is not hard to check that  $\frac{1-2t-t^2}{1+2t-t^2} \geq -t$ ; it follows that the numerator above is positive so that  $j'(t) < 0$ . Further,  $j(t)$  goes to  $+\infty$  as  $t \rightarrow 0+$  and goes to  $-1$  as  $t \rightarrow 1-$ . Then,  $j^{-1}$  exists as a function  $(-1, \infty) \rightarrow \mathbb{R}_+$ ; it follows that Eq. (2.26) has a unique solution when  $J_3 > -1$  and no solutions otherwise. We also see that  $\beta_c \rightarrow 0$  as  $J_3 \rightarrow \infty$ , and  $\beta_c \rightarrow \infty$  as  $J_3 \rightarrow -1$ .

Finally, we check that  $g''(\beta_c) > 0$ . It is enough to check that  $a'(\beta_c) \neq 0$ . We have

$$a'(\beta) = -2(1-t^2)[1+t+(1-t)\tanh(\beta J_3)] - J_3(1+2t-t^2)[1-\tanh^2(\beta J_3)]. \quad (2.28)$$

At  $\beta = \beta_c$ , we have  $\tanh(\beta_c J_3) = \frac{1-2t-t^2}{1+2t-t^2}$ , where  $t = \tanh \beta_c$ . It is then possible to write  $a'(\beta_c)$  as

$$a'(\beta_c) = -\frac{4(1-t^2)(1+t^2+2tJ_3)}{1+2t-t^2}. \quad (2.29)$$

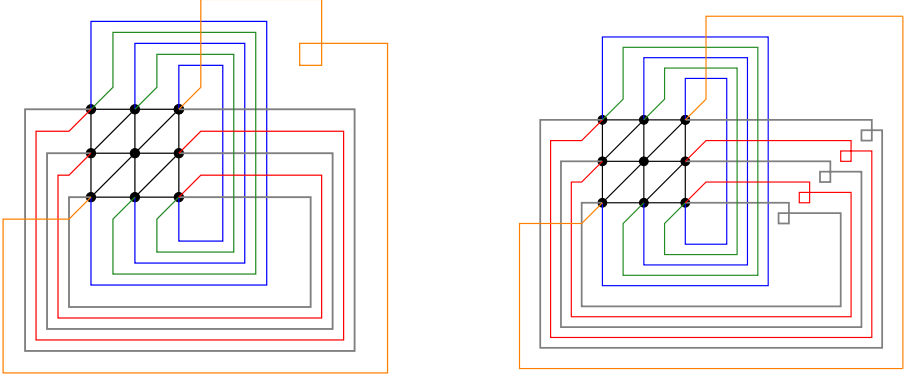


FIGURE 4. Two faithful projections of the graph  $(\mathbb{T}_{3,3}, \mathcal{E}_{3,3})$ . The handles cross at non-vertex locations; some handles cross themselves. The matrix  $K^{(1)}$  is defined on the left graph; the matrix  $K^{(2)}$  is defined on the right graph

This is clearly not 0 since  $t < 1$  and  $J_3 > -1$ .

The condition on  $h$  in Proposition 2.5 clearly holds.  $\square$

### 3. The Kac–Ward Identity

We rely on the extension of the Kac–Ward identity to “faithful projections” of non-planar graphs. It was proposed by Cimasoni [4] and used in [1, 14]. In order to accommodate negative weights, we need two faithful projections for  $\mathbb{T}_{L,M}$  with edges between nearest-neighbours. The graphs are  $G_1$  and  $G_2$ , and they are illustrated in Fig. 4. Here is a full description of the left graph:

- The vertices are  $(i, j)$  with  $1 \leq i \leq L$  and  $1 \leq j \leq M$ .
- There are edges represented by straight lines between  $(i, j)$  and  $(i + 1, j)$  for  $1 \leq i \leq L - 1$ ,  $1 \leq j \leq M$ ; between  $(i, j)$  and  $(i, j + 1)$  for  $1 \leq i \leq L$ ,  $1 \leq j \leq M - 1$ ; and between  $(i, j)$  and  $(i + 1, j + 1)$  for  $1 \leq i \leq L - 1$ ,  $1 \leq j \leq M - 1$ .
- There are edges represented by “handles” (continuous curves with winding number  $-1$ ) between  $(L, j)$  and  $(1, j)$  for  $1 \leq j \leq M$ ; between  $(L, j)$  and  $(1, j + 1)$  for  $1 \leq j \leq M - 1$ ; between  $(i, M)$  and  $(i, 1)$  for  $1 \leq i \leq L$ ; and between  $(i, M)$  and  $(i + 1, 1)$  for  $1 \leq i \leq L - 1$ .
- And there is a self-crossing handle between  $(L, M)$  and  $(1, 1)$  whose winding number is  $-2$ .
- The handles are drawn so that handles starting at  $(i, M)$  only cross the handles starting at  $(L, j)$  (and they cross them exactly once); the self-crossing handle belongs to both groups.

The second graph is similar, except that the oblique handle no longer self-crosses but the other horizontal handles all self-cross.

The Kac–Ward identity involves matrices indexed by directed edges. We denote  $\tilde{\mathcal{E}}_{L,M}$  the edges of  $\mathcal{E}_{L,M}$  with direction. The coupling constants defined in Eq. (2.2) can be extended to directed edges by assigning the same value  $J_e$  to both directions of the same edge; then, we let  $W$  to be the diagonal matrix whose element  $W_{e,e}$  is equal to  $\tanh J_e$ . We now introduce the Kac–Ward matrix  $K^{(1)}$  by

$$K_{e,e'}^{(1)} = 1_{e \triangleright e'} e^{\frac{i}{2}\angle_1(e,e') + \frac{i}{2}\angle_1(e)}, \quad e, e' \in \mathcal{E}_{L,M}. \quad (3.1)$$

Here,  $e \triangleright e'$  means that the endpoint of  $e$  is equal to the starting point of  $e'$  and also that  $e'$  is not equal to the reverse of  $e$  (the matrix is not “backtracking”).  $\angle_1(e, e') : \mathcal{E}_{L,M} \rightarrow (-\pi, \pi]$  is the angle between the end of  $e$  and the start of  $e'$  on the faithful projection  $G_1$ ;  $\angle_1(e) : \mathcal{E}_{L,M} \rightarrow \mathbb{R}$  is the integrated angle along the planar curve that represents the edge  $e$ .

