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INCOMPRESSIBLE PHASE IN LATTICE SYSTEMS OF INTERACTING BOSONS

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Abstract. Phase diagrams for a class of boson lattice models with small hopping and at low temperatures are rigorously described. We show that incompressibility is a general property of systems with strong interactions and conservation of total particle number. More precisely, we prove that the density (as a function of the pressure) exhibits plateaux at zero temperature. As an application, we investigate the Bose-Hubbard model with nearest and next nearest neighbour interactions and show that it exhibits long-range order. For magnetic systems, we show that the zero temperature susceptibility with respect to the magnetic field in the z-direction vanishes, if the Hamiltonian commutes with the z-component of the total spin.

Keywords: Lattice boson systems, incompressibility, zero susceptibility phase, Bose-Hubbard model, Mott insulator, Pirogov-Sinai Theory.

Contents

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1. The Bose-Hubbard model

1.1. **Introduction.** Lattice models of interacting bosons have been considered for different reasons. On the one hand they were used as models capturing important features of such systems as, for instance, ^{4}He absorbed in porous media, or superconductors where Cooper pairs are approximately bosonic quasiparticles. But more importantly, it was suggested that these systems could play an important role in the study of Bose-Einstein condensation[1] and superfluidity in interacting systems.

Widely used is the *Bose-Hubbard model* [FWGF] which describes bosonic particles hopping on a lattice. The basic ingredients are a hopping term for the kinetic energy of the bosons, and an on-site interaction proportional to the number of pairs of bosons at the same site,

$$H = -t \sum_{<x,y>} (a_x^\dagger a_y + a_y^\dagger a_x) + U_0 \sum_x \big(\hat{n}_x^2 - \hat{n}_x\big). \tag{1.1}$$

Here a_x and $a_x^\dagger$ are boson annihilation and creation operators at site x, $\hat{n}_x = a_x^\dagger a_x$ is the operator of the number of particles at the site x, the sum of hopping terms runs over nearest neighbours, and the on-site repulsive potential per pair is $2U_0$.

The zero temperature phase diagram was studied by Fisher *et al.* [FWGF] (with and without an additional random potential); their discussion suggested the phase diagram according to Fig. 1. It consists of domains of incompressible phases with integer densities near the $t = 0$ axis, and a domain of the superfluid phase. Calculations using pertubative techniques indicate that the lobes should be asymmetric [FM]. The nature of the transition between incompressible and superfluid phases is still not understood.

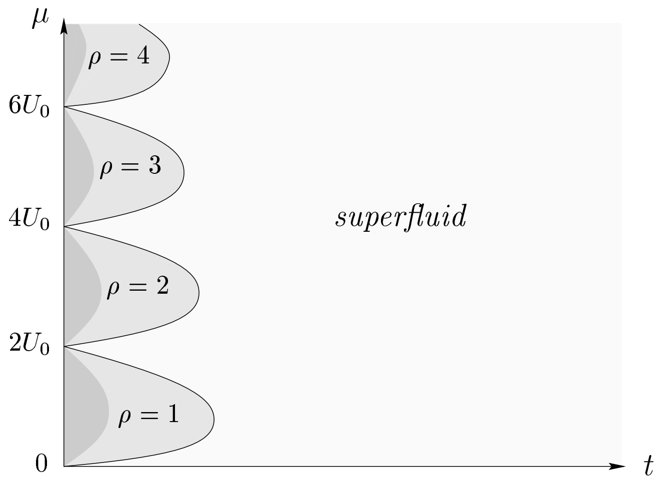


FIGURE 1. Zero temperature phase diagram for the Bose-Hubbard model in two dimensions. Lobes are incompressible phases with integer densities. Our results hold in darker regions near the $t = 0$ axis (and also for low temperatures).

A natural way to extend the Bose-Hubbard model is to introduce longer-range interactions between bosons. Let us consider the Hamiltonian defined on a d-dimensional lattice $\Lambda \subset \mathbb{Z}^d$ ($d \geqslant 2$) by

$$H = -t \sum_{<x,y>} (a_x^\dagger a_y + a_y^\dagger a_x) + U_0 \sum_x \big(\hat{n}_x^2 - \hat{n}_x\big) + \sum_{k=1}^{d} U_k \sum_{\substack{|x-y|=\sqrt{k} \\ |x-y|_\infty \leqslant 1}} \hat{n}_x \hat{n}_y. \tag{1.2}$$

[1]The Indian name “Bose” has to be pronounced “Bosh”; we thank Nilanjana Datta for this crucial information. Please pay attention in the sequel to boshons and boshonic systems.

The ground states are not difficult to find in two extremal cases, $t = 0$ and $t = \infty$ (i.e. setting all the couplings U_k to 0). The first case reduces to a problem of finding the ground states of a classical system. In the latter case, the bosons are independent and a Fourier transform diagonalizes the one-body Hamiltonian associated with the kinetic part; at zero temperature the particles exhibit a Bose-Einstein condensation.

Another useful approximation of (1.2) is the hard-core limit, $U_0 \to \infty$. Introducing the chemical potential μ, we get an anisotropic Heisenberg antiferromagnet with Hamiltonian

$$H_{\text{Heisenberg}} = \sum_{|x-y|=1} \big(U_1 \hat S^3_x \hat S^3_y - 2t \hat S^1_x \hat S^1_y - 2t \hat S^2_x \hat S^2_y \big) + \sum_{k=2}^{d} U_k \sum_{\substack{|x-y|=\sqrt{k} \\ |x-y|_\infty \leqslant 1}} \hat S^3_x \hat S^3_y - h \sum_x \hat S^3_x \tag{1.3}$$

where $h = \mu - 4U_1 - 4U_2$ in $d = 2$, $h = \mu - 6U_1 - 12U_2 - 8U_3$ in $d = 3$, ... The model (1.3) can be — at least in some regions of the phase diagram — directly treated (with the help of [BKU] or [DFF]) as a quantum perturbation of the classical lattice gas model

$$H_{\text{LG}} = \sum_{k=1}^{d} U_k \sum_{|x-y|=\sqrt{k}} n_x n_y - \mu \sum_x n_x, \tag{1.4}$$

with $n_x \in \{0, 1\}$. For large U_0 and small μ/U_0 the phase diagram of (1.4) and the $t = 0$ limit of (1.2) are identical. Since furthermore, (1.3) is the $U_0 \to \infty$ limit of (1.2) for all t, the $t - \mu$ phase diagrams of the full models (1.2) and (1.3) are thus expected to be similar for large U_0. We will meet the model (1.3) once more in Section 2.5 as an example of a situation displaying a phase with vanishing susceptibility (at zero temperature).

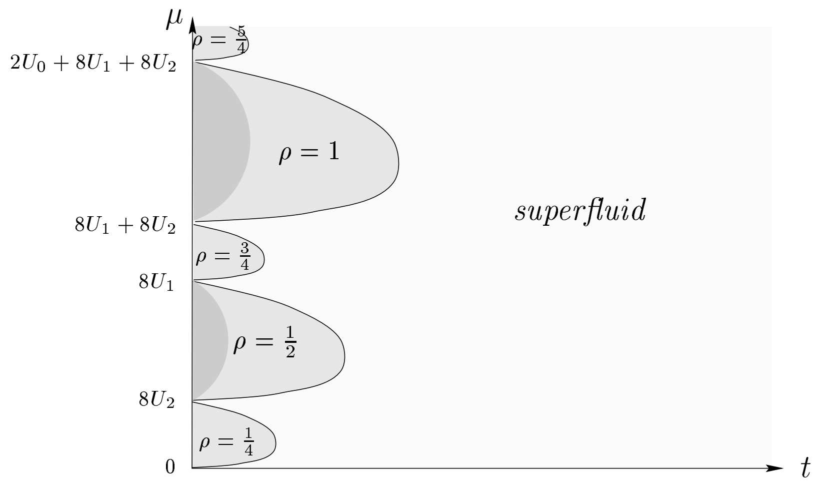


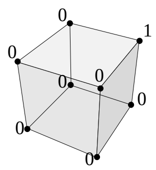
FIGURE 2. Zero temperature phase diagram for the Bose-Hubbard model in two dimensions with nearest and next nearest neighbour interactions. Incompressible (insulating) phases of given density are expected to exist in grey regions. In the darker regions the existence of such phases is rigorously established in the present paper. Supersolid phases might appear between solid and superfluid phases.

The zero temperature phase diagram of the two-dimensional version of (1.2) was considered in [OWBBFS]; in the case of large enough U_0 and $U_1 > 2U_2$ its qualitative shape is depicted in Fig. 2. The translation invariant phases $\rho = n$ were also present for the on-site Bose-Hubbard model. Nearest neighbour interactions are responsible for the occurrence of chessboard phases (with $\rho = n + \frac{1}{2}$). These phases are not translation invariant — the system exhibits symmetry breaking, a phenomenon known as diagonal long-range order or solidity. Finally, phases with quarter integer densities with alternating rows of density n and $n + \frac{1}{2}$ are present because of next nearest neighbour interactions.

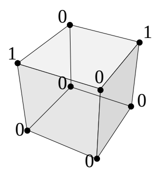
Monte-Carlo simulations have been used to investigate the transition from solid to superfluid [BSZK, OWBBFS]. The aim was to determine whether [with increasing t in (1.2)], the structure factor vanishes exactly when off-diagonal long-range order sets in. If on the contrary there is a range of parameters where the two properties coexist, one has a new phase that is solid and superfluid at the same time — the *supersolid phase*. However results of different calculations seem to be in contradiction and this question is still open.

It is interesting to discuss the degeneracy of the classical ($t = 0$) ground states of (1.2). While the integer and half integer phases have finite degeneracies, the quarter integer phases do not. Taking, e.g., the phase $\rho = 1/4$, there is alternatively an empty row without any boson and a row of staggered (antiferromagnetic) occupation pattern with 0 or 1 boson at each site. The degeneracy is essentially $2^{\frac{1}{2}|\Lambda|^{\frac{1}{2}}}$. In this paper, we will actually prove solidity, incompressibility and absence of superfluidity for the half integer and/or integer phases for small t and large β. Our theorems, however, do not cover quarter densities, since we require that the number of classical ground states is finite.

In three dimensions the model exhibits even more interesting degenerated phases. With well chosen parameters, the classical part of (1.2) has infinitely many ground states such that their restriction to any cube is a configuration of the following form (up to rotations and reflections)



Here all ground configurations have density 1/8 and their number grows as $2^{\frac{1}{4}|\Lambda|^{\frac{2}{3}}}$. For a different choice of parameters, the typical cube is as follows,



The density is locked to 1/4 and the degeneracy is roughly proportional to $2^{\frac{1}{2}|\Lambda|^{\frac{1}{3}}}$.

At non-zero temperature the degeneracy is removed since a finite number of particular configurations of alternated staggered rows have lower excitation energy; this theory of "dominating ground states" preferred by low energy fluctuations was presented in [BS]. One should expect that this domination is stable against perturbations with a small hopping term. On the other hand, it is natural to conjecture that a quantum perturbation itself also removes the degeneracy of the classical ground state, but leads to a different set of configurations. Again, one would expect that this conjectured phase is stable against small perturbations, this time stemming from the thermal fluctuations (the method of [DFFR] should apply here). If a coexistence surface separates the domain in the t, β plane that is dominated by thermal fluctuations from that which is dominated by quantum fluctuations, an interesting transition occurs, driven by the competition between two different kinds of fluctuations. We intend to study this phenomenon further in the future.

In order to state our results in $d = 2$, let us return to the Hamiltonian (1.2). We will assume that $U_0 > 0$, and

$$U_0 > 4U_1 + 4U_2, \qquad U_1 > 2U_2 \geqslant 0 \tag{1.5}$$

(or $U_1 = U_2 = 0$). Under this assumption, it is not difficult to show that the positive real line decomposes into consecutive intervals $Q_0^{(1)}$, H_0, $Q_0^{(2)}$, I_1, $Q_1^{(1)}$, $H_1, \dots$ corresponding to classical ground states with densities $\rho_0 = n + \frac{1}{4}$ for $\mu \in Q_n^{(1)}$, $\rho_0 = n + \frac{1}{2}$ for $\mu \in H_n$, $\rho_0 = n + \frac{3}{4}$ for $\mu \in Q_n^{(2)}$ and $\rho_0 = n$ for $\mu \in I_n$. In I_n, the corresponding classical ground state has period 1, while in H_n and $Q_n^{(i)}$ they are not translation invariant: in H_n the ground states are chessboard like configurations with sublattice occupation numbers n and $n+1$ respectively. For $\mu \in Q_n^{(1)}$ (resp. $Q_n^{(2)}$) the ground states consist of alternating rows of constant and staggered configurations. See Section 4.2 for a proof.

Our main result for the 2 dimensional Bose-Hubbard model is the statement that small quantum and small thermal fluctuations do not destroy this structure. To be more precise, we will show for sufficiently small t and large β (see Fig. 2 for the corresponding regions in the (t, μ) diagram)

- There is no off-diagonal long-range order [PO, Yang] for $\mu \in I_n$ or H_n, i.e.
$$\big|\langle a_x^\dagger a_y \rangle_{\Lambda,\beta}\big| \leqslant O(1)\, \mathrm{e}^{-|x-y|/\xi}\,.$$
- As $\beta \to \infty$, the compressibility vanishes,
$$\Big|\frac{\partial \rho}{\partial \mu}\Big| \leqslant O(1)\, \mathrm{e}^{-c\beta}\,, \quad c > 0,$$
and the density ρ approaches its classical value,
$$|\rho - \rho_0| \leqslant O(1)\, \mathrm{e}^{-c\beta}\,,$$
if $\mu \in I_n$ or H_n. Here, the density ρ is defined as
$$\rho = \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \langle \hat{n}_x \rangle.$$
- For $\mu \in H_n$, the local density $\langle \hat{n}_x \rangle$ is staggered
$$\langle \hat{n}_x \rangle = \rho + (-1)^x \Delta, \quad \Delta > 0$$
where $(-1)^x = (-1)^{x_1 + x_2}$. As a consequence, the structure factor
$$S(k) = \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|^2} \sum_{x,y \in \Lambda} \mathrm{e}^{\mathrm{i}k(x-y)}\, \langle \hat{n}_x \hat{n}_y \rangle$$
is different from zero for $k = (\pi, \pi) \neq 0$.

We will state these results more precisely in the following section, including a precise definition of the corresponding infinite volume quantum Gibbs states.

1.2. **Results.** For given μ, we call *classical ground states* the configurations $\{n_x\}_{x \in \Lambda}$ which appear in the zero-temperature phase diagram for $(t = 0, \mu)$ and we use ρ_0 to denote the corresponding mean density, $\rho_0 = \lim \frac{1}{|\Lambda|} \sum_{x \in \Lambda} n_x$. For each $n \in \mathbb{N}$ consider the disjoint intervals

$$I_n = \{\mu : (2U_0 + 8U_1 + 8U_2)n - 2U_0 < \mu < (2U_0 + 8U_1 + 8U_2)n\},$$
$$H_n = \{\mu : (2U_0 + 8U_1 + 8U_2)n + 8U_2 < \mu < (2U_0 + 8U_1 + 8U_2)n + 8U_1\}$$

and

$$Q_n^{(1)} = \{\mu : (2U_0 + 8U_1 + 8U_2)n < \mu < (2U_0 + 8U_1 + 8U_2)n + 8U_2\},$$

$$Q_n^{(2)} = \{\mu : (2U_0 + 8U_1 + 8U_2)n + 8U_1 < \mu < (2U_0 + 8U_1 + 8U_2)n + 8U_1 + 8U_2\}.$$

Next theorem states that for small t and large β, the mean of the operator

$$\hat{\rho}_\Lambda = \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \hat{n}_x, \tag{1.6}$$

in the quantum state is close (more precisely: exponentially close with respect to β) to the corresponding classical value ρ_0 whenever $\mu \in I_n$ and $\mu \in H_n$. In the same region we get a bound on the compressibility

$$\kappa_T = \frac{\partial}{\partial\mu} \langle \hat{\rho}_\Lambda \rangle \Big|_{\beta,V}. \tag{1.7}$$

Theorem 1.1 (Two-dimensional Bose-Hubbard model)**.** *Assume that the coupling constant satisfy the conditions* (1.5)*, and that the lattice Λ is a rectangle that sides have even length, with periodic boundary conditions. Then for each $\mu \in I_n$, or $\mu \in H_n$, there exists $t_0(\mu)$ and $\beta_0(\mu)$ such that for $\beta \geqslant \beta_0(\mu)$, $t \leqslant t_0(\mu)$*

i) **Incompressibility.**

For some constants $C, C' < \infty$ and $c, c' > 0$, one has

$$\big| \langle \hat{\rho}_\Lambda \rangle_{\Lambda,\beta} - \rho_0 \big| \leqslant C\, \mathrm{e}^{-c\beta} \tag{1.8}$$

for every Λ; recall that $\rho_0 = n$ for $\mu \in I_n$ and $\rho_0 = n + \frac{1}{2}$ for $\mu \in H_n$. Moreover,

$$\Big| \frac{\partial}{\partial\mu} \langle \hat{\rho}_\Lambda \rangle_{\Lambda,\beta} \Big| \leqslant C'\, \mathrm{e}^{-c'\beta}\,. \tag{1.9}$$

When β goes to infinity the compressibility vanishes.

ii) **Absence of superfluidity.**

There is no off-diagonal long-range order. More precisely,

$$\big| \langle a_x^\dagger a_y \rangle_{\Lambda,\beta} \big| \leqslant C''\, \mathrm{e}^{-|x-y|/\xi} \tag{1.10}$$

for some $C'', \xi < \infty$.

iii) **Solid phase.**

For $\mu \in H_n$, and with the boundary conditions taken to be one the two chessboard configurations[2]*, there is long-range order, i.e. with $k = (\pi,\pi)$,*

$$\frac{1}{|\Lambda|^2} \Big| \sum_{x,y\in\Lambda} \mathrm{e}^{\mathrm{i}k(x-y)}\, \langle \hat{n}_x \hat{n}_y \rangle_{\Lambda,\beta} \Big| > \text{const} \tag{1.11}$$

for some positive constant uniform in Λ.