Following [1], we define an average over even subgraphs: If  $f$  is a function on graphs, let

$$\langle f \rangle_{L,M} = \frac{1}{\tilde{Z}_{L,M}} \sum_{\Gamma \subset \mathcal{E}_{L,M} : \partial\Gamma = \emptyset} f(\Gamma) w(\Gamma) \quad (3.2)$$

where the normalisation is  $\tilde{Z}_{L,M} = \sum_{\Gamma \subset \mathcal{E}_{L,M} : \partial\Gamma = \emptyset} w(\Gamma)$ . The definition of the weight is  $w(\Gamma) = \prod_{e \in \Gamma} \tanh J_e$ . The boundary  $\partial\Gamma$  of a graph is the set of vertices whose incidence number is odd; the sum in the right-hand side is over even subgraphs. Notice that  $\tilde{Z}_{L,M}$  is always positive as can be seen from its relation to the Ising partition function, see (3.4).

With these definition, we have the remarkable Kac–Ward identity [1, Theorem 5.1]:

$$\sqrt{\det(1 - K^{(1)}W)} = \tilde{Z}_{L,M} \langle (-1)^{n_0^{(1)}(\Gamma)} \rangle_{L,M}. \quad (3.3)$$

Here,  $n_0^{(1)}(\Gamma)$  is the total number of crossings between all edges of  $\Gamma$  when the graph is projected on  $G_1$ .

It is worth noting that the right side of (3.3) is a multinomial in  $(W_{e,e})$ , something that is not apparent in the left side—there are remarkable cancellations indeed. This allows [1] to prove the identity for small  $(W_{e,e})$ ; the extension to larger values is automatic. The determinant cannot be negative and the sign of the square root cannot change.

We define the matrix  $K^{(2)}$  as in (3.1) but  $\angle_2(e, e')$  and  $\angle_2(e)$  are the corresponding angles on the faithful projection  $G_2$ . Analogously, we define  $n_0^{(2)}(\Gamma)$  for this projection.

The connection with the Ising model is through the high-temperature expansion, see, for example, [7, Section 3.7.3]. The partition function (2.4) is equal to

$$\begin{aligned} Z_{L,M}(J_1, J_2, J_3) &= 2^{LM} \left( \prod_{e \in \mathcal{E}_{L,M}} \cosh J_e \right) \sum_{\Gamma \subset \mathcal{E}_{L,M} : \partial\Gamma = \emptyset} w(\Gamma) \\ &= 2^{LM} \left( \prod_{e \in \mathcal{E}_{L,M}} \cosh J_e \right) \tilde{Z}_{L,M}. \end{aligned} \quad (3.4)$$

The strategy of Aizenman and Warzel [1] is to prove that  $\langle (-1)^{n_0^{(1)}(\Gamma)} \rangle_{L,M} \rightarrow 1$  as  $L, M \rightarrow \infty$ . This can be done when the coupling constants are positive, and small enough so the temperature is higher than the 2D critical temperature. (Then, duality is used to get the formula for low temperatures.) The presence of negative coupling constants necessitates a different approach. We first show in Lemma 3.1 that a combination of the two faithful projections gives the partition function, up to a correction. We then show in Lemma 3.2 that this correction vanishes in the limit  $L \rightarrow \infty$ , for fixed  $M$ . Denote by  $n_h(\Gamma)$  the number of horizontal handles of the subgraph  $\Gamma$ , that is, the number of handles in  $\Gamma$  that connect sites of the form  $(L, i)$  with sites  $(1, j)$ . Note that the total number of horizontal handles of  $\mathcal{E}_{L,M}$  is  $2M$ .

**Lemma 3.1.** *We have*

$$\sqrt{\det(1 - K^{(1)}W)} + \sqrt{\det(1 - K^{(2)}W)} = 2\tilde{Z}_{L,M} \left( 1 - \langle 1_{n_h(\Gamma) \text{ odd}} \rangle_{L,M} \right).$$

*Proof.* From Eq. (3.3), we have

$$\sqrt{\det(1 - K^{(1)}W)} + \sqrt{\det(1 - K^{(2)}W)} = \tilde{Z}_{L,M} \langle (-1)^{n_0^{(1)}(\Gamma)} + (-1)^{n_0^{(2)}(\Gamma)} \rangle_{L,M} \quad (3.5)$$

Let  $n_v(\Gamma)$  be the number of handles in  $\Gamma$  that connect sites of the form  $(i, M)$  with sites  $(j, 1)$  (excluding the handle between  $(L, M)$  and  $(1, 1)$ ) and let  $n_{hv}(\Gamma) = 0, 1$  be the indicator on whether the handle from  $(L, M)$  and  $(1, 1)$  is present. (Notice the asymmetric definition of  $n_v$  and  $n_h$ , as the oblique handle is included in  $n_h$  but not in  $n_v$ .) We have

$$\begin{aligned} 1_{n_0^{(1)}(\Gamma) \text{ odd}} &= 1_{n_h(\Gamma) \text{ odd}} (1_{n_{hv}(\Gamma)=0} 1_{n_v(\Gamma) \text{ odd}} + 1_{n_{hv}(\Gamma)=1} 1_{n_v(\Gamma) \text{ even}}); \\ 1_{n_0^{(2)}(\Gamma) \text{ odd}} &= 1_{n_h(\Gamma) \text{ odd}} (1_{n_{hv}(\Gamma)=0} 1_{n_v(\Gamma) \text{ even}} + 1_{n_{hv}(\Gamma)=1} 1_{n_v(\Gamma) \text{ odd}}). \end{aligned} \quad (3.6)$$

It follows that

$$1_{n_0^{(1)}(\Gamma) \text{ odd}} + 1_{n_0^{(2)}(\Gamma) \text{ odd}} = 1_{n_h(\Gamma) \text{ odd}}. \quad (3.7)$$

By combining the above relation with (3.5), using  $(-1)^{n_0^{(i)}(\Gamma)} = 1 - 2 \cdot 1_{n_0^{(i)}(\Gamma) \text{ odd}}$ , the lemma follows.  $\square$

**Lemma 3.2.** *For any  $J_1, J_2, J_3 \in \mathbb{R}$ , for any  $M \in \mathbb{N}$ , we have*

$$\lim_{L \rightarrow \infty} \langle 1_{n_h(\Gamma) \text{ odd}} \rangle_{L,M} = 0.$$

*Proof.* We condition on the horizontal handles (including possibly the self-crossing ones). We denote by  $\mathfrak{h}$  the set of handles that connect sites in the leftmost and rightmost columns:

$$\mathfrak{h} = \{ \{(1, j_1), (L, j'_1)\}, \dots, \{(1, j_k), (L, j'_k)\} \}. \quad (3.8)$$

Then, we define the support  $\text{supp}_1 \mathfrak{h}$ , resp.  $\text{supp}_L \mathfrak{h}$ , to be the set of vertices of the form  $(1, j_i)$ , resp.  $(L, j'_i)$ , that appear an odd number of times in  $\mathfrak{h}$ . Let  $\text{supp } \mathfrak{h} = \text{supp}_1 \mathfrak{h} \cup \text{supp}_L \mathfrak{h}$ . We let  $\tilde{\mathcal{E}}_{L,M}$  be the set of edges of the cylinder

(not the torus)  $\{1, \dots, L\} \times \mathbb{T}_M$ . With  $1_{\mathfrak{h}} = 1_{\mathfrak{h}}(\Gamma)$  the indicator function that the random graph  $\Gamma$  has set of handles  $\mathfrak{h}$ , we have

$$\begin{aligned} \langle 1_{n_{\mathfrak{h}}(\Gamma) \text{ odd}} \rangle_{L,M} &= \sum_{|\mathfrak{h}| \text{ odd}} \langle 1_{\mathfrak{h}} \rangle_{L,M} \\ &= \sum_{|\mathfrak{h}| \text{ odd}} \left( \prod_{i=1}^k \tanh J_{(1,j_i),(L,j'_i)} \right) \frac{1}{\widetilde{Z}_{L,M}} \sum_{\Gamma \subset \mathcal{E}_{L,M} : \partial\Gamma = \text{supp } \mathfrak{h}} w(\Gamma). \end{aligned} \quad (3.9)$$

We now consider an Ising model on the cylinder  $\{1, \dots, L\} \times \mathbb{T}_M$ . We have

$$\frac{1}{\widetilde{Z}_{L,M}^{\text{cyl}}} \sum_{\Gamma \subset \mathcal{E}_{L,M} : \partial\Gamma = \text{supp } \mathfrak{h}} w(\Gamma) = \left\langle \prod_{x \in \text{supp } \mathfrak{h}} \sigma_x \right\rangle_{L,M}^{\text{cyl}}. \quad (3.10)$$

Notice that the partition function  $\widetilde{Z}_{L,M}^{\text{cyl}}$  is almost equal to  $\widetilde{Z}_{L,M}$ ; either ratio is less than  $e^{2M(|J_1|+|J_3|)}$ . Next we introduce the transfer matrix  $T_{\eta,\eta'}$  between column configurations  $\eta, \eta' \in \{-1, +1\}^M$ :

$$T_{\eta,\eta'} = \exp \left\{ \sum_{i=1}^M \left( J_1 \eta_i \eta'_i + J_2 \eta_i \eta'_{i+1} + J_3 \eta_i \eta'_{i+1} \right) \right\}. \quad (3.11)$$

Here, we defined  $\eta_{M+1} \equiv \eta_1$ . The transfer matrix allows to write the Ising correlations above as

$$\begin{aligned} \left\langle \prod_{x \in \text{supp } \mathfrak{h}} \sigma_x \right\rangle_{L,M}^{\text{cyl}} &= \frac{1}{T^L} \sum_{\eta, \eta'} \langle \eta | T^{L-1} | \eta' \rangle \\ &\quad \left( \prod_{x \in \text{supp } {}_1\mathfrak{h}} \eta_x \right) \left( \prod_{y \in \text{supp } {}_L\mathfrak{h}} \eta'_y \right) e^{J_2 \sum_{i=1}^M \eta'_i \eta'_{i+1}}. \end{aligned} \quad (3.12)$$

The matrix elements of  $T$  are positive; by the Perron–Frobenius theorem there exist vectors  $|v\rangle, |w\rangle$  such that

$$\lim_{L \rightarrow \infty} \frac{T^{L-1}}{\text{Tr} T^L} = \frac{1}{\lambda_{\max}} |v\rangle \langle w|. \quad (3.13)$$

Here,  $\lambda_{\max} > 0$  is the largest eigenvalue of  $T$ . (It depends on  $M$ .) The vectors  $|v\rangle, |w\rangle$  can be decomposed in the basis  $\{|\eta\rangle\}$  of column configurations, and their coefficients have the spin-flip symmetry. Taking the limit  $L \rightarrow \infty$  in (3.12), one gets 0. Indeed, the sum over  $\eta$  is

$$\sum_{\eta} \left( \prod_{x \in \text{supp } {}_1\mathfrak{h}} \eta_x \right) \langle \eta | v \rangle \quad (3.14)$$

which is zero since  $\text{supp } {}_1\mathfrak{h}$  contains an odd number of vertices; the sum over  $\eta'$  also gives zero.  $\square$

Next we seek to calculate the determinants of  $1 - K^{(1)}W$  and  $1 - K^{(2)}W$ . For this, we first make the matrices translation-invariant so we can use the