Remark: When n becomes large, the domains for (t,μ) with corresponding incompressible phases are smaller. In our representation the reason is that the boson damping condition (2.9) is getting weaker when increasing n.

The physical significance of the theorem may be more clear when considering the relation between density and pressure. Recall that $\frac{\partial\rho}{\partial\mu} = \rho\frac{\partial\rho}{\partial p}$ (fixed variables are the temperature and the volume). Then we may derive the existence of plateaux in the graph of Fig. 3.

A system of free bosons without mutual interaction features Bose-Einstein condensation and in particular the claims i) and ii) do not hold. Incompressibility and absence of off-diagonal long-range order are thus a direct effect of the interactions between the particles. This situation is analogous to Mott insulator transition in fermionic systems, where an insulating phase may appear because of the interactions between fermions — in contrast to the situation in band

[2]See Section 2 for a precise definition of the boundary conditions.

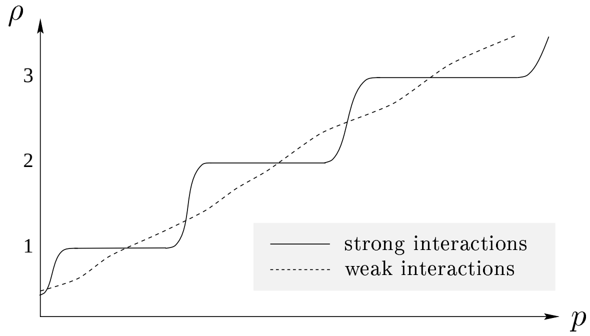


FIGURE 3. Graph of the density as a function of the pressure, at zero temperature (and in the case of the Bose-Hubbard model with only on-site interactions).

theory, where the insulating phase is due to an external periodic potential. So it is generally said that the Bose-Hubbard system forms a Mott insulator in the incompressible phase[3].

The proof of Theorem 1.1 is given in Section 4 and relies on Theorems 2.1 and 2.3 below, and on the contour representation introduced in Section 3.

2. General boson system

A quantum model may be expanded, using for instance the Trotter formula, so as to obtain a classical contour model in one more dimension [Gin]. If the contours have a small probability of occurence, which is the case if the temperature is low and the off diagonal terms in the Hamiltonian small, we can use techniques developped for classical spin systems, namely, the Peierls argument [Pei] (or its extension to asymmetric situations, the Pirogov-Sinai theory [PS, Sin, Zah, BI]; see e.g. [Kot] for an introduction to these ideas), to obtain a rigorous description of the states in the thermodynamic limit, and of the phase diagram. General theory for quantum spin lattice models has been recently proposed in [BKU] and [DFF]. In the latter the sign problem arising for fermions was dealt with, so that the results also apply to lattice fermionic systems. The theory was generalized to situations where the diagonal part of the Hamiltonian has degeneracies that can be removed by non diagonal terms [DFFR, FR].

In the sequel we present an extension of these ideas to the case of boson systems. New difficulties arise here since we are dealing with Hilbert spaces of infinite dimensions (also for finite systems) and with unbounded operators. For technical reasons we assume that the Hamiltonian is of finite range and conserves the total number of particles. The resulting theory will be then used to analyse solidity and absence of superfluidity in the Bose-Hubbard systems with Hamiltonian (1.2).

We consider the lattice $\mathbb{Z}^d$; a classical configuration is a function n: $\mathbb{Z}^d \to \mathbb{N}$, $x \mapsto n_x$. For every finite $A \subset \mathbb{Z}^d$ we define the vector

$$|n_A\rangle = \prod_{x\in A} \bigl(a_x^\dagger\bigr)^{n_x} \, |0\rangle. \tag{2.1}$$

The space spanned on base vectors $|n_\Lambda\rangle$ with finite Λ is the Fock space $\mathcal{H}_\Lambda$. Notice that if $A \subset \Lambda$, we can consider $|n_A\rangle$ as a vector of $\mathcal{H}_\Lambda$. In the following we will always tacitly make this identification. The set of base vectors of $\mathcal{H}_\Lambda$ is denoted by $\mathcal{B}_\Lambda$. We are interested in Bose

[3]We intentionally avoid the term "Mott insulating phase" because *stricto sensu* it is not a phase: the fact that a system can be considered to have Mott insulator behaviour depends actually on the chosen microscopic description.

systems with Hamiltonian of the form

$$H = H^{(0)} + \lambda V, \tag{2.2}$$

where $H^{(0)}$ is diagonal in the occupation number basis. Before writing the precise assumptions, similar to those of [BKU], let us make three important remarks.

Remarks: 1. We consider lattice $\mathbb{Z}^d$ for sake of simplicity, but our results hold for any periodic lattices (with minor changes in constants).

2. The operator $H^{(0)}$ may be periodic rather than translation invariant. However, we need that $\Phi_x(g^{(m)}_{U(x)})$ be independent of x (see below for definitions); in the periodic case we can define new operators by averaging over a cell whose size is given by the least common multiple of the periods. The operator V does not need to be translation invariant, provided that (2.11)–(2.13) are valid for any $A \subset \Lambda$.

3. We might be interested in systems of particles with internal degrees of freedom, such as spins, or systems with different kinds of particles, or mixed systems with localized spins interacting with itinerant particles, or... . To make the link between these systems and our framework of spinless bosons, a way is to split each lattice site according to the internal degrees of freedom and to redefine the Hilbert space and the operators to be compatible with this new picture. The obtained lattice will be different from $\mathbb{Z}^d$ and the operators will not be translation invariant (however periodic), but from remarks 1. and 2. above this is not a relevant problem.

2.1. **Assumptions on diagonal terms.** The Hamiltonian $H^{(0)}$ will be given in terms of classical potentials Φ_A — functions of occupation number configurations n_A on $A \subset \mathbb{Z}^d$, $|A| \leqslant R_0$ (finite range). Namely, let

$$\Phi_x(n) = \sum_{A \ni x} \frac{1}{|A|} \Phi_A(n_A). \tag{2.3}$$

Then $H^{(0)}$ is formally given as

$$H^{(0)}_\Lambda = \sum_{x \in \Lambda} \Phi_x(\hat{n}). \tag{2.4}$$

Actually, we will introduce the operators $H^{(0)}_{q,\Lambda}$ depending on the boundary conditions — see Section 2.3.

The classical potentials $\Phi_A(n_A)$ are supposed to be chosen to satisfy the assumptions that allow to apply Pirogov-Sinai theory to $H^{(0)}$ (see [BKU] for more detailed explanations). Namely, they are supposed to be translation invariant $[\Phi_{A+x}(n_{A+x}) = \Phi_A(n_A)]$ and to depend on a vector parameter $\boldsymbol{\mu} \in \mathcal{U} \subset \mathbb{R}^\nu$. Further, there is a finite number of periodic (occupation number) configurations $g = \{n_x; x \in \mathbb{Z}^d\}$ to be denoted $g^{(1)}$ to $g^{(r)}$ such that for each $\boldsymbol{\mu} \in \mathcal{U}$ the set of periodic ground states $G(\boldsymbol{\mu})$ is included in $G := \{g^{(1)}, \dots, g^{(r)}\}$. Actually, denoting

$$e_m = e_m(\boldsymbol{\mu}) = \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \sum_{A \cap \Lambda \neq \emptyset} \Phi_A(g^{(m)}), \tag{2.5}$$

we have

$$G(\boldsymbol{\mu}) = \{g^{(m)}; e_m(\boldsymbol{\mu}) = e_0(\boldsymbol{\mu}) := \min_{q=1,\dots,r} e_q(\boldsymbol{\mu})\}. \tag{2.6}$$

We also assume (cf [BKU]) that there exists $\boldsymbol{\mu}_0 \in \mathcal{U}$ such that $G(\boldsymbol{\mu}_0) = G$, that $e_m(\boldsymbol{\mu})$ are $\mathcal{C}^1$ functions in $\mathcal{U}$ and that the matrix of derivatives

$$E = \left(\frac{\partial e_m(\boldsymbol{\mu})}{\partial \mu_i} \right) \tag{2.7}$$

has rank $r-1$ for all $\boldsymbol{\mu} \in \mathcal{U}$ with uniform bounds on the inverse of the corresponding submatrices. Next, introducing $U(x) := \{y \in \mathbb{Z}^d \mid \operatorname{dist}(x,y) \leqslant R_0\}$, we assume the *Peierls condition*: there exists γ_0 such that

$$\Phi_x(n_{U(x)}) \geqslant e_0(\boldsymbol{\mu}) + \gamma_0 \tag{2.8}$$

whenever $n_{U(x)} \notin \{g^{(1)}_{U(x)}, \dots, g^{(r)}_{U(x)}\}$, and, in addition, the *boson damping condition*,

$$\Phi_x(n_{U(x)}) \geqslant e_0(\boldsymbol{\mu}) + a(n_x - b), \tag{2.9}$$

for some fixed positive γ_0, a, b. We also need that the derivatives of Φ_x with respect to μ_i are not too big; namely, there must exist constants C_0 and c_0 such that

$$\Big| \frac{\partial}{\partial \mu_i} \Phi_x(n_{U(x)}) \Big| \leqslant C_0 \, \mathrm{e}^{c_0 n_x} . \tag{2.10}$$

Remark that the diagonal Hamiltonian is not bounded [see assumption (2.9)], its domain is in fact not necessarily dense in $\mathcal{H}_\Lambda$, but the operator $\mathrm{e}^{-H^{(0)}_\Lambda}$ is bounded because of (2.8) and its domain is therefore all of $\mathcal{H}_\Lambda$.

2.2. **Assumptions on hopping terms.** Let $\mathcal{A}_0$ be the set of (connected) supports for the quantum perturbations. The term V in (2.2) will be given in terms of the operators $\hat{t}_A$, $A \in \mathcal{A}_0$, that are assumed to be densely defined on $\mathcal{H}_A$ (and thus also on $\mathcal{H}_\Lambda$ for any $\Lambda \supset A$. Further, they are supposed to be of finite range, $|A| \leqslant K$ for some $K < \infty$, translation invariant (identification of operators $\hat{t}_A$ and $\hat{t}_{A+x}$ on $\mathcal{H}_A$ and $\mathcal{H}_{A+x}$, respectively) and commuting with the operators $N_A = \sum_{x \in A} \hat{n}_x$, $[\hat{t}_A, N_A] = 0$. Moreover, denoting $\mathcal{H}_{A,N}$ the subspace of $\mathcal{H}_A$ spanned by the vectors $\{ |n_A\rangle \in \mathcal{B}_A \mid \sum_{x\in A} n_x = N\}$ and $P_{A,N}$ the orthogonal projector $P_{A,N} : \mathcal{H}_A \to \mathcal{H}_{A,N}$, we require that

$$\|\hat{t}_A P_{A,N}\| \leqslant N, \tag{2.11}$$

$$\Big\| \frac{\partial}{\partial \mu_i} \hat{t}_A P_{A,N} \Big\| \leqslant N, \tag{2.12}$$

$$\# \, \{n' \mid \langle n'_\Lambda | \, \hat{t}_A \, |n_\Lambda\rangle \neq 0\} \leqslant k, \tag{2.13}$$

for some $k < \infty$ and for any occupation number configuration. The conditions (2.11)–(2.13) are, clearly, trivially satisfied for standard hopping terms of the form $\hat{t}_{\{x,y\}} = \frac{1}{2} a^\dagger_x a_y + \frac{1}{2} a^\dagger_y a_x$. Notice that $[\hat{t}_A, N_A] = 0$ when $\hat{t}_A$ is a monomial that contains the same number of creation and annihilation operators.

2.3. **Stability of the phase diagram.** Let us first clarify how we will treat the boundary conditions. Whenever n is an occupation number configuration and $\Lambda \subset \mathbb{Z}^d$ is a finite set, we consider the configuration $n^{(q,\Lambda)}$ that equals n in Λ and is identified with the boundary condition $g^{(q)}$ outside of Λ, $n^{(q,\Lambda)} = (n_\Lambda, g^{(q)}_{\Lambda^c})$. Let us introduce the operators

$$H^{(0)}_{q,x} = \sum_{n_\Lambda} \Phi_x(n^{(q,\Lambda)}) \, |n_\Lambda\rangle\langle n_\Lambda| \tag{2.14}$$

and define

$$H_{q,\Lambda} = \sum_{x \in \Lambda} H^{(0)}_{q,x} + \sum_{A \subset \Lambda} \hat{t}_A. \tag{2.15}$$

These operators are acting on $\mathcal{H}_\Lambda$ and the boundary conditions q appear here as parameters.

To be able to include e.g. hard core interactions in $H^{(0)}_{q,x}$, we do not assume that it is densely defined. However, with $\mathcal{D}$ its domain, we can consider $\mathrm{e}^{-\beta H^{(0)}_{q,x}}$ *defined* by

$$\langle n_\Lambda|\, \mathrm{e}^{-\beta H^{(0)}_{q,x}}\, |n'_\Lambda\rangle = \langle n_\Lambda|n'_\Lambda\rangle \begin{cases} \mathrm{e}^{-\beta\langle n_\Lambda|\, H^{(0)}_{q,x}\,|n_\Lambda\rangle} & \text{if } |n_\Lambda\rangle \in \mathcal{D}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.16}$$

In this case, the operator $\mathrm{e}^{-\beta H_{q,\Lambda}}$ is directly defined via Duhamel expansion involving only well defined terms $\mathrm{e}^{-\beta H^{(0)}_{q,x}}$ and $\hat{t}_A$, see (3.8) and (3.9).

The main objects of our interest are the quantum states

$$\langle \cdot \rangle_{q,\Lambda} = \frac{1}{Z_{q,\Lambda}} \mathrm{Tr}_{\,\mathcal{H}_\Lambda} \left(\cdot \; \mathrm{e}^{-\beta H_{q,\Lambda}} \right), \tag{2.17}$$

introduced by taking the operator $H_{q,\Lambda}$ with the boundary condition fixed as $g^{(q)}$ and the normalization factor given by the partition function $Z_{q,\Lambda} = \mathrm{Tr}_{\,\mathcal{H}_\Lambda} \mathrm{e}^{-\beta H_{q,\Lambda}}$.

Our main result is that, deforming slightly the phase diagram of the "classical" Hamiltonian $H^{(0)}$, we get the full phase diagram of $H^{(0)} + \lambda V$ with phases that can be linked to ground configurations $g^{(q)}$. Two theorems repeating almost *verbatim* the corresponding statements from [BKU] can be formulated.

Consider, for each x in $\mathbb{Z}^d$ and any local operator Ψ (operator on $\mathcal{H}_{\mathrm{Supp}\,\Psi}$ for some finite set $\mathrm{Supp}\,\Psi \subset \Lambda$), the translate $t_x(\Psi)$. We will always suppose that the domain of operator Ψ is dense in $\mathcal{H}_\Lambda$ and contains every vector $|n_\Lambda\rangle \in \mathcal{B}_\Lambda$. Ψ may not be symmetric — later on we shall consider in its role the operators $a^\dagger_x$ and a_y for the proof of the absence of off-diagonal long-range order in the Bose-Hubbard model. We shall, however, always suppose that Ψ is a *moderately off-diagonal* operator. Namely, denoting $N_A(n) := \sum_{x\in A} n_x$, we assume that there exists a constant $c_\Psi < \infty$ such that

$$\sum_{|n'_\Lambda\rangle \in \mathcal{B}_\Lambda} |\langle n'_\Lambda|\, \Psi\, |n_\Lambda\rangle| \leqslant \mathrm{e}^{c_\Psi N_{\mathrm{Supp}\,\Psi}(n)} \tag{2.18}$$

for any n such that $|n_\Lambda\rangle \in \mathcal{B}_\Lambda$; $\mathrm{Supp}\,\Psi \subset \Lambda$ is the support of Ψ.

Define, finally, $\Lambda(L)$ as the box

$$\Lambda(L) = \{x \in \mathbb{Z}^d \mid |x_i| \leqslant L \quad \text{for all} \quad i = 1, \dots, d\}. \tag{2.19}$$

Theorem 2.1. *Let $d \geqslant 2$ and let $H^{(0)}$ be a Hamiltonian obeying the assumptions of Section 2.1. Then there are constants $0 < \beta_0 < \infty$ and $0 < \lambda_0 < \infty$, such that for all hopping terms V obeying the assumptions of Section 2.2, all $\beta \geqslant \beta_0$ and all $\lambda \in \mathbb{C}$ with $|\lambda| \leqslant \lambda_0$, there are constants ξ_q and continuously differentiable functions $f_q(\boldsymbol{\mu})$, $q = 1, \dots, r$, such that the following statements hold true whenever*

$$a_q(\beta, \lambda, \boldsymbol{\mu}) \;:=\; \mathrm{Re}\, f_q(\boldsymbol{\mu}) - \min_m \mathrm{Re}\, f_m(\boldsymbol{\mu}) = 0\,. \tag{2.20}$$

i) The infinite volume free energy corresponding to $Z_{q,\Lambda(L)}$ exists and equals f_q,

$$f_q = -\lim_{L\to\infty} \frac{1}{|\Lambda(L)|} \frac{1}{\beta} \log Z_{q,\Lambda(L)} \tag{2.21}$$

ii) The infinite volume limit

$$\langle \Psi \rangle_q = \lim_{L\to\infty} \langle \Psi \rangle_{q,\Lambda(L)} \tag{2.22}$$

exists for all moderately off-diagonal local operators Ψ [i.e. operators satisfying (2.18)].

iii) For all moderately off-diagonal local operators Ψ and Φ, there exists a constant $C_{\Psi,\Phi} < \infty$, such that

$$\big|\langle \Psi t_x(\Phi)\rangle_q - \langle\Psi\rangle_q\langle t_x(\Phi)\rangle_q\big| \leqslant C_{\Psi,\Phi}\, \mathrm{e}^{-|x|/\xi_q}\,. \tag{2.23}$$

iv) The projection operators

$$P^{(q)}_{U(x)} = |g^{(q)}_{U(x)}\rangle\langle g^{(q)}_{U(x)}| \tag{2.24}$$

onto the "classical states" $g^{(q)}_{U(x)}$ obey the bounds

$$\big|\langle P^{(q)}_{U(x)}\rangle_q - 1\big| < \frac{1}{2} \tag{2.25}$$

and

$$\big|\langle P^{(m)}_{U(x)}\rangle_q\big| < \frac{1}{2} \tag{2.26}$$

for all $m \neq q$.
v) There exists a point $\tilde{\boldsymbol{\mu}}_0 \in \mathcal{U}$ such that $a_m(\tilde{\boldsymbol{\mu}}_0) = 0$ for all $m = 1, \dots, r$. For all $\boldsymbol{\mu} \in \mathcal{U}$, the matrix of derivatives

$$F = \Big(\frac{\partial \mathrm{Re}\, f_m(\boldsymbol{\mu})}{\partial \mu_i}\Big) \tag{2.27}$$

has rank $r-1$, and the inverse of the corresponding submatrix is uniformly bounded in $\mathcal{U}$.