Fourier transform. Let us define  $\tilde{K}^{(i)}$ ,  $i = 1, 2$  to be as  $K^{(i)}$ ,  $i = 1, 2$  but omitting the respective integrated angle of the handles:

$$\tilde{K}_{e,e'}^{(i)} = 1_{e \triangleright e'} e^{\frac{i}{2} \angle_i(e,e')} \quad i = 1, 2. \quad (3.15)$$

Actually,  $\tilde{K}_{e,e'}^{(1)} = \tilde{K}_{e,e'}^{(2)}$  and we shall write  $\tilde{K}_{e,e'}$  for either  $\tilde{K}_{e,e'}^{(1)}$  or  $\tilde{K}_{e,e'}^{(2)}$ . Then, we define modified diagonal matrices  $\tilde{W}_{e,e}^{(1)}$  and  $\tilde{W}_{e,e}^{(2)}$ ; matrix elements now depend on the direction of  $e$ :

$$\tilde{W}_{e,e}^{(1)} = \begin{cases} W_{e,e} e^{i\pi/L} & \text{if } e = \rightarrow, \\ W_{e,e} e^{-i\pi/L} & \text{if } e = \leftarrow, \\ W_{e,e} e^{i\pi/M} & \text{if } e = \uparrow, \\ W_{e,e} e^{-i\pi/M} & \text{if } e = \downarrow, \\ W_{e,e} e^{i\pi(\frac{1}{L} + \frac{1}{M})} & \text{if } e = \nearrow, \\ W_{e,e} e^{-i\pi(\frac{1}{L} + \frac{1}{M})} & \text{if } e = \searrow. \end{cases} \quad \tilde{W}_{e,e}^{(2)} = \begin{cases} W_{e,e} & \text{if } e = \rightarrow \text{ or } \leftarrow, \\ W_{e,e} e^{i\pi/M} & \text{if } e = \uparrow \text{ or } \nearrow, \\ W_{e,e} e^{-i\pi/M} & \text{if } e = \downarrow \text{ or } \searrow. \end{cases} \quad (3.16)$$

**Lemma 3.3.** *We have*

$$\det(1 - K^{(1)}W) = \det(1 - \tilde{K}\tilde{W}^{(1)}), \quad \det(1 - K^{(2)}W) = \det(1 - \tilde{K}\tilde{W}^{(2)}).$$

*Proof.* One can expand the determinants as products of directed loops as in [1, Theorem 3.2]. Let  $\gamma = (e_1, \dots, e_k)$  be a directed loop with  $\ell$  handles. (The self-crossing handle between  $(L, M)$  and  $(1, 1)$  is counted twice.) We have

$$\prod_{i=1}^k K_{e_i, e_{i+1}}^{(1)} = (-1)^\ell \prod_{i=1}^k \tilde{K}_{e_i, e_{i+1}}, \quad \prod_{i=1}^k \tilde{W}_{e_i, e_i}^{(1)} = (-1)^\ell \prod_{i=1}^k W_{e_i, e_i}. \quad (3.17)$$

Then, each loop gives the same contribution in  $\det(1 - K^{(1)}W)$  and in  $\det(1 - \tilde{K}\tilde{W}^{(1)})$ . The argument for  $\det(1 - K^{(2)}W)$  is the same, and counting only vertical and oblique handles between sites  $(i, M)$  and  $(j, 1)$ ,  $1 \leq i, j \leq L$ .  $\square$

**Lemma 3.4.** *Let  $\mathbb{T}_L^* = \frac{2\pi}{L} \mathbb{T}_L$  and recall that  $\tilde{\mathbb{T}}_L = \mathbb{T}_L^* + \frac{\pi}{L}$ .*

(a) *With  $k_3 = k_1 + k_2$ , we have*

$$\det(1 - \tilde{K}\tilde{W}^{(1)}) = \prod_{k_1 \in \tilde{\mathbb{T}}_L} \prod_{k_2 \in \tilde{\mathbb{T}}_M} \left[ \prod_{i=1}^3 (1 + \tanh^2 J_i) + 8 \prod_{i=1}^3 \tanh J_i - 2 \sum_{i=1}^3 \tanh J_i (1 - \tanh^2 J_{i+1}) (1 - \tanh^2 J_{i+2}) \cos k_i \right].$$

(b) *Again with  $k_3 = k_1 + k_2$ , we have*

$$\det(1 - \tilde{K}\tilde{W}^{(2)}) = \prod_{k_1 \in \mathbb{T}_L^*} \prod_{k_2 \in \tilde{\mathbb{T}}_M} \left[ \prod_{i=1}^3 (1 + \tanh^2 J_i) + 8 \prod_{i=1}^3 \tanh J_i - 2 \sum_{i=1}^3 \tanh J_i (1 - \tanh^2 J_{i+1}) (1 - \tanh^2 J_{i+2}) \cos k_i \right].$$

*Proof.* For (a), we label the set of directed edges as  $(x, \alpha)$  where  $x \in \mathbb{T}_{L,M}$  and  $\alpha \in A$ , with

$$A = \{\rightarrow, \leftarrow, \uparrow, \downarrow, \nearrow, \swarrow\}. \quad (3.18)$$

The Fourier coefficients are  $(k, \alpha)$  with  $k \in \mathbb{T}_{L,M}^* = \mathbb{T}_L^* \times \mathbb{T}_M^*$ . The Fourier transform is represented by the unitary matrix  $U$ :

$$U_{(k,\alpha),(x,\beta)} = \frac{1}{\sqrt{LM}} e^{-ikx} \delta_{\alpha,\beta}, \quad (3.19)$$

for  $x \in \mathbb{T}_{L,M}$ ,  $k \in \mathbb{T}_{L,M}^*$ , and  $\alpha, \beta \in A$ . Since  $\widetilde{W}_{e,e}$  depends only on  $\alpha \in A$ , we have

$$(U\widetilde{W}^{(1)}U^{-1})_{(k,\alpha),(k',\beta)} = \widetilde{W}_{\alpha}^{(1)} \delta_{k,k'} \delta_{\alpha,\beta}. \quad (3.20)$$