Remarks. 1. In a standard fashion of the Pirogov-Sinai theory, the statement v) of the Theorem implies that the phase diagram of the quantum system has the same structure as the zero temperature phase diagram of the classical sytem, with a $\nu - (r-1)$ dimensional coexistence surface $\tilde{S}_0$ where all states are stable, r different $\nu-(r-1)+1$ dimensional surfaces $\tilde{S}_n$ ending in $\tilde{S}_0$ where all states but the state m are stable, … .

2. The bounds (2.25) and (2.26) can be made arbitrarily sharp (by taking β sufficiently large and λ sufficiently small). Then, whenever $a_q = 0$ (i.e. q is stable), the quantum states $\langle\cdot\rangle_q$ are small perturbations of the corresponding classical states.

3. It may happen that for some q we have $a_q(\beta,\lambda,\boldsymbol{\mu}) > 0$ for all $\beta < \infty$, but $\lim_{\beta\to\infty} a_q(\beta,\lambda,\boldsymbol{\mu}) = 0$. In this case we define the zero-temperature free energy to be

$$f_q = -\lim_{L\to\infty} \frac{1}{|\Lambda(L)|} \lim_{\beta\to\infty} \frac{1}{\beta} \log Z_{q,\Lambda(L)}.$$

Our future expansions work in this case, because "in a finite volume, any q-contour is small" (this sentence will get a meaning later).

Consider the torus $\Lambda_{\mathrm{per}}(L) = \big(\mathbb{Z}/(2L+1)\mathbb{Z}\big)^d$ and the corresponding Hamiltonian

$$H_{\mathrm{per},\Lambda(L)} = \sum_{x\in\Lambda_{\mathrm{per}}(L)} H^{(0)}_x + \lambda \sum_{A\subset\Lambda_{\mathrm{per}}(L)} \hat{t}_A\,. \tag{2.28}$$

Introducing the quantum state with periodic boundary conditions as

$$\langle\cdot\rangle_{\mathrm{per},\Lambda(L)} = \frac{1}{Z_{\mathrm{per},\Lambda(L)}} \mathrm{Tr}_{\mathcal{H}_{\Lambda(L)}}\big(\cdot\, \mathrm{e}^{-\beta H_{\mathrm{per},\Lambda(L)}}\big), \tag{2.29}$$

where

$$Z_{\mathrm{per},\Lambda(L)} = \mathrm{Tr}_{\mathcal{H}_{\Lambda(L)}} \mathrm{e}^{-\beta H_{\mathrm{per},\Lambda(L)}} \tag{2.30}$$

we state the following.

Theorem 2.2. *Let $H^{(0)}$, V, β and λ as in Theorem 2.1. Assume in addition that λ is real. Then the infinite volume state with periodic boundary conditions,*

$$\langle\Psi\rangle_{\text{per}} = \lim_{L\to\infty} \langle\Psi\rangle_{\text{per},\Lambda(L)}, \tag{2.31}$$

L being multiple of the lcm of the periods of the configurations $g^{(1)}, \ldots, g^{(r)}$, exists for all local operators Ψ satisfying (2.18). *Moreover, it is a convex combination with equal weights of the stable states,*

$$\langle\Psi\rangle_{\text{per}} = \sum_{q\in Q(\boldsymbol{\mu})} \frac{1}{|Q(\boldsymbol{\mu})|} \langle\Psi\rangle_q. \tag{2.32}$$

Here

$$Q(\boldsymbol{\mu}) = \{q \in \{1,\ldots,r\} \mid a_q(\boldsymbol{\mu}) = 0\}. \tag{2.33}$$

2.4. **Incompressibility.** When studying a system in the framework of the grand-canonical ensemble, it is often convenient to make an abuse of notation by defining a Hamiltonian depending on the chemical potential. In the following we will take even more liberty and refer to the diagonal operator $H^{(0)}_\Lambda - \mu N_\Lambda$ as the "classical part" of the Hamiltonian, and for given μ define the "classical ground states" $g^{(q)}$, $1 \leqslant q \leqslant r(\mu)$ as to be the occupation number configurations minimizing the energy of $H^{(0)}_\Lambda - \mu N_\Lambda$.

The key property, responsible for the occurence of an incompressible phase is the conservation of the total number of particles (together with the Peierls condition). Notice that in this section the particles conservation is a crucial assumption, while it was only a technical one for Theorems 2.1 and 2.2.

Our incompressibility theorem should not be mixed up with the uniform density theorem of [LLM], although there is some overlap. The latter uses special symmetries of the system and shows uniformity of the density with respect to coupling constants *and temperature*, for a class of models of Hubbard type (the "classical ground states" may be infinitely degenerate, and in this case our results do not apply). However it is not uniform with respect to the chemical potential, because only for special values of the latter the system has the necessary symmetries; the compressibility coefficient does not vanish in general.

Beside conservation of the total number of particles, precise assumptions are as follows. The Hamiltonian is $H_\Lambda = H^{(0)}_\Lambda - \mu N_\Lambda + \lambda V$, with $H^{(0)}_\Lambda$ given by (2.4). The chemical potential μ plays a role of vector parameter $\boldsymbol{\mu}$, however, we do not require the condition (2.7) concerning the topology of the phase diagram. Inequalities (2.8)–(2.10) must be satisfied by $\Phi_x = \sum_{A\ni x} \frac{1}{|A|}\Phi_A - \mu \hat{n}_x$, and the quantum perturbation still obeys (2.11)–(2.13).

The last assumption concerns the classical ground states; it requires that their number $|G(\mu)|$ is finite and that the ground state density

$$\rho_0 = \lim_{\Lambda\nearrow\mathbb{Z}^d} \frac{1}{|\Lambda|} \sum_{x\in\Lambda} g^{(q)}_x \tag{2.34}$$

is independent of q, $q \in G(\mu)$.

Theorem 2.3. *For any chemical potential such that the above assumptions hold, and L as in Theorem 2.2, there exist constants $\beta_0 < \infty$, $\lambda_0 > 0$ and $C, C' < \infty$, $c, c' > 0$, independent of L, such that for any $\beta \geqslant \beta_0$ and $\lambda \leqslant \lambda_0$,*

$$\Big|\Big\langle \frac{1}{|\Lambda(L)|} \sum_{x\in\Lambda(L)} \hat{n}_x - \rho_0 \Big\rangle_{\text{per},\Lambda(L)}\Big| \leqslant C\,\mathrm{e}^{-c\beta} \tag{2.35}$$

and

$$\Big|\frac{\partial}{\partial\mu}\Big\langle\frac{1}{|\Lambda(L)|}\sum_{x\in\Lambda(L)}\hat{n}_x\Big\rangle_{\mathrm{per},\Lambda(L)}\Big|\leqslant C'\,\mathrm{e}^{-c'\beta}\,. \tag{2.36}$$

Remarks: 1. The theorem should be still valid for any specified configuration outside Λ with bounded single site occupation number, as well as for free boundary conditions; however, the proof would be more tedious.

2. This theorem may be generalized for diagonal observables that commute with the Hamiltonian. Such observables may include the total spin (see next section), the total number of particles of given species, or of given spin, etc ... (of course, everything depends on the considered system).

3. For one-dimensional systems Theorem 2.3 is still valid, except that β_0 now depends on L. Of course, $\lim_{L\to\infty}\beta_0(L)\to\infty$, which means that the result has a meaning only for the ground state.

2.5. **The case of magnetic systems.** A related result for magnetic systems can be formulated. Typical Hamiltonians in this case consist of a "classical term" involving the spin operators in a given direction, say the 3rd direction, $\{\hat{S}^{(3)}_x\}_{x\in\Lambda}$, including an external magnetic field h, and of a "quantum perturbation" with remaining coordinate of spin operators. Namely,

$$H_{\text{spin system}}=H^{(0)}_\Lambda-h\sum_{x\in\Lambda}S^{(3)}_x+\lambda\sum_{A\subset\Lambda}V_A. \tag{2.37}$$

The Hilbert space of this system is a tensorial product of finite dimensional Hilbert spaces for one localized spin particle. The norm of operators V_A has to decrease exponentially with A.

We require that the Hamiltonian conserves the total spin in the 3rd direction, i.e. for any $A\subset\Lambda$,

$$[V_A,\sum_{x\in\Lambda}\hat{S}^{(3)}_x]=0. \tag{2.38}$$

Then the magnetic susceptibility vanishes at zero temperature. More precisely,

Theorem 2.4. *For any spin model with Hamiltonian* (2.37) *commuting with the total spin in the 3rd direction, and that satisfy the assumptions of Sections 2.1 and 2.2 of* [BKU], *or Section 2.2 of* [DFF], *— without the splitting condition for the phase diagram, but still with a finite number of ground states — we have for $\beta\geqslant\beta_0$ and $\lambda\leqslant\lambda_0$,*

$$\Big|\Big\langle\frac{1}{|\Lambda(L)|}\sum_{x\in\Lambda(L)}\hat{S}^{(3)}_x-m_0\Big\rangle_{\mathrm{per},\Lambda(L)}\Big|\leqslant K\,\mathrm{e}^{-k\beta}\,, \tag{2.39}$$

$$\Big|\frac{\partial}{\partial h}\Big\langle\frac{1}{|\Lambda(L)|}\sum_{x\in\Lambda(L)}\hat{S}^{(3)}_x\Big\rangle_{\mathrm{per},\Lambda(L)}\Big|\leqslant K'\,\mathrm{e}^{-k'\beta}\,, \tag{2.40}$$

for constants $K,K'<\infty$ and $k,k'>0$ (independent of $\Lambda\subset\mathbb{Z}^d$, $d\geqslant 2$); the magnetization of all the classical ground states is supposed to be the same and is denoted by m_0.

As an application, we can consider the anisotropic Heisenberg antiferromagnet (1.3). The zero-temperature phase diagram exhibits phases with zero susceptibility.

Of course, fermion systems present similar properties. To prove it using our contour representation requires to take into account a new difficulty, namely the *anti*commutation relations for the creation and annihilation operators; they yield a sign that does not obviously factorize with respect to the contours. But it does, as was shown in [DFF], and so the incompressibility can be considered to be established in the case of fermions.

3. Derivation of the classical contour representation

In this section we show the equivalence of our system with a classical model in one more dimension in exactly the same manner as in [BKU]. The classical model may be written as a *contour model*, whose contours have complicated weights satisfying exponential decay with respect to their length, when the temperature is low and the perturbation small enough.

Before stating the results of this section in Proposition 3.1 below, we set the notation by recalling some definitions from [BKU]. Let $M \in \mathbb{N}$ and $\tilde{\beta} > 0$ be such that $M\tilde{\beta} = \beta$ — the discretization of the additional continuous dimension, as we shall see. Setting $\mathbb{L}_\Lambda = \Lambda \times \{1, 2, \ldots, M\}$ and $C(x, t) \subset \mathbb{R}^{d+1}$ to denote the closed unit cube centered in $(x, t - \frac{1}{2})$, we introduce the "lattice" $\mathbb{T}_\Lambda = \bigcup_{(x,t)\in\mathbb{L}_\Lambda} C(x, t)$.

We view $\mathbb{T}_\Lambda$ as a cylinder by imposing periodic boundary conditions along the extra dimension (i.e. we assume that for all $x \in \Lambda$, the cubes $C(x, 1)$ and $C(x, M)$ are neighbours). A *contour* Y is a pair $(\operatorname{Supp} Y, \alpha_Y)$, where $\operatorname{Supp} Y \subset \mathbb{T}_\Lambda$ is a (non-empty) connected set of cubes and α_Y is a labelling of elementary faces F of $\partial \operatorname{Supp} Y$, $\alpha_Y(F) = 1, \ldots, r$, that is constant on the boundary of each connected component of $\mathbb{T}_\Lambda \setminus \operatorname{Supp} Y$. We write $|Y|$ for the length of the contour Y, i.e. the number of elementary cubes contained in $\operatorname{Supp} Y$. A set of contours $\{Y_1, \ldots, Y_k\}$ is *admissible* if the contours are mutually disjoint and if the labelling is constant on the boundary of each connected component of $\left[\bigcup_{i=1}^k \operatorname{Supp} Y_i\right]^{\mathrm{c}}$. This set is said to be *compatible with the boundary conditions* q if the external connected components (those touching $\mathbb{T}_{\mathbb{Z}^d} \setminus \mathbb{T}_\Lambda$) of $\left[\bigcup_{i=1}^k \operatorname{Supp} Y_i\right]^{\mathrm{c}}$ have the label equal to q. The horizontal faces centered at (x, t) will be refered to as $P(x, t)$ (P for "plaquette").

We define $\mathbb{T}_\Lambda^\Psi = \bigcup_{(x,t)\in\mathbb{L}_\Lambda} C(x, t)$, with periodic boundary conditions along the time direction for all $x \in \Lambda$ not belonging to $\operatorname{Supp} \Psi$ (i.e. we assume that for all $x \in \Lambda \setminus \operatorname{Supp} \Psi$: $C(x, 1)$ and $C(x, M)$ are neighbours). In other words, think of $\mathbb{T}_\Lambda^\Psi$ as the cylinder $\mathbb{T}_\Lambda$ that is cut along $\operatorname{Supp} \Psi$ at $t = 0$. The "boundary" $S(\Psi) \subset \mathbb{T}_\Lambda^\Psi$ in time direction is

$$S(\Psi) = \bigcup_{x\in\operatorname{Supp}\Psi} P(x, 0) \bigcup_{x\in\operatorname{Supp}\Psi} P(x, M);$$

notice that $P(x, 0) \equiv P(x, M)$ whenever $x \notin \operatorname{Supp} \Psi$. The admissibility and compatibility with the boundary conditions of a set of contours in $\mathbb{T}_\Lambda$ is defined in the same way as above.

A Ψ-*contour* Y_Ψ now is a pair $(\operatorname{Supp} Y_\Psi, \alpha_{Y_\Psi})$ where $\operatorname{Supp} Y_\Psi \subset \mathbb{T}_\Lambda^\Psi$ is a union of cubes such that each connected component intersects $S(\Psi)$, possibly $\operatorname{Supp} Y_\Psi = \emptyset$, and the labelling α_{Y_Ψ} is constant on boundary faces of each connected components of the complement $[\operatorname{Supp} Y_\Psi]^{\mathrm{c}}$.

We are now ready for the definition of the equivalent classical contour model.

Proposition 3.1. *There exists a function* ρ: $\{Y \mid \operatorname{Supp} Y \subset \mathbb{T}_\Lambda\} \longrightarrow \mathbb{C}$ *such that*
i) the partition function of the Hamiltonian (2.2) *can be written as*

$$Z_{q,\Lambda} = \sum_{\{Y_1,\ldots,Y_k\}} \prod_{i=1}^k \rho(Y_i) \prod_{m=1}^r \mathrm{e}^{-\tilde{\beta} e_m(\boldsymbol{\mu})|W_m|}, \tag{3.1}$$

where the sum is over admissible sets of contours in $\mathbb{T}_\Lambda$ *compatible with the boundary conditions* q*; the set* W_m *is the union of the connected components of* $\left[\bigcup_{i=1}^k \operatorname{Supp} Y_i\right]^c$ *with labels* m *on their boundaries,* $|W_m|$ *is the number of elementary cubes contained in* W_m.

ii) For any $\gamma \in \mathbb{R}$, there exist $\tilde\beta_0 < \infty$ and $\lambda_0 > 0$ such that if $\tilde\beta \in [\tilde\beta_0, 2\tilde\beta_0]$ and $\lambda \leqslant \lambda_0$ the following bound is valid for any Y:

$$|\rho(Y)| \leqslant \mathrm{e}^{-(\tilde\beta e_0(\boldsymbol{\mu})+\gamma)|Y|}. \tag{3.2}$$

iii) For any $\gamma' \in \mathbb{R}$, there exist $\tilde\beta_0' < \infty$ and $\lambda_0' > 0$ such that if $\tilde\beta \in [\tilde\beta_0', 2\tilde\beta_0']$ and $\lambda \leqslant \lambda_0'$ we have

$$\Bigl|\frac{\partial}{\partial\mu_i}\rho(Y)\Bigr| \leqslant (C_0\tilde\beta + 1)\,\mathrm{e}^{-(\tilde\beta e_0(\boldsymbol{\mu})+\gamma')|Y|}. \tag{3.3}$$

If Ψ is a moderately off-diagonal local operator whose domain contains $\mathcal{B}_\Lambda$, there exists a function ρ_Ψ: $\{Y_\Psi \mid \operatorname{Supp} Y_\Psi \subset \mathbb{T}_\Lambda^\Psi\} \longrightarrow \mathbb{C}$ such that
iv)

$$Z_{q,\Lambda}^\Psi := \operatorname{Tr}_{\mathcal{H}_\Lambda} \Psi\, \mathrm{e}^{-\beta H_{q,\Lambda}} = \sum_{\{Y_\Psi, Y_1,\dots,Y_k\}} \rho_\Psi(Y_\Psi) \prod_{i=1}^k \rho(Y_i) \prod_{m=1}^r \mathrm{e}^{-\tilde\beta e_m(\boldsymbol{\mu})|W_m|}. \tag{3.4}$$

As before, the sum is over admissible sets of contours, compatible with the boundary condition q; ρ is the same function as in i); W_m is the union of the connected components of $\bigl[\bigcup_{i=1}^k \operatorname{Supp} Y_i \cup \operatorname{Supp} Y_\Psi\bigr]^c$ with labels m on their boundaries.
v) If Ψ satisfies the bound (2.18), then for any $\gamma_\Psi \in \mathbb{R}$ there exist $\tilde\beta_{0,\Psi} < \infty$ and $\lambda_{0,\Psi} > 0$ such that if $\tilde\beta \in [\tilde\beta_{0,\Psi}, 2\tilde\beta_{0,\Psi}]$ and $\lambda \leqslant \lambda_{0,\Psi}$ we have

$$|\rho_\Psi(Y_\Psi)| \leqslant C_\Psi\, \mathrm{e}^{-(\tilde\beta e_0(\boldsymbol{\mu})+\gamma_\Psi)|Y_\Psi|} \tag{3.5}$$

with $C_\Psi = \mathrm{e}^{c_\Psi b|\operatorname{Supp}\Psi|}$ [b is the constant from the bound (2.9)].

The rest of the section is the proof of this proposition. We begin by expanding $Z_{q,\Lambda}^\Psi$ to obtain explicit expressions for ρ_Ψ and ρ; hence part iv) will be proven, and also part i) that can be viewed as a special case of iv) with $\Psi = \mathbb{1}$ [i.e., formally, $\operatorname{Supp}\Psi = \emptyset$ and there is no summation over Y_Ψ in (3.4)]. Similarly v) implies ii) and therefore the following proofs of iv), v), and iii) are sufficient.

Proof of Proposition 3.1 iv). We proceed with the expansion of $Z_{q,\Lambda}^\Psi$ as in [BKU]. We first write

$$Z_{q,\Lambda}^\Psi = \operatorname{Tr}\Psi\bigl[\mathrm{e}^{-\tilde\beta H_{q,\Lambda}}\bigr]^M. \tag{3.6}$$

We will interpret this as a discretization of the additional "time" dimension (the idea was introduced in [Bor]), we can insert the expansion of unity $\mathbb{1}_{\mathcal{H}_\Lambda} = \sum_{n_\Lambda} |n_\Lambda\rangle\langle n_\Lambda|$ into (3.6) to get (we omit here and in the following the index Λ in n_Λ)

$$Z_{q,\Lambda}^\Psi = \sum_{n^{(0)},\dots,n^{(M)}} \langle n^{(M)}|\,\Psi\,|n^{(0)}\rangle\langle n^{(0)}|\,\mathrm{e}^{-\tilde\beta H_{q,\Lambda}}\,|n^{(1)}\rangle \dots \langle n^{(M-1)}|\,\mathrm{e}^{-\tilde\beta H_{q,\Lambda}}\,|n^{(M)}\rangle. \tag{3.7}$$

Expansion with the help of Duhamel formula yields[4]

$$\mathrm{e}^{-\tilde\beta H_{q,\Lambda}} = \sum_{\boldsymbol{m}} \Bigl[\prod_{A\in\mathcal{A}_0} \frac{(-\lambda)^{m_A}}{m_A!} \int_0^{\tilde\beta} \mathrm{d}\tau_A^1 \dots \mathrm{d}\tau_A^{m_A}\Bigr] T(\boldsymbol{\tau},\boldsymbol{m}), \tag{3.8}$$

where $\boldsymbol{m}$ is a multiindex $\boldsymbol{m} : \mathcal{A}_0 \to \{0,1,\dots\}$ with finite $m = \sum_{A\in\mathcal{A}_0} m_A$, $\boldsymbol{\tau} = \{\tau_A^1,\dots,\tau_A^{m_A} | A \in \mathcal{A}_0\} \in [0,\tilde\beta]^m$, and $T(\boldsymbol{\tau},\boldsymbol{m})$: $\mathcal{H}_\Lambda \to \mathcal{H}_\Lambda$ is defined as follows. First we set $(s_1,\dots,s_m)$ to be

[4]In the case $H_{q,\Lambda}^{(0)}$ is not densely defined, this is actually the defining equation for $\mathrm{e}^{-\tilde\beta H_{q,\Lambda}}$ as explained in Section 2.3.

a permutation of $(\tau^1_{A_1},\ldots,\tau^{m_{A_1}}_{A_1},\ldots,\tau^1_{A_k},\ldots,\tau^{m_{A_k}}_{A_k})$ such that $s_1 \leqslant \ldots \leqslant s_m$, and $(\tilde t_1,\ldots,\tilde t_m)$ be the same permutation of $(\hat t_{A_1},\ldots,\hat t_{A_1},\ldots,\hat t_{A_k},\ldots,\hat t_{A_k})$ where each $\hat t_A$ appears exactly m_A times. Then

$$T(\boldsymbol{\tau},\boldsymbol{m}) = \mathrm{e}^{-s_1 H^{(0)}_{q,\Lambda}}\,\tilde t_1\,\mathrm{e}^{-(s_2-s_1)H^{(0)}_{q,\Lambda}}\,\tilde t_2\ldots\tilde t_m\,\mathrm{e}^{-(\tilde\beta-s_m)H^{(0)}_{q,\Lambda}} \tag{3.9}$$

[with $H^{(0)}_{q,\Lambda} := \sum_{x\in\Lambda} H^{(0)}_{q,x}$, and $H^{(0)}_{q,x}$ is given in (2.14)]. Equations (3.7)–(3.8) can be interpreted in the following manner on the $(d+1)$-dimensional cylinder $\mathbb{T}^{\Psi}_{\Lambda}$. We call a *time slice* the union of cubes $\cup_{x\in\Lambda} C(x,t)$ for a given $t\in\{1,\ldots,M\}$; at the intersection between each pair of time slices at t and $t+1$ we define an occupation number configuration $n^{(t)}_{\Lambda}$, and use $\boldsymbol{n}$ to denote the $(M+1)$-tuple $(n^{(0)},n^{(1)},\ldots,n^{(M)})$. In each time slice we choose sets of sites $A_1,\ldots,A_k$ together with times $\tau^1,\ldots,\tau^k$. Then $Z^{\Psi}_{q,\Lambda}$ is an integral over all the possibilities of choosing $\boldsymbol{n}$, and the sets and times in each time slice,

$$Z^{\Psi}_{q,\Lambda} = \sum_{\boldsymbol{n}} \langle n^{(M)}|\,\Psi\,|n^{(0)}\rangle \prod_{t=1}^{M}\sum_{\boldsymbol{m}^{(t)}}\Big[\prod_{A\in\mathcal{A}_0}\frac{(-\lambda)^{m^{(t)}_A}}{m^{(t)}_A!}\int_0^{\tilde\beta}\mathrm{d}\tau^1_A\ldots\mathrm{d}\tau^{m^{(t)}_A}_A\Big]\langle n^{(t-1)}|\,T(\boldsymbol{\tau},\boldsymbol{m}^{(t)})\,|n^{(t)}\rangle \tag{3.10}$$

Notice that the matrix element is zero whenever $n^{(t-1)}$ and $n^{(t)}$ do not match on $\Lambda\backslash B^{(t)}$, where $B^{(t)} = \bigcup_{A\in\mathcal{A}_0, m^{(t)}_A\neq 0} A$, for $t=1,\ldots,M$. Further, $n^{(M)}$ and $n^{(0)}$ have to match on $\Lambda\backslash\operatorname{Supp}\Psi$. To allow these configurations to differ on $\operatorname{Supp}\Psi$ was actually the reason for cutting $\mathbb{T}^{\Psi}_{\Lambda}$ along $(\operatorname{Supp}\Psi, t=0)$.

For a given $\boldsymbol{n}$ we introduce

$$E(\boldsymbol{n}) = \bigcup_{\substack{(x,t):n^{(t)}_{U(x)}\neq g^{(m)}_{U(x)}\\ m=1,\ldots,r}} C(x,t).$$

We call *excited cubes* the cubes belonging to $\cup_{t=1}^{M}\big(\cup_{x\in\bar B^{(t)}} C(x,t)\big)\cup E(\boldsymbol{n})$ with $\bar B^{(t)} = \cup_{x\in B^{(t)}} U(x)$, while we say that a cube $C(x,t)\not\subset E(\boldsymbol{n})$ is *in the ground state* m if $n^{(t)}_{U(x)} = g^{(m)}_{U(x)}$. The complement of the set of excited cubes has the property that each connected subset contains cubes in the same ground state. Therefore for given $\boldsymbol{n}$ and $B^{(t)}, 1\leqslant t\leqslant M$, we may introduce contours $Y_\Psi, Y_1,\ldots,Y_k$ by decomposing the set of excited cubes into maximal connected components. The components intersecting $S(\Psi)$ are identified with $\operatorname{Supp} Y_\Psi$, while the other ones form $\operatorname{Supp} Y_1,\ldots,\operatorname{Supp} Y_k$. We define $\alpha_Y(F)$ to be m if either F is a face shared by a cube in $\operatorname{Supp} Y$ and a cube in the ground state m, $1\leqslant m\leqslant r$, or it is a plaquette from $S(\Psi)$ bordering a non excited cube in the ground state m.

Taking $\boldsymbol{B} = (B^{(1)},\ldots,B^{(M)})$ and abreviating by Supp the set $\operatorname{Supp} Y_\Psi \cup \big[\cup_{j=1}^{k}\operatorname{Supp} Y_j\big]$, we get the partition function as a sum over admissible sets of contours (and compatible with the boundary condition q)

$$Z^{\Psi}_{q,\Lambda} = \sum_{\{Y_\Psi,Y_1,\ldots,Y_k\}}\ \sum_{\substack{\boldsymbol{n}\\ E(\boldsymbol{n})\subset\mathrm{Supp}}} \langle n^{(M)}|\,\Psi\,|n^{(0)}\rangle$$
$$\sum_{\boldsymbol{B}:(\mathrm{Supp},\boldsymbol{n})}\prod_{t=1}^{M}\Big[\sum_{\boldsymbol{m}^{(t)}:B^{(t)}}\prod_{A\in\mathcal{A}_0}\frac{(-\lambda)^{m^{(t)}_A}}{m^{(t)}_A!}\int_0^{\tilde\beta}\mathrm{d}\tau^1_A\ldots\mathrm{d}\tau^{m^{(t)}_A}_A\Big]\langle n^{(t-1)}|\,T(\boldsymbol{\tau},\boldsymbol{m}^{(t)})\,|n^{(t)}\rangle. \tag{3.11}$$

Here, the sum over $\boldsymbol{n}$ is restricted to those $\boldsymbol{n}$ compatible with the labels α_Y, $Y \in \{Y_\Psi, Y_1, \dots, Y_k\}$, the sum over $\boldsymbol{B} : (\mathrm{Supp}, \boldsymbol{n})$ restricts to $\boldsymbol{B}$ that satisfy the condition

$$\mathrm{Supp} \setminus E(\boldsymbol{n}) \subset \bigcup_{t=1}^{M} \bigcup_{x \in \bar{B}^{(t)}} C(x,t) \subset \mathrm{Supp},$$

and $\boldsymbol{m}^{(t)} : B^{(t)}$ stands for the condition $\bigcup_{A \in \mathcal{A}_0, m_A^{(t)} \neq 0} A = B^{(t)}$.

Operators in the Hamiltonian obey the commutation rules

$$[H_x^{(0)}, H_y^{(0)}] = 0, \tag{3.12a}$$

$$[H_x^{(0)}, \hat{t}_A] \quad = 0 \qquad \text{if } U(x) \cap A = \emptyset, \tag{3.12b}$$

$$[\hat{t}_A, \hat{t}_{A'}] \quad = 0 \qquad \text{if } A \cap A' = \emptyset. \tag{3.12c}$$

Consider the matrix element in (3.11). On the one hand the operators $H_x^{(0)}$ with $x \notin \mathrm{Supp}$ commute with all the other operators; their contribution may be extracted from the rest yielding the factor $\mathrm{e}^{-\tilde{\beta} \sum_m e_m(\boldsymbol{\mu})|W_m|}$. On the other hand, two operators with support on two different contours also commute. Considering time slices of contours $D_j^{(t)} = \{x \in \Lambda \mid C(x,t) \subset \mathrm{Supp}\, Y_j\}$, $j = \Psi, 1, \dots, k$, and with $\alpha_j^{(t)}$ the labelling of the external vertical faces of $D_j^{(t)}$, we introduce below the operator $T_{\alpha_j^{(t)}, D_j^{(t)}}(\boldsymbol{\tau}, \boldsymbol{m}^{(t)})$, densely defined in $\mathcal{H}_{D_j^{(t)}}$, in such a way that it contains the information stored in the labelling of the external faces of $D_j^{(t)}$. This will allow us to separate the contributions from different contours, because we will actually have

$$\langle n^{(t-1)} | \, T(\boldsymbol{\tau}, \boldsymbol{m}^{(t)}) \, | n^{(t)} \rangle = \mathrm{e}^{-\tilde{\beta} \sum_{m=1}^r e_m(\boldsymbol{\mu})|W_m^{(t)}|} \prod_{j \in \{\Psi, 1, \dots, k\}} \langle n_{D_j^{(t)}}^{(t-1)} | \, T_{\alpha_j^{(t)}, D_j^{(t)}}(\boldsymbol{\tau}, \boldsymbol{m}^{(t)}) \, | n_{D_j^{(t)}}^{(t)} \rangle, \tag{3.13}$$

with $W_m^{(t)}$ being the set of cubes in the ground state m and located on the slice at time t. To define $T_{\alpha_j^{(t)}, D_j^{(t)}}(\boldsymbol{\tau}, \boldsymbol{m}^{(t)})$ we consider a modified (contour) configuration so that Y_j is the unique contour in $\mathbb{T}_\Lambda^\Psi$; the exterior of $\mathrm{Supp}\, Y_j$ decomposes in $\bigcup_{m=1}^r W_m$, and we define the Hamiltonian $H_{\alpha_j^{(t)}, D_j^{(t)}}^{(0)}$ (with domain included in $\mathcal{H}_{D_j^{(t)}}$) by the relation [c.f. (2.14)]

$$H_{\alpha_j^{(t)}, D_j^{(t)}}^{(0)} = \sum_{n_{D_j^{(t)}}} \sum_{x \in D_j^{(t)}} \Phi_x(n^{\alpha_j^{(t)}, D_j^{(t)}}) \, |n_{D_j^{(t)}}\rangle \langle n_{D_j^{(t)}}| \tag{3.14}$$

where $n^{\alpha_j^{(t)}, D_j^{(t)}}$ is the configuration

$$n_x^{\alpha_j^{(t)}, D_j^{(t)}} = \begin{cases} n_x & \text{if } x \in D_j^{(t)} \\ g_x^{(m)} & \text{if } C(x,t) \in W_m \end{cases}$$

(clearly, $n^{\alpha_j^{(t)}, D_j^{(t)}}$ is defined unambiguously). Then $T_{\alpha_j^{(t)}, D_j^{(t)}}(\boldsymbol{\tau}, \boldsymbol{m}^{(t)})$ is defined the same way as $T(\boldsymbol{\tau}, \boldsymbol{m}^{(t)})$ by replacing $H^{(0)}_{q,\Lambda}$ with $H^{(0)}_{\alpha_j^{(t)}, D_j^{(t)}}$ in (3.9). We thus get

$$Z^{\Psi}_{q,\Lambda} = \sum_{\{Y_\Psi, Y_1, \dots, Y_k\}} \mathrm{e}^{-\tilde\beta \sum_{m=1}^r e_m(\boldsymbol{\mu})|W_m|} \sum_{\substack{\boldsymbol{n} \\ E(\boldsymbol{n}) \subset \mathrm{Supp}}} \langle n^{(M)}|\, \Psi \,|n^{(0)}\rangle \prod_{j \in \{\Psi, 1, \dots, k\}}$$
$$\sum_{\boldsymbol{B}_j : (\mathrm{Supp}\, Y_j, \boldsymbol{n})} \prod_{t=1}^{M} \Big[\sum_{\boldsymbol{m}^{(t)} : B_j^{(t)}} \prod_{A \in \mathcal{A}_0} \frac{(-\lambda)^{m_A^{(t)}}}{m_A^{(t)}!} \int_0^{\tilde\beta} \mathrm{d}\tau_A^1 \dots \mathrm{d}\tau_A^{m_A^{(t)}} \Big] \langle n^{(t-1)}_{D_j^{(t)}}|\, T_{\alpha_j^{(t)}, D_j^{(t)}}(\boldsymbol{\tau}, \boldsymbol{m}^{(t)}) \,|n^{(t)}_{D_j^{(t)}}\rangle \tag{3.15}$$

[with the same constraint on the sum over $\boldsymbol{n}$ as in (3.11)].