Further, straightforward Fourier calculations give

$$(U\widetilde{K}U^{-1})_{(k,\alpha),(k',\beta)} = \delta_{k,k'} \sum_{x \in \mathbb{T}_{L,M}} e^{ikx} \widetilde{K}_{(0,\alpha),(x,\beta)} \equiv \delta_{k,k'} \widehat{K}_{\alpha,\beta}(k), \quad (3.21)$$

with the matrix  $\widehat{K}(k)$  given by

$$\widehat{K}(k) = \begin{pmatrix} e^{ik_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-ik_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{ik_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-ik_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{i(k_1+k_2)} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-i(k_1+k_2)} \end{pmatrix} \\ \times \begin{pmatrix} 1 & 0 & e^{i\frac{\pi}{4}} & e^{-i\frac{\pi}{4}} & e^{i\frac{\pi}{8}} & e^{-i\frac{3\pi}{8}} \\ 0 & 1 & e^{-i\frac{\pi}{4}} & e^{i\frac{\pi}{4}} & e^{-i\frac{3\pi}{8}} & e^{i\frac{\pi}{8}} \\ e^{-i\frac{\pi}{4}} & e^{i\frac{\pi}{4}} & 1 & 0 & e^{-i\frac{\pi}{8}} & e^{i\frac{3\pi}{8}} \\ e^{i\frac{\pi}{4}} & e^{-i\frac{\pi}{4}} & 0 & 1 & e^{i\frac{3\pi}{8}} & e^{-i\frac{\pi}{8}} \\ e^{-i\frac{\pi}{8}} & e^{i\frac{3\pi}{8}} & e^{i\frac{\pi}{8}} & e^{-i\frac{3\pi}{8}} & 1 & 0 \\ e^{i\frac{3\pi}{8}} & e^{-i\frac{\pi}{8}} & e^{-i\frac{3\pi}{8}} & e^{i\frac{\pi}{8}} & 0 & 1 \end{pmatrix}. \quad (3.22)$$

Let us define

$$\widehat{W}^{(1)}(k) := \begin{pmatrix} t_1 e^{ik_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & t_2 e^{-ik_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & t_2 e^{ik_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & t_2 e^{-ik_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & t_3 e^{i(k_1+k_2)} & 0 \\ 0 & 0 & 0 & 0 & 0 & t_3 e^{-i(k_1+k_2)} \end{pmatrix} \quad (3.23)$$

where  $t_i = \tanh J_i$ ,  $i = 1, 2, 3$ . Then,

$$\begin{aligned} \det(1 - \widehat{K}\widetilde{W}^{(1)}) &= \det(1 - \widetilde{W}^{(1)}\widehat{K}) = \prod_{k \in \mathbb{T}_{L,M}^*} \det\left[1 - \widehat{W}^{(1)}\left(k + \left(\frac{\pi}{L}, \frac{\pi}{M}\right)\right)\widehat{K}(0)\right] \\ &= \prod_{k_1 \in \widetilde{\mathbb{T}}_L} \prod_{k_2 \in \widetilde{\mathbb{T}}_M} \det\left[1 - \widehat{W}^{(1)}((k_1, k_2))\widehat{K}(0)\right]. \end{aligned} \quad (3.24)$$



The first identity follows from a loop expansion, see [1, Theorem 3.2]. A calculation of the determinant by grouping the terms according to  $k_1 + k_2, k_1, k_2$  yields

$$\begin{aligned} \det[1 - \widehat{W}^{(1)}((k_1, k_2))\widehat{K}(0)] &= \prod_{i=1}^3 (1 + \tanh^2 J_i) + 8 \prod_{i=1}^3 \tanh J_i \\ &\quad - 2 \sum_{i=1}^3 \tanh J_i (1 - \tanh^2 J_{i+1})(1 - \tanh^2 J_{i+2}) \cos k_i \end{aligned} \quad (3.25)$$

where  $k_3 = k_1 + k_2$ . This gives (a).

The proof of (b) is similar. □

**Corollary 3.5.**

- (a) The determinants are nonnegative,  $\det(1 - \widetilde{K}\widetilde{W}^{(1)}) \geq 0$  and  $\det(1 - \widetilde{K}\widetilde{W}^{(2)}) \geq 0$ .  
 (b) Taking the logarithms, dividing by  $L$ , we have as  $L \rightarrow \infty$

$$\begin{aligned} \lim_{L \rightarrow \infty} \frac{1}{L} \log \det(1 - \widetilde{K}\widetilde{W}^{(1)}) &= \lim_{L \rightarrow \infty} \frac{1}{L} \log \det(1 - \widetilde{K}\widetilde{W}^{(2)}) \\ &= \int_{[-\pi, \pi]} dk_1 \sum_{k_2 \in \widetilde{\mathbb{T}}_M} \log \left[ \prod_{i=1}^3 (1 + \tanh^2 J_i) + 8 \prod_{i=1}^3 \tanh J_i \right. \\ &\quad \left. - \sum_{i=1}^3 2 \tanh J_i (1 - \tanh^2 J_{i+1})(1 - \tanh^2 J_{i+2}) \cos k_i \right]. \end{aligned}$$

*Proof.* (a) By Eq. (3.3) and Lemma 3.3, we obtain that both square roots of the above determinants are real. (b) This is a consequence of Lemma 3.4; taking the logarithm we obtain Riemann sums. □

*Proof of Theorem 2.1.* (a) From the high-temperature expansion (3.4), we observe that the finite-volume free energy with periodic boundary conditions satisfies

$$-f_{L,M}(J_1, J_2, J_3) = \log 2 + \log \left[ \prod_{i=1}^3 \cosh J_i \right] + \frac{1}{LM} \log [\widetilde{Z}_{L,M}]. \quad (3.26)$$