The sum over $\boldsymbol{n}$ concerns only sites belonging to $\cup_t \cup_j \big(D_j^{(t)} \cap D_j^{(t+1)}\big)$ — the rest being constrained by the labels of the contours. This sum factorizes, and therefore the contribution of a contour is independent of the occurence of other contours and it is possible to write

$$Z^{\Psi}_{q,\Lambda} = \sum_{\{Y_\Psi, Y_1, \dots, Y_k\}} \mathrm{e}^{-\tilde\beta \sum_{m=1}^r e_m(\boldsymbol{\mu})|W_m|} \rho_\Psi(Y_\Psi) \prod_{j=1}^{k} \rho(Y_j), \tag{3.16}$$

where we introduced the function ρ_Ψ by

$$\rho_\Psi(Y_\Psi) = \sum_{\substack{\boldsymbol{n} \\ E(\boldsymbol{n}) \subset \mathrm{Supp}\, Y_\Psi}} \langle n^{(M)}|\, \Psi \,|n^{(0)}\rangle \sum_{\boldsymbol{B} : (\mathrm{Supp}\, Y_\Psi, \boldsymbol{n})} \prod_{t=1}^{M}$$
$$\Big[\sum_{\boldsymbol{m}^{(t)} : B^{(t)}} \prod_{A \in \mathcal{A}_0} \frac{(-\lambda)^{m_A^{(t)}}}{m_A^{(t)}!} \int_0^{\tilde\beta} \mathrm{d}\tau_A^1 \dots \mathrm{d}\tau_A^{m_A^{(t)}} \Big] \langle n^{(t-1)}_{D_\Psi^{(t)}}|\, T_{\alpha_\Psi^{(t)}, D_\Psi^{(t)}}(\boldsymbol{\tau}, \boldsymbol{m}^{(t)}) \,|n^{(t)}_{D_\Psi^{(t)}}\rangle \tag{3.17}$$

with the constraint that the sum over $\boldsymbol{n}$ be compatible with the label α_{Y_Ψ}. The expression for $\rho(Y)$ is the same except that $\langle n^{(M)}|\, \Psi \,|n^{(0)}\rangle$ is replaced by $\langle n^{(M)}|n^{(0)}\rangle$. This concludes proof of Proposition 3.1 iv).

□

Proof of Proposition 3.1 v). Recall the decomposition $\mathrm{Supp}\, Y_\Psi = \cup_{t=1}^M D_\Psi^{(t)}$. We introduce

$$\tilde\rho_{Y_\Psi, t}(\boldsymbol{n}) = \sum_{B : (D_\Psi^{(t)}, \boldsymbol{n})} \sum_{m=0}^{\infty} (-\lambda)^m \sum_{\substack{A_1, \dots, A_m \\ \cup_{i=1}^m A_i = B}} \int_{0<\tau^1<\dots<\tau^m<\tilde\beta} \mathrm{d}\tau^1 \dots \mathrm{d}\tau^m \langle n^{(t-1)}|\, T_{q, D_\Psi^{(t)}}(\boldsymbol{\tau}, \boldsymbol{A}) \,|n^{(t)}\rangle \tag{3.18}$$

where we defined, similarly as in (3.9) (recall that $H^{(0)}_{q,A} = \sum_{x \in A} H^{(0)}_{q,x}$)

$$T_{q, D_\Psi^{(t)}}(\boldsymbol{\tau}, \boldsymbol{A}) = \mathrm{e}^{-\tau^1 H^{(0)}_{q, D_\Psi^{(t)}}} \hat t_{A_1} \,\mathrm{e}^{-(\tau^2 - \tau^1) H^{(0)}_{q, D_\Psi^{(t)}}} \hat t_{A_2} \dots \hat t_{A_m} \,\mathrm{e}^{-(\tilde\beta - \tau^m) H^{(0)}_{q, D_\Psi^{(t)}}}.$$

We therefore have

$$\rho_\Psi(Y_\Psi) = \sum_{\substack{\boldsymbol{n} \\ E(\boldsymbol{n}) \subset \mathrm{Supp}\, Y_\Psi}} \langle n^{(M)}|\, \Psi \,|n^{(0)}\rangle \prod_{t=1}^{M} \tilde\rho_{Y_\Psi, t}(\boldsymbol{n}), \tag{3.19}$$

with the sum over $\boldsymbol{n}$ being compatible with α_{Y_Ψ}.

The operator $H^{(0)}_{q,D^{(t)}_\Psi}$ may be written as $H^{(0)}_{q,D^{(t)}_\Psi\setminus\bar B}+H^{(0)}_{q,\bar B}$ and a consequence of the commutation rules (3.12) is that $H^{(0)}_{q,D^{(t)}_\Psi\setminus\bar B}$ commutes with $\hat t_{A_i}$, $i=1,\dots,m$. The minimum value for m in (3.18) is $\frac{|B|}{K}$; indeed if m is smaller it is impossible to find $A_1,\dots,A_m$ with their union equalling B. Omitting then the constraint $\cup_{i=1}^m A_i=B$, we bound (3.18) by

$$|\tilde\rho_{Y_\Psi,t}(\boldsymbol n)|\leqslant \sum_{B:(D^{(t)}_\Psi,\boldsymbol n)} \mathrm e^{-\tilde\beta\langle n^{(t)}|\,H^{(0)}_{q,D^{(t)}_\Psi\setminus\bar B}\,|n^{(t)}\rangle} \sum_{m\geqslant\frac{|B|}{K}}|\lambda|^m \int_{0<\tau^1<\dots<\tau^m<\tilde\beta}\mathrm d\tau^1\dots\mathrm d\tau^m \sum_{\substack{A_1,\dots,A_m\\ A_i\subset B}}\bigl|\langle n^{(t-1)}|\,T_{q,\bar B}(\boldsymbol\tau,\boldsymbol A)\,|n^{(t)}\rangle\bigr|. \quad (3.20)$$

Let us recall that $N_A(n)=\sum_{x\in A}n_x$ and introduce $\mathcal B_{\bar B,N}=\{n\in\mathcal B_\Lambda\mid N_{\bar B}(n)=N\}$. The boson damping condition (2.9) implies that for any $n\in\mathcal B_\Lambda$ we have

$$\mathrm e^{-\tau\langle n|\,H^{(0)}_{\bar B}\,|n\rangle}\leqslant \mathrm e^{-\tau e_0(\boldsymbol\mu)|\bar B|}\;\mathrm e^{-\tau a(N_{\bar B}(n)-b|\bar B|)}.$$

Inserting the expansion of unity into (3.20) and using the previous inequality we easily get

$$\begin{aligned}\sum_{\substack{A_1,\dots,A_m\\ A_i\subset B}}&\bigl|\langle n^{(t-1)}|\,T_{q,\bar B}(\boldsymbol\tau,\boldsymbol A)\,|n^{(t)}\rangle\bigr|\\ &\leqslant \mathrm e^{-\tilde\beta e_0(\boldsymbol\mu)|\bar B|}\;\mathrm e^{-\tilde\beta a(N_{\bar B}(n^{(t)})-b|\bar B|)}\prod_{i=1}^m\max_{n'\in\mathcal B_{\bar B,N}}\sum_{A_i\subset B}\sum_{n''\in\mathcal B_{\bar B,N}}|\langle n''|\,\hat t_{A_i}\,|n'\rangle|\\ &\leqslant \mathrm e^{-\tilde\beta e_0(\boldsymbol\mu)|\bar B|}\;\mathrm e^{-\tilde\beta a(N_{\bar B}(n^{(t)})-b|\bar B|)}\Bigl[\max_{n'\in\mathcal B_{\bar B,N}}k\sum_{A\subset B}N_A(n')\Bigr]^m. \quad (3.21)\end{aligned}$$

We here used the assumptions (2.11) and (2.13) for the last estimate. Each n'_x, $x\in B$, is counted a finite number of times in the last sum, because $|A|\leqslant K$; using $c_d(K)$ to denote the finite bound for this number, we have $\sum_{A\subset B}N_A(n')\leqslant c_d(K)N_{\bar B}(n')$.

Substituting this into (3.21) and then (3.21) in (3.20) we obtain

$$|\tilde\rho_{Y_\Psi,t}(\boldsymbol n)|\leqslant \sum_{B:(D^{(t)}_\Psi,\boldsymbol n)} \mathrm e^{-\tilde\beta\langle n^{(t)}|\,H^{(0)}_{q,D^{(t)}_\Psi\setminus\bar B}\,|n^{(t)}\rangle} \sum_{m\geqslant\frac{|B|}{K}}\frac{1}{m!}\Bigl[|\lambda|kc_d(K)N_{\bar B}(n^{(t)})\tilde\beta\Bigr]^m\;\mathrm e^{-\tilde\beta e_0(\boldsymbol\mu)|\bar B|}\;\mathrm e^{-\tilde\beta a(N_{\bar B}(n^{(t)})-b|\bar B|)}. \quad (3.22)$$

We use now the inequality $\sum_{n\geqslant m}\frac{a^n}{n!}\leqslant\frac{a^m}{m!}\,\mathrm e^a$, $m\in\mathbb N$, and assume that $|\lambda|$ is small enough in order that $|\lambda|kc_d(K)\tilde\beta\leqslant 1$; replacing (3.22) in (3.19), with $\lceil\frac{|B|}{K}\rceil$ standing for the integer immediately bigger than or equal to $\frac{|B|}{K}$

$$|\rho_\Psi(Y_\Psi)|\leqslant \sum_{\substack{\boldsymbol n\\ E(\boldsymbol n)\subset\mathrm{Supp}\,Y_\Psi}}|\langle n^{(M)}|\,\Psi\,|n^{(0)}\rangle|\prod_{t=1}^M\sum_{B:(D^{(t)}_\Psi,\boldsymbol n)}\mathrm e^{-\tilde\beta\langle n^{(t)}|\,H^{(0)}_{q,D^{(t)}_\Psi\setminus\bar B}\,|n^{(t)}\rangle}\;\mathrm e^{-\tilde\beta e_0(\boldsymbol\mu)|\bar B|}\;\mathrm e^{-\tilde\beta a(N_{\bar B}(n^{(t)})-b|\bar B|)}\Bigl(|\lambda|kc_d(K)\tilde\beta\Bigr)^{\frac{|B|}{K}}\frac{N_{\bar B}(n^{(t)})^{\lceil\frac{|B|}{K}\rceil}}{\lceil\frac{|B|}{K}\rceil!}\,\mathrm e^{N_{\bar B}(n^{(t)})}. \quad (3.23)$$

Having chosen $n^{(M)}$, we use the assumption (2.18) on Ψ to get

$$\sum_{n^{(0)}} |\langle n^{(M)}|\,\Psi\,|n^{(0)}\rangle| \leqslant \mathrm{e}^{c_\Psi N_{\mathrm{Supp}\,\Psi}(n^{(M)})}\,;$$

this last expression is certainly smaller than $\mathrm{e}^{c_\Psi |\,\mathrm{Supp}\,\Psi| b}\,\mathrm{e}^{c_\Psi \sum_{t=1}^{M} N_{D_\Psi^{(t)}}(n^{(t)})}$ (because of (2.9), $g_x^{(m)} \leqslant b$ for any x, m). Now summing first over $\boldsymbol{B} = (B^{(1)}, \ldots, B^{(M)})$ such that $\cup_{x\in\bar{B}^{(t)}} C(x,t) \subset D_\Psi^{(t)}$, $1 \leqslant t \leqslant M$, we obtain an upper bound by replacing the sum over $\boldsymbol{n}$ by a sum over $n^{(t)}_{D_\Psi^{(t)}}$, $1 \leqslant j \leqslant s$, with the only condition that if $x \in D_\Psi^{(t)}\setminus\bar{B}^{(t)}$, we have $n^{(t)}_{U(x)} \neq g^{(q)}_{U(x)}$ for any $1 \leqslant q \leqslant r$. Finally, we simply bound $\frac{1}{\lceil\frac{|B|}{K}\rceil!} N_{\bar{B}}(n^{(t)})^{\lceil\frac{|B|}{K}\rceil}$ by $\mathrm{e}^{N_{\bar{B}}(n^{(t)})}$ to get

$$|\rho_\Psi(Y_\Psi)| \leqslant \mathrm{e}^{c_\Psi |\,\mathrm{Supp}\,\Psi| b} \prod_{t=1}^{M} \sum_{B,\bar{B}\subset D_\Psi^{(t)}} \sum_{\substack{n_{D_\Psi^{(t)}},\, n_{U(x)}\neq g^{(m)}_{U(x)} \\ x\in D_\Psi^{(t)}\setminus\bar{B},\, 1\leqslant m\leqslant r}} \mathrm{e}^{-\tilde{\beta}\langle n|\,H^{(0)}_{D_\Psi^{(t)}\setminus\bar{B}}\,|n\rangle}$$
$$\mathrm{e}^{c_\Psi N_{D_\Psi^{(t)}}(n)}\,\mathrm{e}^{-\tilde{\beta} e_0(\boldsymbol{\mu})|\bar{B}|}\,\mathrm{e}^{-\tilde{\beta} a\left(N_{\bar{B}}(n) - b|\bar{B}|\right)} \left(|\lambda| k c_d(K)\tilde{\beta}\right)^{\frac{|B|}{K}} \mathrm{e}^{2N_{\bar{B}}(n)}\,. \quad (3.24)$$

For a given N the number of configurations on $\bar{B}$ with N bosons is equal to $\binom{N+|\bar{B}|}{|\bar{B}|} \leqslant \mathrm{e}^{N+|\bar{B}|}$; since $|\bar{B}| \leqslant |B| R_0^d$, and using the Peierls condition (2.8) and the boson damping condition (2.9), we get

$$|\rho_\Psi(Y_\Psi)| \leqslant \mathrm{e}^{c_\Psi b|\,\mathrm{Supp}\,\Psi|}\,\mathrm{e}^{-\tilde{\beta} e_0(\boldsymbol{\mu})|Y_\Psi|} \prod_{t=1}^{M} \sum_{B,\bar{B}\subset D_\Psi^{(t)}} \left(|\lambda| k c_d(K)\tilde{\beta}\right)^{\frac{|\bar{B}|}{KR_0^d}} \mathrm{e}^{(\tilde{\beta} ab+1)|\bar{B}|}$$
$$\Bigg[\prod_{x\in D_\Psi^{(t)}\setminus\bar{B}} \Big[\sum_{n_x=0}^{2b-1} \mathrm{e}^{-\tilde{\beta}\gamma_0 + c_\Psi n_x} + \sum_{n_x=2b}^{\infty} \mathrm{e}^{-\tilde{\beta} a(n_x - b) + c_\Psi n_x}\Big]\Bigg] \sum_{N=0}^{\infty} \mathrm{e}^{-N[\tilde{\beta} a - c_\Psi - 3]}\,. \quad (3.25)$$

The last point is to remark that for any $\gamma_\Psi < \infty$, there exist $\tilde{\beta}_0$, λ_0 such that for any $\tilde{\beta} \in [\tilde{\beta}_0, 2\tilde{\beta}_0[$ and $\lambda \leqslant \lambda_0$ we have

$$\sum_{n_x=0}^{2b-1} \mathrm{e}^{-\tilde{\beta}\gamma_0 + c_\Psi n_x} + \sum_{n_x=2b}^{\infty} \mathrm{e}^{-\tilde{\beta} a(n_x - b) + c_\Psi n_x} \leqslant \tfrac{1}{2}\,\mathrm{e}^{-\gamma_\Psi} \quad (3.26)$$

and

$$\Big[\sum_{N=0}^{\infty} \mathrm{e}^{-N[\tilde{\beta} a - c_\Psi - 3]}\Big] \left(|\lambda| k c_d(K)\tilde{\beta}\right)^{\frac{1}{KR_0^d}} \mathrm{e}^{\tilde{\beta} ab+1} \leqslant \tfrac{1}{2}\,\mathrm{e}^{-\gamma_\Psi}\,. \quad (3.27)$$

The number of sets B such that $\bar{B} \subset D_\Psi^{(t)}$ is smaller than $2^{|D_\Psi^{(t)}|}$, so that we finally get

$$|\rho_\Psi(Y_\Psi)| \leqslant \mathrm{e}^{c_\Psi b|\,\mathrm{Supp}\,\Psi|}\,\mathrm{e}^{-\tilde{\beta} e_0(\boldsymbol{\mu})|Y_\Psi|}\,\mathrm{e}^{-\gamma_\Psi |Y_\Psi|}\,, \quad (3.28)$$

which establishes thus the bound (3.5).