Using Lemma 3.1, Lemma 3.2 and Lemma 3.3, we see that the free energy on the infinite cylinder is

$$\begin{aligned} -f_M(J_1, J_2, J_3) &= \log 2 + \log \left[ \prod_{i=1}^3 \cosh J_i \right] \\ &\quad + \lim_{L \rightarrow \infty} \frac{1}{LM} \log \left[ \sqrt{\det(1 - \widetilde{K}\widetilde{W}^{(1)})} + \sqrt{\det(1 - \widetilde{K}\widetilde{W}^{(2)})} \right] \end{aligned} \quad (3.27)$$

By Corollary 3.5(a), we have

$$\begin{aligned} \log \sqrt{\det(1 - \widetilde{K}\widetilde{W}^{(1)})} &\leq \log \left[ \sqrt{\det(1 - \widetilde{K}\widetilde{W}^{(1)})} + \sqrt{\det(1 - \widetilde{K}\widetilde{W}^{(2)})} \right] \\ &\leq \max_{i=1,2} \log \sqrt{\det(1 - \widetilde{K}\widetilde{W}^{(i)})} + \log 2. \end{aligned} \quad (3.28)$$

Dividing by  $L$ , all terms above converge to the same limit as  $L \rightarrow \infty$  by Corollary 3.5(b). We get

$$\begin{aligned} -f(J_1, J_2, J_3) &= \log 2 + \log \left[ \prod_{i=1}^3 \cosh J_i \right] \\ &+ \frac{1}{4\pi M} \int_{[0, 2\pi]} dk_1 \sum_{k_2 \in \widetilde{\mathbb{T}}_M} \log \left[ \prod_{i=1}^3 (1 + \tanh^2 J_i) \right. \\ &\left. + 8 \prod_{i=1}^3 \tanh J_i - \sum_{i=1}^3 2 \tanh J_i (1 - \tanh^2 J_{i+1})(1 - \tanh^2 J_{i+2}) \cos k_i \right]. \end{aligned} \quad (3.29)$$

In order to get the expression of Theorem 2.1, one should use the hyperbolic identities  $1 + \tanh^2 x = \frac{\cosh(2x)}{\cosh^2 x}$  and  $\tanh x = \frac{\sinh 2x}{2 \cosh^2 x}$  and extract a factor  $(\prod_i \cosh J_i)^{-1}$ .  $\square$

## 4. The 1D Quantum Ising Model

One application of the cylinder formula of Theorem 2.1(a) deals with the one-dimensional quantum Ising model. It is well known that it can be mapped to a classical model in  $1 + 1$  dimensions, the extra dimension being the continuous interval  $[0, \beta]$  with periodic boundary conditions. A phase transition is only possible when both dimensions are infinite, which necessitates taking the limit of zero-temperature  $\beta \rightarrow \infty$ . The free energy of the quantum Ising model was first computed by Pfeuty [22] using the fermionic method of [24]. The results of this section are not new, but the Kac–Ward approach may have more appeal to some readers.

We consider the chain  $\{1, \dots, L\}$  with periodic boundary conditions. The Hilbert space is  $\mathcal{H}_L = \otimes_{i=1}^L \mathbb{C}^2$ . Let  $S^{(1)}$  and  $S^{(3)}$  denote the spin operators on  $\mathbb{C}^2$  whose matrices are

$$S^{(1)} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S^{(3)} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (4.1)$$

Then, we denote  $S_i^{(j)}$  the spin operators at site  $i \in \mathbb{Z}$ . With  $h \in \mathbb{R}$  the magnetic field, the Hamiltonian is

$$H_L = - \sum_{i=1}^L S_i^{(3)} S_{i+1}^{(3)} - h \sum_{i=1}^L S_i^{(1)}. \quad (4.2)$$

Here, the site  $i = L + 1$  is defined as  $i = 1$ . The partition function is

$$Z_L^{\text{qu}}(\beta, h) = \text{Tr}_{\mathcal{H}_\Lambda} e^{-\beta H_L}. \quad (4.3)$$

The finite-volume free energy is

$$f_L^{\text{qu}}(\beta, h) = -\frac{1}{\beta L} \log Z_L^{\text{qu}}(\beta, h). \quad (4.4)$$

Notice the division by  $\beta$ , which allows to get the ground state energy by taking the limit  $\beta \rightarrow \infty$ .

**Theorem 4.1.** *The infinite-volume free energy of the one-dimensional quantum Ising model is equal to*

$$\begin{aligned} f^{\text{qu}}(\beta, h) &= \lim_{L \rightarrow \infty} f_L^{\text{qu}}(\beta, h) = -\frac{1}{\beta} \log 2 \\ &\quad - \frac{1}{2\pi\beta} \int_{-\pi}^{\pi} dk \log \cosh\left(\frac{\beta}{4} \sqrt{1 + 4h^2 + 4h \cos k}\right). \end{aligned}$$

We prove this theorem by invoking the well-known fact that the  $d$ -dimensional quantum Ising model is equivalent to a  $(d + 1)$ -dimensional classical Ising model, the extra dimension being continuous; see Proposition 4.2. We check in Proposition 4.3 that the continuum limit can be taken *after* the infinite-volume limit. This allows to make direct use of Theorem 2.1. The remaining step is to take the continuum limit, and it is not entirely straightforward; the proof of Theorem 4.1 can be found at the end of this section.

**Proposition 4.2.** *Let us define coupling constants  $J_1^{(n)}, J_2^{(n)}$  by*

$$J_1^{(n)} = \frac{\beta}{4n}, \quad J_2^{(n)} = -\frac{1}{2} \log \frac{\beta h}{2n}.$$

*Then, we have the identity*

$$Z_L^{\text{qu}}(\beta, h) = \lim_{n \rightarrow \infty} Z_{L,n}^{\text{qu}}(\beta, h)$$

*with*

$$Z_{L,n}^{\text{qu}}(\beta, h) = \exp\left\{\frac{1}{2}Ln \log \frac{\beta h}{2n}\right\} Z_{L,n}(J_1^{(n)}, J_2^{(n)}).$$

*Here,  $Z_{L,n}(J_1^{(n)}, J_2^{(n)})$  is the partition function defined in Eq. (2.4) with  $J_3 = 0$ .*