□

Proof of Proposition 3.1 iii). Recall that $\rho(Y)$ may be obtained from $\rho_\Psi(Y_\Psi)$ by replacing $\langle n^{(M)}|\,\Psi\,|n^{(0)}\rangle$ with $\langle n^{(M)}|n^{(0)}\rangle$ in (3.19); therefore

$$\frac{\partial}{\partial\mu_i}\rho(Y) = \sum_{\substack{\boldsymbol{n}\\ E(\boldsymbol{n})\subset \operatorname{Supp} Y}} \sum_{t=1}^{M} \Big[\frac{\partial}{\partial\mu_i}\tilde\rho_{Y,t}(\boldsymbol{n})\Big] \prod_{\substack{t'=1\\ t'\neq t}}^{M} \tilde\rho_{Y,t'}(\boldsymbol{n}). \tag{3.29}$$

Using now (3.18) we express $\frac{\partial}{\partial\mu_i}\tilde\rho_{Y,t}(\boldsymbol{n})$ in terms of the derivative of $T_{q,D^{(t)}}(\boldsymbol{\tau},\boldsymbol{A})$. Namely,

$$\begin{aligned}\frac{\partial}{\partial\mu_i}T_{q,D^{(t)}}(\boldsymbol{\tau},\boldsymbol{A}) = &-\sum_{l=0}^{m}(\tau^l-\tau^{l-1})\,\mathrm{e}^{-\tau^1 H^{(0)}_{Y,t}}\,\hat t_{A_1}\dots\hat t_{A_{l-1}}\Big(\frac{\partial}{\partial\mu_i}H^{(0)}_{Y,t}\Big)\\ &\qquad\qquad \mathrm{e}^{-(\tau^l-\tau^{l-1})H^{(0)}_{Y,t}}\,\hat t_{A_l}\dots\mathrm{e}^{-(\tilde\beta-\tau^m)H^{(0)}_{Y,t}}\\ &+\sum_{l=1}^{m}\mathrm{e}^{-\tau^1 H^{(0)}_{Y,t}}\,\hat t_{A_1}\dots\mathrm{e}^{-(\tau^l-\tau^{l-1})H^{(0)}_{Y,t}}\Big(\frac{\partial}{\partial\mu_i}\hat t_{A_l}\Big)\,\mathrm{e}^{-(\tau^{l+1}-\tau^l)H^{(0)}_{Y,t}}\dots\mathrm{e}^{-(\tilde\beta-\tau^m)H^{(0)}_{Y,t}}.\end{aligned} \tag{3.30}$$

We proceed as above using $|\langle n|\,\frac{\partial}{\partial\mu_i}H^{(0)}_{Y,t}\,|n\rangle| \leqslant |D^{(t)}|C_0\,\mathrm{e}^{c_0 N_{D^{(t)}}(n)}$ [in view of (2.10)] and $\sum_{l=0}^{m}(\tau^l-\tau^{l-1})=\tilde\beta$, while $\frac{\partial}{\partial\mu_i}\hat t_{A_l}$ obeys the same bound as $\hat t_{A_l}$, see (2.12). Proposition 3.1 iii) is then straightforward.

□

4. Proofs of Theorems 1.1, 2.1, 2.2 and 2.3

Proofs of Theorems 2.1 and 2.2. They are exactly the same as the ones of Theorems 2.1 and 2.2 in [BKU] and will not be repeated here. Indeed, they follow from Lemmas 5.1 and 6.1 of [BKU] which are valid because of Proposition 3.1 — the only modification is to replace $||\Psi||$ in Lemma 6.1 by $\mathrm{e}^{c_\Psi b|\operatorname{Supp}\Psi|}$.[5] □

4.1. **Proof of the incompressibility.**

The key point in the proof of the incompressibility [the bounds (2.35) and (2.36)] will rely on the special form that characterizes the expansion (3.4) in the case where Ψ is the total number of particles. Namely, taking into account the conservation of the total number of particles, the probability that the configuration has a different number of bosons than that of the corresponding ground state may be related to the presence of a contour that winds around the cylinder. Since the length of such a contour is of the order β, and its weight is damped, we obtain that this probability is exponentially small in β.

We say that a contour *winds* around the cylinder $\mathbb{T}_\Lambda$ if its support intersects each time slice. Notice that this definition is not equivalent to the usual topological one; actually, it is only a consequence. However, it is enough for our purpose, and the proof is simpler in that case.

Lemma 4.1. *For any Hamiltonian satisfying the assumptions of Theorem 2.3, we have for any stable phase q:*

[5] We assumed in [BKU] that Ψ was symmetric; however the proof never used this property and hence Lemma 6.1 applies to non-symmetric operators as well.

i) There exists a weight $\rho_{\circlearrowleft}(Y)$, with the property $\rho_{\circlearrowleft}(Y) = 0$ if Y is not a winding contour, and such that the following expansion holds.

$$\operatorname{Tr}\Big[\sum_{x\in\Lambda}\big(\hat{n}_x - g_x^{(q)}\big)\,\mathrm{e}^{-\beta H_{q,\Lambda}}\Big] = \sum_{Y}\rho_{\circlearrowleft}(Y)\sum_{\{Y_1,\dots,Y_k\}:Y}\prod_{i=1}^{k}\rho(Y_i)\prod_{m=1}^{r}\mathrm{e}^{-\tilde{\beta}e_m(\boldsymbol{\mu})|W_m|}\,. \tag{4.1}$$

The last sum over admissible sets of contours $\{Y_1,\dots,Y_k\}$, with support on $\mathbb{T}_\Lambda$ and compatible with boundary conditions q, has to be compatible with Y.
ii) There exists $\rho_{\Psi,\circlearrowleft}$ such that for any moderately off-diagonal local operator Ψ, $\operatorname{Supp}\Psi\subset\Lambda$, we have

$$\begin{aligned}\operatorname{Tr}\Big[\sum_{x\in\Lambda}\big(\hat{n}_x - g_x^{(q)}\big)\Psi\,\mathrm{e}^{-\beta H_{q,\Lambda}}\Big] &= \sum_{Y_\Psi}\rho_{\Psi,\circlearrowleft}(Y_\Psi)\sum_{\{Y_1,\dots,Y_k\}:Y_\Psi}\prod_{i=1}^{k}\rho(Y_i)\prod_{m=1}^{r}\mathrm{e}^{-\tilde{\beta}e_m(\boldsymbol{\mu})|W_m|}\\ &\quad+\sum_{Y}\rho_{\circlearrowleft}(Y)\sum_{\{Y_\Psi,Y_1,\dots,Y_k\}:Y}\rho_\Psi(Y_\Psi)\prod_{i=1}^{k}\rho(Y_i)\prod_{m=1}^{r}\mathrm{e}^{-\tilde{\beta}e_m(\boldsymbol{\mu})|W_m|}\,,\end{aligned} \tag{4.2}$$

and $\rho_{\Psi,\circlearrowleft}(Y_\Psi)$ (resp. $\rho_{\circlearrowleft}(Y)$) is zero whenever Y_Ψ (resp. Y) does not wind.
iii) The weights $\rho_{\circlearrowleft}$ and $\rho_{\Psi,\circlearrowleft}$ are damped, i.e. they satisfy similar decay properties as in Proposition 3.1 ii) and v).

Remark: the constants β_0 and λ_0 of the assumptions, as well as the constants in the bounds (3.2) and (3.5) will actually differ from those of Theorem 2.1 and Proposition 3.1.

Proof of Lemma 4.1 ii). [Part i) of the Lemma is a consequence of part ii)].
First we expand the matrix element $\langle n|\,\Psi\,\mathrm{e}^{-\beta H_{q,\Lambda}}\,|n\rangle$ as in Section 3. This yields

$$\langle n|\,\Psi\,\mathrm{e}^{-\beta H_{q,\Lambda}}\,|n\rangle = \sum_{\{Y_\Psi,Y_1,\dots,Y_k\}}\mathbf{I}\big[n\sim\{Y_j\}\big]\rho_\Psi^n(Y_\Psi)\prod_{i=1}^{k}\rho^n(Y_i)\prod_{m=1}^{r}\mathrm{e}^{-\tilde{\beta}e_m(\boldsymbol{\mu})|W_m|} \tag{4.3}$$

where $\rho_\Psi^n(Y_\Psi) = \rho_{\Psi P_\Lambda(n)}(Y_\Psi)$, with $P_A(n)$ the projector onto the vector $|n_A\rangle$, i.e. $P_A(n) = |n_A\rangle\langle n_A|$ and $\rho^n(Y_i)$ is the same as (3.17) where we replace $\langle n^{(M)}|\,\Psi\,|n^{(0)}\rangle$ by $\langle n^{(M)}|\,P_\Lambda(n)\,|n^{(0)}\rangle$. The indicator function $\mathbf{I}\big[n\sim\{Y_j\}\big]$ equals 1 if the set of contours is compatible with n, and equals 0 otherwise.
Then we have

$$\begin{aligned}\langle n|\sum_{x\in\Lambda}\big(\hat{n}_x - g_x^{(q)}\big)\Psi\,\mathrm{e}^{-\beta H_{q,\Lambda}}\,|n\rangle &= \sum_{x\in\Lambda}\big(n_x - g_x^{(q)}\big)\langle n|\,\Psi\,\mathrm{e}^{-\beta H_{q,\Lambda}}\,|n\rangle\\ &= \sum_{\{Y_\Psi,Y_1,\dots,Y_k\}}\mathbf{I}\big[n\sim\{Y_j\}\big]\sum_{x\in\Lambda}\big(n_x - g_x^{(q)}\big)\rho_\Psi^n(Y_\Psi)\prod_{i=1}^{k}\rho^n(Y_i)\prod_{m=1}^{r}\mathrm{e}^{-\tilde{\beta}e_m(\boldsymbol{\mu})|W_m|}\,.\end{aligned} \tag{4.4}$$

With $\operatorname{Int}Y_j$ being the union of all components of $\mathbb{T}_\Lambda^\Psi\setminus\operatorname{Supp}Y_j$ that are not touching the complement $\mathbb{T}_{\Lambda^c}^\Psi$, and $\operatorname{Int}_m Y_j$ being the union of components of $\operatorname{Int}Y_j$ with external labelling m, we define 0-time slices

$$\operatorname{Supp}_0 Y_j = D_j^{(M)}\cap D_j^{(1)} \tag{4.5a}$$

$$\operatorname{Int}_{m,0}Y_j = \operatorname{Int}_m Y_j\cap[\mathbb{Z}^d\times\{t=0\}]. \tag{4.5b}$$

We view these sets as included in $\mathbb{Z}^d$.

For given n and $\{Y_\Psi, Y_1, \dots, Y_k\}$, admissible set of contours compatible with the boundary condition q (and also compatible with n, i.e. $\mathbf{I}\big[n \sim \{Y_j\}\big] = 1$), we have the equality

$$\sum_{x\in\Lambda}\big(n_x - g_x^{(q)}\big) = \sum_{j\in\{\Psi,1,\dots,k\}} \Big[\sum_{x\in\operatorname{Supp}_0 Y_j} \big(n_x - g_x^{(\alpha_{Y_j}^{\rm ext})}\big) + \sum_{m=1}^{r} \sum_{x\in\operatorname{Int}_{m,0} Y_j} \big(g_x^{(m)} - g_x^{(\alpha_{Y_j}^{\rm ext})}\big)\Big], \tag{4.6}$$

with $\alpha_{Y_j}^{\rm ext}$ denoting the labelling of the external faces of the contour Y_j. Let us introduce the new weights $\tilde\rho^n(Y_j)$, $j \in \{1,\dots,k\}$, and $\tilde\rho_\Psi^n(Y_\Psi)$ by

$$\tilde\rho^n(Y_j) = \Big[\sum_{x\in\operatorname{Supp}_0 Y_j} \big(n_x - g_x^{(\alpha_{Y_j}^{\rm ext})}\big) + \sum_{m=1}^{r} \sum_{x\in\operatorname{Int}_{m,0} Y_j} \big(g_x^{(m)} - g_x^{(\alpha_{Y_j}^{\rm ext})}\big)\Big] \rho^n(Y_j) \tag{4.7a}$$

$$\tilde\rho_\Psi^n(Y_\Psi) = \Big[\sum_{x\in\operatorname{Supp}_0 Y_\Psi} \big(n_x - g_x^{(\alpha_{Y_\Psi}^{\rm ext})}\big) + \sum_{m=1}^{r} \sum_{x\in\operatorname{Int}_{m,0} Y_\Psi} \big(g_x^{(m)} - g_x^{(\alpha_{Y_\Psi}^{\rm ext})}\big)\Big] \rho_\Psi^n(Y_\Psi). \tag{4.7b}$$

Then

$$\operatorname{Tr}\Big[\sum_{x\in\Lambda}\big(\hat n_x - g_x^{(q)}\big)\Psi\, \mathrm{e}^{-\beta H_{q,\Lambda}}\Big] = \sum_n \Big(\sum_{x\in\Lambda}\big(n_x - g_x^{(q)}\big)\Big)\langle n|\,\Psi\,\mathrm{e}^{-\beta H_{q,\Lambda}}\;|n\rangle =$$

$$= \sum_{\{Y_\Psi,Y_1,\dots,Y_k\}} \sum_n \mathbf{I}\big[n\sim\{Y_j\}\big]\Big\{\tilde\rho_\Psi^n(Y_\Psi)\prod_{i=1}^k \rho^n(Y_i) + \rho_\Psi^n(Y_\Psi)\sum_{j=1}^k \tilde\rho^n(Y_j)\prod_{i\neq j}\rho^n(Y_i)\Big\}\prod_{m=1}^r \mathrm{e}^{-\tilde\beta e_m(\boldsymbol{\mu})|W_m|}. \tag{4.8}$$

Remark that the summation over all configurations satisfying $\mathbf{I}\big[n\sim\{Y_j\}\big] = 1$ is equivalent to successive sums over $n_{\operatorname{Supp}_0 Y_j}$, $j \in \{\Psi, 1, \dots, k\}$, each sum satisfying the local condition to be compatible with the labels α_{Y_j}. We thus obtain

$$\begin{aligned}\operatorname{Tr}\big[\dots\big] = \sum_{\{Y_\Psi,Y_1,\dots,Y_k\}} \Big\{\sum_{n_{\operatorname{Supp}_0 Y_\Psi}\sim\alpha_{Y_\Psi}} \tilde\rho_\Psi^n(Y_\Psi)\Big\}\Big\{\prod_{i=1}^k \sum_{n_{\operatorname{Supp}_0 Y_i}\sim\alpha_{Y_i}} \rho^n(Y_i)\Big\}\prod_{m=1}^r \mathrm{e}^{-\tilde\beta e_m(\boldsymbol{\mu})|W_m|} \\ + \sum_Y \Big\{\sum_{n_{\operatorname{Supp}_0 Y}\sim\alpha_Y} \tilde\rho^n(Y)\Big\} \sum_{\{Y_\Psi,Y_1,\dots,Y_k\}:Y} \Big\{\sum_{n_{\operatorname{Supp}_0 Y_\Psi}\sim\alpha_{Y_\Psi}} \rho_\Psi^n(Y_\Psi)\Big\} \\ \Big\{\prod_{i=1}^k \sum_{n_{\operatorname{Supp}_0 Y_i}\sim\alpha_{Y_i}} \rho^n(Y_i)\Big\}\prod_{m=1}^r \mathrm{e}^{-\tilde\beta e_m(\boldsymbol{\mu})|W_m|}.\end{aligned} \tag{4.9}$$

Defining

$$\rho_{\Psi,\circlearrowleft}(Y_\Psi) = \sum_{n_{\operatorname{Supp}_0 Y_\Psi}\sim\alpha_{Y_\Psi}} \tilde\rho_\Psi^n(Y_\Psi) \quad\text{and}\quad \rho_\circlearrowleft(Y) = \sum_{n_{\operatorname{Supp}_0 Y}\sim\alpha_Y} \tilde\rho^n(Y),$$

and since

$$\sum_{n_{\operatorname{Supp}_0 Y_i}\sim\alpha_{Y_i}} \rho^n(Y_i) = \rho(Y_i), \qquad \sum_{n_{\operatorname{Supp}_0 Y_\Psi}\sim\alpha_{Y_\Psi}} \rho_\Psi^n(Y_\Psi) = \rho_\Psi(Y_\Psi),$$

we get (4.2). It remains to check that $\tilde\rho^n(Y_j)$ [resp. $\tilde\rho_\Psi^n(Y_\Psi)$] is zero whenever Y_j [resp. Y_Ψ] does not wind, and part ii) of the lemma will be proven.

Let us consider a contour Y such that its exterior is in the ground state m. Taking into account that $\tilde\rho^n(Y)$ depends only on $n_{\operatorname{Supp}_0 Y}$, it is enough to consider the situation where Y

is the unique contour (in this case the boundary conditions are m). For a given n (compatible with Y), the factor in brackets in (4.7) equals

$$\sum_{x\in\operatorname{Supp}_0 Y}\bigl(n_x-g_x^{(m)}\bigr)+\sum_{m'=1}^{r}\sum_{x\in\operatorname{Int}_{m',0}Y}\bigl(g_x^{(m')}-g_x^{(m)}\bigr),$$

and clearly vanishes whenever $\sum_{x\in\Lambda}n_x=\sum_{x\in\Lambda}g_x^{(m)}$. If, on the contrary, this last equality is not true, and if Y does not wind, we show that $\rho^n(Y)=0$ [which in turn implies $\tilde\rho^n(Y)=0$, see (4.7)]. Indeed, if Y does not wind, then for some t, the sum over $\boldsymbol{n}$ in (3.17) is constrained to $n^{(t)}=g^{(m)}$, and only terms like

$$\langle n|\prod_{t'=1}^{t}T_{\alpha_j^{(t')},D_j^{(t')}}(\boldsymbol{\tau},\boldsymbol{m}^{(t')})\,|g^{(m)}\rangle$$

contribute in (3.17). But since

$$[T_{\alpha_j^{(t')},D_j^{(t')}}(\boldsymbol{\tau},\boldsymbol{m}^{(t')}),N_\Lambda]=0 \tag{4.10}$$

for any t', the above terms are zero whenever $\sum_{x\in\Lambda}n_x\neq\sum_{x\in\Lambda}g_x^{(m)}$.

Thus $\tilde\rho^n(Y)=0$ for any non winding contour Y.

To verify the same for $\tilde\rho^n_\Psi(Y_\Psi)$, we take advantage of the assumption that $[\Psi,N_{\operatorname{Supp}\Psi}]=0$, that we can assume without loss of generality. Indeed, if it were not true, then we might decompose

$$\Psi=\Psi^{(0)}+\Psi^{(1)},$$

where $\Psi^{(0)}$ conserves the total number of particles, and $\Psi^{(1)}$ does not. Then, since

$$\operatorname{Tr}\Bigl[\sum_{x\in\Lambda}\bigl(\hat n_x-g_x^{(q)}\bigr)\Psi^{(1)}\,\mathrm{e}^{-\beta H_{q,\Lambda}}\Bigr]=0,$$

we might only consider $\Psi^{(0)}$. □

Proof of Lemma 4.1 iii). The weight $\rho_\circlearrowleft(Y)$ may be written as

$$\rho_\circlearrowleft(Y)=\sum_{n_{\operatorname{Supp}_0 Y}\sim\alpha_Y}\sum_{x\in\operatorname{Supp}_0 Y}(n_x-g_x^{(\alpha_Y^{\rm ext})})\rho^n(Y)+\rho(Y)\sum_{m=1}^{r}\sum_{x\in\operatorname{Int}_{m,0}Y}(g_x^{(m)}-g_x^{(\alpha_Y^{\rm ext})}). \tag{4.11}$$

We compare the first expression of the RHS with the definition (3.17) of $\rho_\Psi(Y_\Psi)$. This is identical if we set

$$\Psi=\sum_{x\in\operatorname{Supp}_0 Y}(\hat n_x-g_x^{(\alpha_Y^{\rm ext})}),$$

and $\operatorname{Supp}\Psi=\operatorname{Supp}_0 Y$. Ψ is a diagonal operator which is not bounded; but since it diverges linearly with the number of bosons on $\operatorname{Supp}\Psi$, it is moderately off-diagonal and we can use Proposition 3.1 v). For the second expression of the RHS we observe that the sums are bounded by const $\cdot\,|\operatorname{Int}_0 Y|\leqslant \mathrm{e}^{\text{const}|\operatorname{Supp}Y|}$, and we use Proposition 3.1 ii). □

We come now to the proof of Theorem 2.3. In order to use Theorem 2.1, we extend the Hamiltonian by adding “external fields”, so as to satisfy the condition of splitting of degeneracy [see (2.7)],

$$H'=H+\sum_{i=1}^{r-1}\mu_i\sum_{A\subset\Lambda}|g_A^{(i)}\rangle\langle g_A^{(i)}|\,. \tag{4.12}$$

Here, the subsets A are such that $\operatorname{diam} A \leqslant R_0$. Then the matrix E defined in (2.7) has indeed the rank $r-1$; conditions (2.8)–(2.10) clearly hold, as it is also the case for (2.11)–(2.13).

Proof of Theorem 2.3. From Theorem 2.2 we know that the states with periodic boundary conditions decompose into states with classical configurations as boundary conditions. Therefore, it is enough to consider the case with boundary conditions q.

The expectation value of the density may be written

$$\Big\langle \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \hat{n}_x - \rho_0 \Big\rangle_{q,\Lambda} = \frac{1}{|\Lambda|} \frac{\operatorname{Tr} \big[\sum_{x\in\Lambda} (\hat{n}_x - g_x^{(q)})\, \mathrm{e}^{-\beta H_{q,\Lambda}} \big]}{\operatorname{Tr}\, \mathrm{e}^{-\beta H_{q,\Lambda}}} \tag{4.13}$$

The first step consists in expanding these traces by using (4.1) and (3.1). Then similarly as in [BKU] (see Section 6) we consider the contour Y of (4.1) and the contours encircling it as a single contour $\mathcal{Y}$. We generalize the notion of interiors of a contour to $\mathcal{Y}$, namely $\operatorname{Int}\mathcal{Y}$ is the union of the components of $\mathbb{T}_\Lambda \setminus \cup_{Y\in\mathcal{Y}} \operatorname{Supp} Y$ that do not touch $\mathbb{T}_{\Lambda^c}$, and $\operatorname{Int}_m \mathcal{Y}$ is the union of components of $\operatorname{Int}\mathcal{Y}$ with labels m on the boundaries. Then

$$\Big\langle \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \hat{n}_x - \rho_0 \Big\rangle_{q,\Lambda} = \frac{1}{|\Lambda|} \sum_{\mathcal{Y}} \rho_{\circlearrowleft}(\mathcal{Y}) \frac{\sum_{\{Y_1,\dots,Y_k\}:\mathcal{Y}} \prod_{i=1}^k \rho(Y_i)\, \mathrm{e}^{-\tilde\beta \sum_{m=1}^{r(\mu)} e_m |W_m|}}{\sum_{\{Y_1,\dots,Y_k\}} \prod_{i=1}^k \rho(Y_i)\, \mathrm{e}^{-\tilde\beta \sum_{m=1}^{r(\mu)} e_m |W_m|}}, \tag{4.14}$$

where

$$\rho_{\circlearrowleft}(\mathcal{Y}) = \rho_{\circlearrowleft}(Y) \prod_{Y'\in\mathcal{Y}, Y'\neq Y} \rho(Y), \tag{4.15}$$

and the constraint $\{Y_1,\dots,Y_k\} : \mathcal{Y}$ means that $[\operatorname{Supp} Y_i \cup \operatorname{Int} Y_i] \cap \operatorname{Supp}\mathcal{Y} = \emptyset$ for any $i \in \{1,\dots,k\}$ (the support of $\mathcal{Y}$ is $\operatorname{Supp}\mathcal{Y} = \cup_{Y\in\mathcal{Y}} \operatorname{Supp} Y$).

The second step is to perform the Pirogov-Sinai transformations to the weights of the contours; namely, writing $Z_q(W)$ for the partition function in $W \subset \mathbb{T}_\Lambda$,

$$Z_q(W) = \sum_{\{Y_1,\dots,Y_k\}} \prod_{i=1}^k \rho(Y_i) \prod_{m=1}^{r(\mu)} \mathrm{e}^{-\tilde\beta e_m |W_m|}, \tag{4.16}$$

with the sum running over admissible sets of contours in W, compatible with the boundary conditions q, and the W_m are sets of cubes of W in the ground state m, we introduce

$$K_q(Y) = \rho(Y)\, \mathrm{e}^{\tilde\beta e_q |Y|} \prod_{m=1}^{r(\mu)} \frac{Z_m(\operatorname{Int}_m Y)}{Z_q(\operatorname{Int}_m Y)}, \tag{4.17a}$$

$$K_{q,\circlearrowleft}(\mathcal{Y}) = \rho_{\circlearrowleft}(\mathcal{Y})\, \mathrm{e}^{\tilde\beta e_q |\mathcal{Y}|} \prod_{m=1}^{r(\mu)} \frac{Z_m(\operatorname{Int}_m \mathcal{Y})}{Z_q(\operatorname{Int}_m \mathcal{Y})}, \tag{4.17b}$$

where $|\mathcal{Y}|$ is the number of elementary cubes contained in $\operatorname{Supp}\mathcal{Y}$. Then we can write

$$\Big\langle \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \hat{n}_x - \rho_0 \Big\rangle_{q,\Lambda} = \frac{1}{|\Lambda|} \sum_{\mathcal{Y}} K_{q,\circlearrowleft}(\mathcal{Y}) \frac{\sum_{\{Z_1,\dots,Z_k\}:\mathcal{Y}} \prod_{i=1}^k K_q(Z_i)}{\sum_{\{Z_1,\dots,Z_k\}} \prod_{i=1}^k K_q(Z_i)}, \tag{4.18}$$

where the Z_i are contours with external labelling fixed to q and the sums over sets of contours $\{Z_1,\dots,Z_k\}$ are restricted to sets of non intersecting contours (the condition $\{Z_1,\dots,Z_k\} : \mathcal{Y}$ means that $\operatorname{Supp}\mathcal{Y}$ does not intersect $\cup_{i=1}^k \operatorname{Supp} Z_i$, and is not contained in $\cup_{i=1}^k \operatorname{Int} Z_i$).

The weights $K_q(Y)$ and $K_{q,\circlearrowleft}(\mathcal{Y})$ satisfy a damping condition, if the phase q is stable. The proof uses the exponential decay of $\rho(Y)$ and $\rho_{\circlearrowleft}(\mathcal{Y})$ and proceeds by induction on the size of the contours [Zah, BI]; see Lemmas 5.1 ii) and 6.1 ii) of [BKU].

In the third step we apply the cluster expansion (see [KP]; actually, a simpler proof may be found in [Dob], although not covering the whole statement) to the numerator and the denominator. With C to denote clusters, i.e. sets of contours with external labelling q, and $\operatorname{Supp} C$, $\operatorname{Int} C$, being the union of respectively the supports and the interiors of the contours of C, we get

$$\Big\langle \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \hat{n}_x - \rho_0 \Big\rangle_{q,\Lambda} = \frac{1}{|\Lambda|} \sum_{\mathcal{Y}} K_{q,\circlearrowleft}(\mathcal{Y}) \exp\{-\sum_{C, C\not\sim\mathcal{Y}} \Phi^{\mathrm{T}}(C)\}, \tag{4.19}$$

where the summation is over clusters "not compatible with $\mathcal{Y}$": $C \not\sim \mathcal{Y}$ stands for the condition

$$[\operatorname{Int} C \cup \operatorname{Supp} C] \cap \operatorname{Supp} \mathcal{Y} \neq \emptyset.$$

Precise definition of the truncated function Φ^{T} is unimportant here (note that $\Phi^{\mathrm{T}}(C) = 0$ if $\operatorname{Supp} C$ is not a connected set). We only mention the following property; for any γ, we can choose $\tilde{\beta}$ large and λ small enough (both depending on γ), in such a way that we have

$$\sum_{\mathcal{Y}\ni x, |\mathcal{Y}| \geqslant \delta} |K_{q,\circlearrowleft}(\mathcal{Y})| \exp\{\sum_{C, C\not\sim\mathcal{Y}} |\Phi^{\mathrm{T}}(C)|\} \leqslant \mathrm{e}^{-\gamma\delta} \tag{4.20}$$

for any δ. Using the fact that $\mathcal{Y}$ contains a winding contour, whose length is bigger than (or equal to) $M = \beta/\tilde{\beta}$, we get the bound (2.35).

The second bound is obtained in a similar manner. The derivative of the density with respect to the chemical potential leads to the fluctuations of the number of particles

$$\frac{\partial}{\partial\mu} \langle \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \hat{n}_x \rangle_{q,\Lambda} = \frac{\beta}{|\Lambda|} \sum_{x,y\in\Lambda} \big\langle \big(\hat{n}_x - \langle \hat{n}_x \rangle_{q,\Lambda}\big)\big(\hat{n}_y - \langle \hat{n}_y \rangle_{q,\Lambda}\big) \big\rangle_{q,\Lambda}. \tag{4.21}$$

Observing that

$$\big\langle \big(\hat{n}_x - \langle \hat{n}_x \rangle_{q,\Lambda}\big)\big(\hat{n}_y - \langle \hat{n}_y \rangle_{q,\Lambda}\big) \big\rangle_{q,\Lambda} = \big\langle \big(\hat{n}_x - g_x^{(q)}\big)\big(\hat{n}_y - \langle \hat{n}_y \rangle_{q,\Lambda}\big) \big\rangle_{q,\Lambda}$$

(the difference between the two expressions is obviously zero), we may write

$$\frac{\partial}{\partial\mu} \langle \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \hat{n}_x \rangle_{q,\Lambda} = \frac{\beta}{|\Lambda|} \sum_{y\in\Lambda} \Big[\langle \sum_{x\in\Lambda} (\hat{n}_x - g_x^{(q)}) \hat{n}_y \rangle_{q,\Lambda} - \langle \sum_{x\in\Lambda} (\hat{n}_x - g_x^{(q)}) \rangle_{q,\Lambda} \langle \hat{n}_y \rangle_{q,\Lambda} \Big]. \tag{4.22}$$

Let us define

$$\check{n}_y = \langle \sum_{x\in\Lambda} (\hat{n}_x - g_x^{(q)}) \hat{n}_y \rangle_{q,\Lambda} - \langle \sum_{x\in\Lambda} (\hat{n}_x - g_x^{(q)}) \rangle_{q,\Lambda} \langle \hat{n}_y \rangle_{q,\Lambda}; \tag{4.23}$$

in the following we show that $|\check{n}_y| \leqslant \mathrm{e}^{-a\beta}$ for some constant $a > 0$.

First we consider the term

$$\langle \sum_{x\in\Lambda} (\hat{n}_x - g_x^{(q)}) \hat{n}_y \rangle_{q,\Lambda} = \frac{1}{Z_{q,\Lambda}} \operatorname{Tr} \Big[\sum_{x\in\Lambda} (\hat{n}_x - g_x^{(q)}) \hat{n}_y \, \mathrm{e}^{-\beta H_{q,\Lambda}} \Big]. \tag{4.24}$$

From Lemma 4.1 ii),

$$\begin{aligned} \langle \sum_{x\in\Lambda} (\hat{n}_x - g_x^{(q)}) \hat{n}_y \rangle_{q,\Lambda} = \frac{1}{Z_{q,\Lambda}} \Big[& \sum_{Y_{\hat{n}_y}} \rho_{\hat{n}_y,\circlearrowleft}(Y_{\hat{n}_y}) \sum_{\{Y_1,\dots,Y_k\}:Y_{\hat{n}_y}} \prod_{i=1}^{k} \rho(Y_i) \prod_{m=1}^{r(\mu)} \mathrm{e}^{-\tilde{\beta} e_m |W_m|} \\ & + \sum_{Y} \rho_{\circlearrowleft}(Y) \sum_{\{Y_{\hat{n}_y}, Y_1,\dots,Y_k\}:Y} \rho_{\hat{n}_y}(Y_{\hat{n}_y}) \prod_{i=1}^{k} \rho(Y_i) \prod_{m=1}^{r(\mu)} \mathrm{e}^{-\tilde{\beta} e_m |W_m|} \Big]. \end{aligned} \tag{4.25}$$

The strategy is the same as above. In (4.25) we sum over all admissible sets of contours, one of them being a $\hat{n}_y$-contour, and one of them (possibly the same) having a winding weight. For a given set, we define $\mathcal{Y}_{\hat{n}_y}$ to be $Y_{\hat{n}_y}$ and all the contours surrounding it. Then we consider the contour with the winding weight, as well as the contours surrounding it. If its interior contains $\operatorname{Supp} \mathcal{Y}_{\hat{n}_y}$, we denote it by $\mathcal{Y}'$, and we define the corresponding weight

$$\rho'_{\circlearrowleft}(\mathcal{Y}') = \begin{cases} \rho_{\hat{n}_y,\circlearrowleft}(Y_{\hat{n}_y}) \prod_{Y' \in \mathcal{Y}', Y' \neq Y_{\hat{n}_y}} \rho(Y') & \text{if } \mathcal{Y}' = \mathcal{Y}_{\hat{n}_y}, \\ \rho_{\circlearrowleft}(Y) \rho_{\hat{n}_y}(Y_{\hat{n}_y}) \prod_{Y' \in \mathcal{Y}', Y' \neq Y, Y_{\hat{n}_y}} \rho(Y') & \text{otherwise.} \end{cases} \tag{4.26}$$

If it does not contain $\operatorname{Supp} \mathcal{Y}_{\hat{n}_y}$, we denote it by $\mathcal{Y}$, and the weight is given by (4.15).

With

$$\rho_{\hat{n}_y}(\mathcal{Y}_{\hat{n}_y}) = \rho_{\hat{n}_y}(Y_{\hat{n}_y}) \prod_{Y \in \mathcal{Y}_{\hat{n}_y}, Y \neq Y_{\hat{n}_y}} \rho(Y),$$

we can rewrite (4.25),

$$\begin{aligned} \Big\langle \sum_{x \in \Lambda} (\hat{n}_x - g_x^{(q)}) \hat{n}_y \Big\rangle_{q,\Lambda} = \frac{1}{Z_{q,\Lambda}} \Big[\sum_{\mathcal{Y}'} \rho'_{\circlearrowleft}(\mathcal{Y}') \sum_{\{Y_1,\dots,Y_k\}:\mathcal{Y}'} \prod_{i=1}^{k} \rho(Y_i) \prod_{m=1}^{r(\mu)} \mathrm{e}^{-\tilde{\beta} e_m |W_m|} \\ + \sum_{\mathcal{Y}} \rho_{\circlearrowleft}(\mathcal{Y}) \sum_{\mathcal{Y}_{\hat{n}_y}:\mathcal{Y}} \rho_{\hat{n}_y}(\mathcal{Y}_{\hat{n}_y}) \sum_{\{Y_1,\dots,Y_k\}:\mathcal{Y} \cup \mathcal{Y}_{\hat{n}_y}} \prod_{i=1}^{k} \rho(Y_i) \prod_{m=1}^{r(\mu)} \mathrm{e}^{-\tilde{\beta} e_m |W_m|} \Big] \end{aligned} \tag{4.27}$$

where the constraint $\mathcal{Y}_{\hat{n}_y} : \mathcal{Y}$ means that their supports do not intersect, nor do their interiors, and the two sums over sets of contours are such that $\{Y_1, \dots, Y_k\} \cup \tilde{\mathcal{Y}}$ is admissible and compatible with the boundary conditions q, and moreover for any i, $\operatorname{Int} Y_i \cap \operatorname{Supp} \tilde{\mathcal{Y}} = \emptyset$ ($\tilde{\mathcal{Y}} = \mathcal{Y}'$ in the first sum and $\tilde{\mathcal{Y}} = \mathcal{Y} \cup \mathcal{Y}_{\hat{n}_y}$ in the second one).