*Proof.* By the Lie–Trotter formula,

$$\begin{aligned} \text{Tr } e^{-\beta H_L} &= \lim_{n \rightarrow \infty} \text{Tr} \left( e^{\frac{\beta}{n} \sum_{i=1}^L S_i^{(3)} S_{i+1}^{(3)}} \prod_{i=1}^L \left(1 + \frac{\beta h}{n} S_i^{(1)}\right) \right)^n \\ &= \lim_{n \rightarrow \infty} \sum_{\sigma^{(1)}, \dots, \sigma^{(n)}} \exp \left\{ \frac{\beta}{4n} \sum_{i=1}^L \sum_{k=1}^n \sigma_i^{(k)} \sigma_{i+1}^{(k)} \right\} \\ &\quad \prod_{i=1}^L \prod_{k=1}^n \langle \sigma_i^{(k)} | \left(1 + \frac{\beta h}{n} S^{(1)}\right) | \sigma_i^{(k+1)} \rangle. \end{aligned} \quad (4.5)$$

We now observe that

$$\langle \sigma | (1 + \frac{\beta h}{n} S^{(1)}) | \sigma' \rangle = e^{-J_2^{(n)} + J_2^{(n)} \sigma \sigma'} . \quad (4.6)$$

Inserting this identity in (4.5), we get the proposition.  $\square$

Next we check that we can exchange the infinite-volume and the continuum limits for the free energy. Let us define

$$f_{L,n}^{\text{qu}}(\beta, h) = -\frac{1}{L} \log \text{Tr} \left( e^{\frac{\beta}{n} \sum_{i=1}^L S_i^{(3)} S_{i+1}^{(3)}} e^{\frac{\beta h}{n} \sum_{i=1}^L S_i^{(1)}} \right)^n . \quad (4.7)$$

We already know that  $f_L^{\text{qu}}(\beta, h) = \lim_{n \rightarrow \infty} f_{L,n}^{\text{qu}}(\beta, h)$  for fixed  $L$ .

**Proposition 4.3.**

- (a) For fixed  $n$ , the limit  $L \rightarrow \infty$  of  $f_{L,n}^{\text{qu}}(\beta, h)$  exists (and is denoted  $f_{\infty,n}^{\text{qu}}(\beta, h)$ ).
- (b) We have

$$f^{\text{qu}}(\beta, h) = \lim_{L \rightarrow \infty} \lim_{n \rightarrow \infty} f_{L,n}^{\text{qu}}(\beta, h) = \lim_{n \rightarrow \infty} \lim_{L \rightarrow \infty} f_{L,n}^{\text{qu}}(\beta, h).$$

*Proof.* Since the trace of the Lie–Trotter product can be written as a classical partition function, see Proposition 4.2, we can proceed as with the usual proofs of thermodynamic limits, see [7], and we easily obtain (a).

The first equality in (b) is clear. For the second equality, we use the following estimates, which again follow from estimates on the classical partition function:

$$Z_{L,n}^{\text{qu}}(\beta, h)^k e^{-\frac{\beta k}{2}} \leq Z_{kL,n}^{\text{qu}}(\beta, h) \leq Z_{L,n}^{\text{qu}}(\beta, h)^k e^{\frac{\beta k}{2}} . \quad (4.8)$$

Taking  $k \rightarrow \infty$ , we get

$$f_{L,n}^{\text{qu}}(\beta, h) + \frac{1}{2L} \geq f_{\infty,n}^{\text{qu}}(\beta, h) \geq f_{L,n}^{\text{qu}}(\beta, h) - \frac{1}{2L} . \quad (4.9)$$

The rest of the proof is a standard  $\frac{\varepsilon}{3}$  argument. For any  $\varepsilon > 0$ , we can find  $L = L(\varepsilon)$  large enough so that for all  $n$ , we have

$$|f^{\text{qu}}(\beta, h) - f_L^{\text{qu}}(\beta, h)| \leq \frac{\varepsilon}{3}, \quad |f_{\infty,n}^{\text{qu}}(\beta, h) - f_{L,n}^{\text{qu}}(\beta, h)| \leq \frac{\varepsilon}{3} . \quad (4.10)$$

Then, we can find  $n_0 = n_0(\varepsilon)$  such that  $|f_L^{\text{qu}}(\beta, h) - f_{L,n}^{\text{qu}}(\beta, h)| \leq \frac{\varepsilon}{3}$  for all  $n \geq n_0$ . Then,

$$\begin{aligned} |f^{\text{qu}}(\beta, h) - f_{\infty,n}^{\text{qu}}(\beta, h)| &\leq |f^{\text{qu}}(\beta, h) - f_L^{\text{qu}}(\beta, h)| \\ &\quad + |f_L^{\text{qu}}(\beta, h) - f_{L,n}^{\text{qu}}(\beta, h)| + |f_{L,n}^{\text{qu}}(\beta, h) - f_{\infty,n}^{\text{qu}}(\beta, h)| \leq \varepsilon . \end{aligned} \quad (4.11)$$

This holds for any  $\varepsilon > 0$  provided  $n$  is large enough. This proves the second identity in (b).  $\square$

*Proof of Theorem 4.1.* We need the following identity:

$$\begin{aligned} \sum_{k_2 \in \widetilde{\mathbb{T}}_M} \log[\coth(2J_2) - \cos k_2] \\ = -M \log 2 + M \log \coth J_2 + 2 \log(1 + (\coth J_2)^{-M}) . \end{aligned} \quad (4.12)$$

It can be obtained by taking the limit  $J_1, J_3 \rightarrow 0$  in Theorem 2.1(a), as the expression converges to the free energy of the 1D Ising model in  $\mathbb{T}_M$ . The latter

is easily calculated with the 1D transfer matrices, yielding  $-\log(2 \cosh J_2) - \frac{1}{M} \log(1 + \tanh^M J_2)$ . We can substitute  $a = \coth(2J_2)$  in the left side of Eq. (4.12), and  $\coth J_2 = a + \sqrt{a^2 - 1}$  in the right side.