We now apply the Pirogov-Sinai transformations; namely, $K_q(Y)$ and $K_{q,\circlearrowleft}(\mathcal{Y})$ are defined by (4.17) and analogously

$$K'_{q,\circlearrowleft}(\mathcal{Y}') = \rho'_{\circlearrowleft}(\mathcal{Y}')\, \mathrm{e}^{\tilde{\beta} e_q |\mathcal{Y}'|} \prod_{m=1}^{r(\mu)} \frac{Z_m(\operatorname{Int}_m \mathcal{Y}')}{Z_q(\operatorname{Int}_m \mathcal{Y}')}, \tag{4.28a}$$

$$K_{q,\hat{n}_y}(\mathcal{Y}_{\hat{n}_y}) = \rho_{\hat{n}_y}(\mathcal{Y}_{\hat{n}_y})\, \mathrm{e}^{\tilde{\beta} e_q |\mathcal{Y}_{\hat{n}_y}|} \prod_{m=1}^{r(\mu)} \frac{Z_m(\operatorname{Int}_m \mathcal{Y}_{\hat{n}_y})}{Z_q(\operatorname{Int}_m \mathcal{Y}_{\hat{n}_y})}. \tag{4.28b}$$

Then, with $\{Z_1, \dots, Z_k\}$ being non-intersecting contours with $\alpha_{Z_i}^{\text{ext}} = q$, and the constraints indicating that they cannot encircle $\mathcal{Y}'$, we obtain an expression convenient for applying cluster expansion, namely

$$\begin{aligned} \Big\langle \sum_{x \in \Lambda} (\hat{n}_x - g_x^{(q)}) \hat{n}_y \Big\rangle_{q,\Lambda} &= \frac{\mathrm{e}^{-\tilde{\beta} e_q |\Lambda|}}{Z_{q,\Lambda}} \Big[\sum_{\mathcal{Y}'} K'_{q,\circlearrowleft}(\mathcal{Y}') \sum_{\{Z_1,\dots,Z_k\}:\mathcal{Y}'} \prod_{i=1}^{k} K_q(Y_i) \\ &\qquad + \sum_{\mathcal{Y}} K_{q,\circlearrowleft}(\mathcal{Y}) \sum_{\mathcal{Y}_{\hat{n}_y}:\mathcal{Y}} K_{q,\hat{n}_y}(\mathcal{Y}_{\hat{n}_y}) \sum_{\{Z_1,\dots,Z_k\}:\mathcal{Y} \cup \mathcal{Y}_{\hat{n}_y}} \prod_{i=1}^{k} K_q(Y_i) \Big] \\ &= \sum_{\mathcal{Y}'} K'_{q,\circlearrowleft}(\mathcal{Y}')\, \mathrm{e}^{-\sum_{C \not\sim \mathcal{Y}'} \Phi^T(C)} + \sum_{\mathcal{Y}} K_{q,\circlearrowleft}(\mathcal{Y}) \sum_{\mathcal{Y}_{\hat{n}_y}:\mathcal{Y}} K_{q,\hat{n}_y}(\mathcal{Y}_{\hat{n}_y})\, \mathrm{e}^{-\sum_{C \not\sim [\mathcal{Y} \cup \mathcal{Y}_{\hat{n}_y}]} \Phi^T(C)}. \end{aligned} \tag{4.29}$$

A similar calculation leads to

$$\Big\langle\sum_{x\in\Lambda}(\hat{n}_x - g_x^{(q)})\Big\rangle_{q,\Lambda}\langle\hat{n}_y\rangle_{q,\Lambda} = \Big[\sum_{\mathcal{Y}} K_{q,\circlearrowleft}(\mathcal{Y})\,\mathrm{e}^{-\sum_{C\not\sim\mathcal{Y}}\Phi^T(C)}\Big]\Big[\sum_{\mathcal{Y}_{\hat{n}_y}} K_{q,\hat{n}_y}(\mathcal{Y}_{\hat{n}_y})\,\mathrm{e}^{-\sum_{C\not\sim\mathcal{Y}_{\hat{n}_y}}\Phi^T(C)}\Big]. \tag{4.30}$$

We now compute the difference $\check{n}_y$, and we find

$$\begin{aligned}\check{n}_y &= \sum_{\mathcal{Y}'} K'_{q,\circlearrowleft}(\mathcal{Y}')\,\mathrm{e}^{-\sum_{C\not\sim\mathcal{Y}'}\Phi^T(C)} \\ &\quad - \sum_{\mathcal{Y},\mathcal{Y}_{\hat{n}_y}:[\operatorname{Supp}\mathcal{Y}\cup\operatorname{Int}\mathcal{Y}]\cap[\operatorname{Supp}\mathcal{Y}_{\hat{n}_y}\cup\operatorname{Int}\mathcal{Y}_{\hat{n}_y}]\neq\emptyset} K_{q,\circlearrowleft}(\mathcal{Y})\,\mathrm{e}^{-\sum_{C\not\sim\mathcal{Y}}\Phi^T(C)}\,K_{q,\hat{n}_y}(\mathcal{Y}_{\hat{n}_y})\,\mathrm{e}^{-\sum_{C\not\sim\mathcal{Y}_{\hat{n}_y}}\Phi^T(C)} \\ &\quad + \sum_{\mathcal{Y}} K_{q,\circlearrowleft}(\mathcal{Y})\,\mathrm{e}^{-\sum_{C\not\sim\mathcal{Y}}\Phi^T(C)} \sum_{\mathcal{Y}_{\hat{n}_y}:\mathcal{Y}} K_{q,\hat{n}_y}(\mathcal{Y}_{\hat{n}_y})\,\mathrm{e}^{-\sum_{C\not\sim\mathcal{Y}_{\hat{n}_y}}\Phi^T(C)}\Big[\mathrm{e}^{\sum_{C,C\not\sim\mathcal{Y}\text{ and }C\not\sim\mathcal{Y}_{\hat{n}_y}}\Phi^T(C)} - 1\Big].\end{aligned} \tag{4.31}$$

In the above equation, each term contains at least one winding contour, with length bigger than or equal to $\beta/\tilde{\beta}$, and somehow connected to the site y (because $(y, t = 0)$ is the support of the operator $\hat{n}_y$) — in the expression of the last line, the connection is through clusters intersecting both $\mathcal{Y}$ and $\mathcal{Y}_{\hat{n}_y}$; note that a sum over clusters with a minimal length yields an exponentially decreasing term with respect to that length [KP]. It is thus not hard to establish that $|\check{n}_y| \leqslant \mathrm{e}^{-a\beta}$ for some constant $a > 0$.
Remark: one-dimensional systems. Winding contours have not necessarily connected supports. However, we artificially connect them with a path smaller than L, i.e. smaller than the cardinality of the support, because $M \geqslant L$ and $|\operatorname{Supp} Y_{\circlearrowleft}| \geqslant M$. Then we can work with these new contours as usual.

□

4.2. Proofs for the Bose-Hubbard model.

Proof of Theorem 1.1 i). Let us first establish the Peierls condition (2.8). The classical part of the (two-dimensional) Bose-Hubbard Hamiltonian (1.2) — with chemical potential — may be written as a potential over plaquettes of 4 sites,

$$H_\Lambda^{(0)}(\boldsymbol{n}) = \sum_{P\subset\Lambda} H_P(n_P) \tag{4.32}$$

where

$$H_P(n_P) = \sum_{x\in P}\tfrac14(U_0 n_x^2 - U_0 n_x - \mu n_x) + \tfrac12 U_1 \sum_{\substack{x,y\in P\\|x-y|=1}} n_x n_y + U_2 \sum_{\substack{x,y\in P\\|x-y|=\sqrt2}} n_x n_y. \tag{4.33}$$

With n an integer we introduce new variables, $n_x = n + m_x$, and with $H_P(n, m_P) = H_P(n_P)$, a straightforward calculation leads to

$$H_P(n, m_P) = C_n + \sum_{x\in P}\tfrac14(U_0 m_x^2 - U_0 m_x - \mu_n m_x) + \tfrac12 U_1 \sum_{\substack{x,y\in P\\|x-y|=1}} m_x m_y + U_2 \sum_{\substack{x,y\in P\\|x-y|=\sqrt2}} m_x m_y, \tag{4.34}$$

where we defined $C_n = \frac14 U_0(n^2 - n) - \frac14\mu n + 4n^2(U_1 + U_2)$, and

$$\mu_n = \mu - (2U_0 + 8U_1 + 8U_2)n. \tag{4.35}$$

In the following, we show that, for a given n,

a) if $\mu_n \in [-2U_0, 0]$, $m_P = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ minimizes $H_P(n, m_P)$,
b) if $\mu_n \in [0, 8U_2]$, $m_P = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ (and the three obtained by rotation) minimizes $H_P(n, m_P)$,
c) if $\mu_n \in [8U_2, 8U_1]$, $m_P = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ (and the other obtained by rotation) minimizes $H_P(n, m_P)$,
d) if $\mu_n \in [8U_1, 8U_1 + 8U_2]$, $m_P = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ (and the three obtained by rotation) minimizes $H_P(n, m_P)$.

Clearly, from this and (4.35) we obtain the classical ground states for all $\mu \geqslant 0$ [and in the case $\mu < 0$, we see immediately in (4.33) that $n_x = 0$, for any x, minimizes $H_P(n_P)$].

For the point a), let us introduce a such that $\mu_n = -2U_0(a + \frac{1}{2})$; it is easy to check that

$$H_P(n, m_P) = C_n' + (\tfrac{1}{4}U_0 - U_1 - U_2) \sum_{x \in P} (m_x + a)^2 + \tfrac{1}{4}U_1 \sum_{\substack{x,y \in P \\ |x-y|=1}} (m_x + m_y + a)^2 + \tfrac{1}{2}U_2 \sum_{\substack{x,y \in P \\ |x-y|=\sqrt{2}}} (m_x + m_y + a)^2, \quad (4.36)$$

and this is minimum for $m_x = 0$, for any $x \in P$, when $a \in [-\frac{1}{2}, \frac{1}{2}]$, i.e. $\mu_n \in [-2U_0, 0]$.

Moreover, we obtain a Peierls condition if $\mu_n \neq -2U_0, 0$.

Point c) is similar; we define a such that $\frac{1}{4}\mu_n = U_1 + U_2 - 2a(U_1 - U_2)$; in this case

$$H_P(n, m_P) = C_n'' + (\tfrac{1}{4}U_0 - U_1 + U_2) \sum_{x \in P} (m_x - \tfrac{1}{2})^2 + (\tfrac{1}{4}U_1 - \tfrac{1}{2}U_2) \sum_{\substack{x,y \in P \\ |x-y|=1}} (m_x + m_y - 1 + a)^2 + U_2 \Big(\sum_{x \in P} m_x - 2 + a\Big)^2. \quad (4.37)$$

$\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is ground state when $a \in [-\frac{1}{2}, \frac{1}{2}]$, i.e. $\mu_n \in [8U_2, 8U_1]$ (recall that $U_1 > 2U_2$). The Peierls condition is also straightforward.

Finally, we show that $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is ground state for $\mu_n \in [0, 8U_2]$ and $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ for $\mu_n \in [-2U_0 - 8U_2, -2U_0]$. With $\varepsilon = +1$ in the first case and $\varepsilon = -1$ in the second case, we have

$$H_P(n, m_P) = C_n^\varepsilon + (\tfrac{1}{4}U_0 - U_1 + U_2) \sum_{x \in P} (m_x - \tfrac{1}{2}\varepsilon)^2 + (\tfrac{1}{4}U_1 - \tfrac{1}{2}U_2) \sum_{\substack{x,y \in P \\ |x-y|=1}} (m_x + m_y - \tfrac{1}{2}\varepsilon)^2 + U_2 \Big(\sum_{x \in P} m_x - \varepsilon + a\Big)^2 \quad (4.38)$$

where $a = \frac{1}{2} - \mu_n/8U_2$ in the first case, and $a = -\frac{1}{2} - (\mu_n + 2U_0)/8U_2$ in the second case; the condition $a \in [-\frac{1}{2}, \frac{1}{2}]$ yields the intervals for μ_n.

Theorem 1.1 i) is then a special case of Theorem 2.3.

□

For parts ii) and iii) we use Theorem 2.1. The splitting condition may be fulfilled by adding "external fields" with projectors onto classical ground configurations, see the paragraph introducing the proof of Theorem 2.3.

Proof of Theorem 1.1 ii). From Theorem 2.2,

$$\langle a_x^\dagger a_y \rangle_{\text{per}} = \frac{1}{Q(\boldsymbol{\mu})} \sum_q \langle a_x^\dagger a_y \rangle_q = \frac{1}{Q(\boldsymbol{\mu})} \sum_q \big[\langle a_x^\dagger a_y \rangle_q - \langle a_x^\dagger \rangle_q \langle a_y \rangle_q \big], \quad (4.39)$$

since $\langle a_x^\dagger \rangle_q = \langle a_y \rangle_q = 0$. The conclusion is immediate by Theorem 2.1 iii) — $a_x^\dagger$ and a_y are indeed moderately off-diagonal, see (2.18). □

Proof of Theorem 1.1 iii). From the definition of the structure factor

$$|S(k)| = \frac{1}{|\Lambda|^2}\Big|\sum_{x,y\in\Lambda} \mathrm{e}^{\mathrm{i}k(x-y)}\, \langle \hat{n}_x\rangle_{q,\Lambda}\langle \hat{n}_y\rangle_{q,\Lambda} + \sum_{x,y\in\Lambda} \mathrm{e}^{\mathrm{i}k(x-y)}\, \Big(\langle \hat{n}_x\hat{n}_y\rangle_{q,\Lambda} - \langle \hat{n}_x\rangle_{q,\Lambda}\langle \hat{n}_y\rangle_{q,\Lambda}\Big)\Big|$$

$$\geqslant \frac{1}{|\Lambda|^2}\Big|\sum_{x,y\in\Lambda} \mathrm{e}^{\mathrm{i}k(x-y)}\, \langle \hat{n}_x\rangle_{q,\Lambda}\langle \hat{n}_y\rangle_{q,\Lambda}\Big| - \frac{1}{|\Lambda|^2}\sum_{x,y\in\Lambda}\Big| \mathrm{e}^{\mathrm{i}k(x-y)}\, \Big(\langle \hat{n}_x\hat{n}_y\rangle_{q,\Lambda} - \langle \hat{n}_x\rangle_{q,\Lambda}\langle \hat{n}_y\rangle_{q,\Lambda}\Big)\Big|. \quad (4.40)$$

Using Theorem 2.1 iii) we easely find the following bound for the last term

$$\frac{1}{|\Lambda|}\frac{\text{const}}{1-\mathrm{e}^{-1/\xi_q}} =: \frac{C_1}{|\Lambda|}$$

with const depending only on the dimension d. Therefore

$$|S(k)| \geqslant \Big|\frac{1}{|\Lambda|}\sum_{x\in\Lambda} \mathrm{e}^{\mathrm{i}kx}\, \langle \hat{n}_x\rangle_{q,\Lambda}\Big|^2 - \frac{C_1}{|\Lambda|} \quad (4.41)$$

and it is sufficient to show that the Fourier transform of $\langle \hat{n}_x\rangle_{q,\Lambda}$ is bigger than a constant independent of the lattice.

$$\frac{1}{|\Lambda|}\Big|\sum_{x\in\Lambda} \mathrm{e}^{\mathrm{i}kx}\, \langle \hat{n}_x\rangle_{q,\Lambda}\Big| \geqslant \frac{1}{|\Lambda|}\Big|\sum_{x\in\Lambda} \mathrm{e}^{\mathrm{i}kx}\, \langle \hat{n}_x P^{(q)}_{U(x)}\rangle_{q,\Lambda}\Big| - \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\Big|\langle \hat{n}_x(1-P^{(q)}_{U(x)})\rangle_{q,\Lambda}\Big|. \quad (4.42)$$

From Cauchy-Schwarz inequality

$$\langle AB\rangle \leqslant \langle A^2\rangle^{\frac{1}{2}}\langle B^2\rangle^{\frac{1}{2}}$$

for any symmetric operators A, B whose domains contain $\mathcal{B}_\Lambda$. Then

$$\frac{1}{|\Lambda|}\Big|\sum_{x\in\Lambda} \mathrm{e}^{\mathrm{i}kx}\, \langle \hat{n}_x\rangle_{q,\Lambda}\Big| \geqslant \frac{1}{|\Lambda|}\Big|\sum_{x\in\Lambda} \mathrm{e}^{\mathrm{i}kx}\, \frac{1}{Z_{q,\Lambda}}\sum_{n\in\mathcal{B}_\Lambda} n_x\langle n|\, P^{(q)}_{U(x)}\, \mathrm{e}^{-\beta H_{q,\Lambda}}\, |n\rangle\Big|$$

$$- \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\Big|\langle \hat{n}_x^2\rangle_{q,\Lambda}\Big|^{\frac{1}{2}}\Big|\langle(1-P^{(q)}_{U(x)})\rangle_{q,\Lambda}\Big|^{\frac{1}{2}}. \quad (4.43)$$

Observing that in the sum over $\mathcal{B}_\Lambda$, $n_x = g^{(q)}_x$ because of the action of the projector $P^{(q)}_{U(x)}$, and using Theorem 2.1 ii) and iv) for the second expression

$$\frac{1}{|\Lambda|}\Big|\sum_{x\in\Lambda} \mathrm{e}^{\mathrm{i}kx}\, \langle \hat{n}_x\rangle_{q,\Lambda}\Big| \geqslant \frac{1}{|\Lambda|}\Big|\sum_{x\in\Lambda} \mathrm{e}^{\mathrm{i}kx}\, g^{(q)}_x\langle P^{(q)}_{U(x)}\rangle_{q,\Lambda}\Big| - C_2 \quad (4.44)$$

where C_2 is a constant as small as we may need if the temperature is sufficiently low and the hopping perturbation small.

$$\frac{1}{|\Lambda|}\Big|\sum_{x\in\Lambda} \mathrm{e}^{\mathrm{i}kx}\, \langle \hat{n}_x\rangle_{q,\Lambda}\Big| \geqslant \frac{1}{|\Lambda|}\Big|\sum_{x\in\Lambda} \mathrm{e}^{\mathrm{i}kx}\, g^{(q)}_x\Big| - \frac{1}{|\Lambda|}\sum_{x\in\Lambda} g^{(q)}_x\Big|\langle P^{(q)}_{U(x)}\rangle_{q,\Lambda} - 1\Big| - C_2 \quad (4.45)$$

and the proof is achieved since $g^{(q)}_x$ is a chessboard type configuration and $k = (\pi,\pi)$, and the last sum is bounded by a small constant because of Theorem 2.1 iv).

□

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