By Propositions 4.2 and 4.3, the free energy of the quantum Ising model is the limit  $n \rightarrow \infty$  of

$$\begin{aligned} f_{\infty,n}^{\text{qu}}(\beta, h) = & -\frac{n}{2} \log \frac{2\beta h}{n} - \frac{n}{2} \log \sinh(-\log \frac{\beta h}{2n}) \\ & - \frac{1}{4\pi} \int_{-\pi}^{\pi} dk_1 \sum_{k_2 \in \tilde{\mathbb{T}}_n} \log \left[ \cosh \frac{\beta}{2n} \coth(-\log \frac{\beta h}{2n}) \right. \\ & \left. - \frac{\sinh \frac{\beta}{2n}}{\sinh(-\log \frac{\beta h}{2n})} \cos k_1 - \cos k_2 \right]. \end{aligned} \quad (4.13)$$

We now use

$$\begin{aligned} \cosh \frac{\beta}{2n} &= 1 + \frac{1}{2} \left( \frac{\beta}{2n} \right)^2 + O\left(\frac{1}{n^4}\right). \\ \coth(-\log \frac{\beta h}{2n}) &= 1 + 2 \left( \frac{\beta h}{2n} \right)^2 + O\left(\frac{1}{n^4}\right). \\ \sinh \frac{\beta}{2n} &= \frac{\beta}{2n} + O\left(\frac{1}{n^3}\right). \\ \sinh(-\log \frac{\beta h}{2n}) &= \frac{1}{2} \left( \frac{2n}{\beta h} \right) (1 + O(\frac{1}{n^2})). \end{aligned} \quad (4.14)$$

Inserting in the previous expression for  $f_n(\beta, h)$  we obtain

$$\begin{aligned} f_{\infty,n}^{\text{qu}}(\beta, h) = & -\frac{n}{2} \log 2 + O\left(\frac{1}{n}\right) - \frac{1}{4\pi} \int_{-\pi}^{\pi} dk_1 \sum_{k_2 \in \tilde{\mathbb{T}}_n} \\ & \log \left[ 1 + \frac{1}{2} \left( \frac{\beta}{2n} \right)^2 \varepsilon(h, k_1)^2 + O\left(\frac{1}{n^3}\right) - \cos k_2 \right], \end{aligned} \quad (4.15)$$

where we introduced

$$\varepsilon(h, k_1) = \sqrt{1 + 4h^2 + 4h \cos k_1}. \quad (4.16)$$

We now use the identity (4.12) with  $a = 1 + \frac{1}{2} \left( \frac{\beta}{2n} \right)^2 \varepsilon(h, k_1)^2 + O\left(\frac{1}{n^3}\right)$ , in which case we have  $a + \sqrt{a^2 - 1} = 1 + \frac{\beta}{2n} \varepsilon(h, k_1) + O\left(\frac{1}{n^2}\right)$ . We get

$$\begin{aligned} f_{\infty,n}^{\text{qu}}(\beta, h) = & -\frac{n}{2} \log 2 + O\left(\frac{1}{n}\right) \\ & - \frac{1}{4\pi} \int_{-\pi}^{\pi} dk_1 \left\{ -n \log 2 + n \log \left( 1 + \frac{\beta}{2n} \varepsilon(h, k_1) + O\left(\frac{1}{n^2}\right) \right) \right. \\ & \left. + 2 \log \left( 1 + \left( 1 + \frac{\beta}{2n} \varepsilon(h, k_1) + O\left(\frac{1}{n^2}\right) \right)^{-n} \right) \right\} \\ = & O\left(\frac{1}{n}\right) - \frac{1}{4\pi} \int_{-\pi}^{\pi} dk_1 \left\{ \frac{\beta}{2} \varepsilon(h, k_1) + 2 \log \left( 1 + e^{-\frac{\beta}{2} \varepsilon(h, k_1)} \right) \right\}. \end{aligned} \quad (4.17)$$

Replacing  $\frac{\beta}{2} \varepsilon(h, k_1)$  by  $2 \log e^{\frac{\beta}{4} \varepsilon(h, k_1)}$  and combining the logarithms, we obtain the expression of Theorem 4.1.  $\square$

We finally discuss the “quantum phase transition” of the quantum Ising model. The free energy  $f^{\text{qu}}(\beta, h)$  of the one-dimensional model is clearly analytic for all  $\beta > 0, h \in \mathbb{R}$  (and in a complex neighbourhood), but interesting

behaviour can happen in the zero-temperature limit. Namely, we consider the ground state energy

$$e_0(h) = \lim_{\beta \rightarrow \infty} f^{\text{qu}}(\beta, h). \quad (4.18)$$

From Theorem 4.1, we get the exact expression

$$e_0(h) = -\frac{1}{8\pi} \int_{-\pi}^{\pi} dk \sqrt{1 + 4h^2 + 4h \cos k}. \quad (4.19)$$

One can check that the derivative of  $e_0$  is continuous. The second derivative is

$$\begin{aligned} e_0''(h) = & -\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{dk}{\sqrt{1 + 4h^2 + 4h \cos k}} \\ & + \frac{1}{2\pi} \int_{-\pi}^{\pi} dk \frac{(2h + \cos k)^2}{(1 + 4h^2 + 4h \cos k)^{3/2}}. \end{aligned} \quad (4.20)$$

The integrals are well behaved except possibly at  $h = \pm \frac{1}{2}$ . While the second integral has a limit as  $h \rightarrow \pm \frac{1}{2}$ , the first integral diverges logarithmically. Precisely, we can check that

$$e_0''(h) \sim \frac{1}{2\pi} \log |h \pm \frac{1}{2}| \quad (4.21)$$

around  $h = -\frac{1}{2}$  and  $h = \frac{1}{2}$ . As is well known, there are multiple ground states when  $|h| < \frac{1}{2}$  that display long-range order; there is a single disordered ground state when  $|h| > \frac{1}{2}$ . More information about the quantum Ising model can be found in the recent works [2, 3, 6, 8, 12, 19, 26].

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## Declarations

**Conflict of interest** None.

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