From Bose-Einstein Condensation to Probability Theory

BY DANIEL UELTSCHI

The Bose-Einstein condensation is a phase transition that takes place in certain large systems of quantum particles. It is believed to give rise to such striking physical phenomena as superfluidity and superconductivity. Besides atomic physics and condensed matter theory, it is also present in turbulence and optics, two strong research areas at The University of Arizona. The theoretical discovery of the Bose-Einstein condensation is an intriguing tale in the history of science and I cannot resist the temptation to retrace it briefly.

In 1924, the unknown Indian physicist Satyendra Nath Bose (pronounced "Bosh") submits an article to the Proceedings of the Royal Society, which rejects it. Nothing unusual, and Bose should have accepted the opinions of established academics. He did not, however, going so far as to send his manuscript to Albert Einstein, the leading physicist of his time... and Einstein took the time to study it, and even translated it himself into German and recommended its publication in the journal *Zeitschrift für Physik.* The article of Bose showed that, by using certain rules for counting "particle configurations"—rules that are referred to nowadays as the "symmetry principle" for identical bosonic particles—it is possible to get the equation of black-body radiation. Shortly afterwards, Einstein wrote his own articles, pointing out that an ideal (i.e. non-interacting) gas of bosonic particles undergoes a "condensation" where the quantum state of lower energy becomes macroscopically occupied. In addition, he gave an exact formula for the critical

temperature, namely $T_c = \frac{2\pi\hbar^2}{mk_B}\left(\rho/\zeta\left(\frac{3}{2}\right)\right)^{\frac{2}{3}}$ where m is

the particle mass, $\hbar$ is Planck's constant, k_B is Boltzmann's constant, ρ is the density,

and $\zeta\left(\frac{3}{2}\right) \approx 2.612$ is Riemann zeta function at $\frac{3}{2}$.

The concept of Bose-Einstein condensation was born. It took many years for physicists to believe that these calculations reflect a genuine physical phenomenon. And it took a full 70 years for experimental physicists

to catch up and create a "Bose condensate" in a laboratory, in 1995. Clearly, theoretical physics scored here an impressive success, and this is why the 2001 Nobel prize of physics was awarded to the experimentalists as a second-place consolation prize.[1]

My current study of interacting bosonic systems is motivated by Feynman's path integral approach, which dates back from the 1950s. Quantum particles are not represented by points in the space, but by "Brownian motions," i.e. by an erratic random walk that keeps changing its direction. Bringing in the symmetry principle, one is led to a representation that involves points in the space, permutations of these points, and Brownian bridges between the points. If we neglect the particle interactions, we can integrate the contribution of Brownian bridges and we obtain "spatial permutations." This notion seems interesting in its own right and I want to describe it in more detail.

Usually, a permutation of n elements is a bijection from {1, : : :, n} to {1, . . . , n}. But here, a *spatial* permutation involves a set of points $x_1, : : :, x_2 \in \mathbb{R}^d$ and a permutation π of n elements. See Fig. 1 for an illustration. We now consider a probability distribution on these spatial permutations. Let us fix the point locations. The probability of the permutation π is proportional to

$$\prod_{i=1}^{n} e^{-\xi(x_i - x_{\pi(i)})}.$$

That is, to each permutation jump $x_i \mapsto x_{\pi(i)}$ there is a factor $e^{-\xi(x_i - x_{\pi(i)})}$. The most relevant choice for ξ, the one that shows up in the quantum Bose gas, is $\xi(x) = \frac{1}{4\beta}|x|^2$.

Here, $\beta = 1/k_B T$ is proportional to the inverse temperature of the system. The effect of these factors is to force all jumps to be small—a point cannot jump to the other side of the box in one step. But although all permutation jumps are "local," the possibility exists for global behavior: a permutation cycle may involve a lot of points that cover a large portion of space. The main question is whether, as the size of the system and the number of points becomes infinite (with fixed density, i.e. with fixed mean-distance between points), some points find themselves in cycles of infinite length. It is expected that a transition to

infinite cycles takes place when the density increases (or when the temperature is lowered) and this transition has something to do with the Bose-Einstein condensation described above.

My interest in this topic goes back to 2001 when I was a postdoc in Princeton, but I really started working on it during my years at UA. I clarified and extended the very fine study of A. Sütő, a mathematical-physicist from Hungary, who had shown that the occurrence of infinite cycles coincides with Bose-Einstein condensation in the ideal gas. Since then I have been joined by several collaborators: D. Gandolfo and J. Ruiz of Marseille, my colleague V. Betz of Warwick University, and John Kerl of UA. John is currently working on lattice permutations as part of his PhD thesis, under the supervision of T. Kennedy.

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Thanks to the help of those collaborators, we have made good progress, especially in proving the existence of a critical density above which infinite (macroscopic) cycles appear. This holds for many jump functions, and also when certain simple interactions between permutation jumps are included. We noticed that some of the problems appeared in studies by mathematical biologists and probabilists. One ambitious goal is to calculate the effects of particle interactions on the critical density of Bose-Einstein condensation. This question has not been fully resolved by physicists; the present approach should allow for some useful analytical steps and more precise numerical simulations. At UA, John is developing numerical techniques that should help answering many questions in the future.

Figure 1. A typical "spatial permutation." There are seven cycles with more than one point (check it!). The box is supposed to be huge, but the permutation jumps have finite length. The main question is about the occurrence of infinitely long permutation cycles.

He also identified a few problems about transition probabilities between spatial permutations that belong to the area of discrete mathematics, and for which we hope to make good progress.

I am currently spending a useful and very pleasant semester in Arizona. My collaborators V. Betz, S. Poghosyan, and R. Seiringer, who came to Tucson, enjoyed the Arizona-Sonora Desert Museum and the Pima Air & Space Museum. My favorite activities include hiking (Sabino and Bear Canyon, Wasson Peak, and hopefully Mount Wrightson soon), playing street hockey (Want to join? We are looking for new players: www.pimastreethockey.com), and enjoying the beautiful campus of UA.

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[1] Of course, this remark should not be taken seriously. The two experimental realizations of Bose-Einstein condensation by Cornell and Wieman, and by Ketterle, were extremely difficult. They involved creating equilibrium systems of many bosons at very low temperatures (around 2000 ^{87}Rb atoms below 170 nK, and 5×10^5 ^{23}Na atoms below 2μK). These physicists fully deserve their Nobel Prize.

Burn the Midnight Oil

"Lecturing during the day and running a project during the evening is an exhausting amount of work for the speakers. Having some help with the project makes their job a little more manageable, provides the students an extra person to talk with about the project, and is an opportunity for

the assistant to get some experience working with students in an advisory role," Savitt said.

Ulmer said the school has always been about change, adding that in the beginning especially, "there was a fair amount of experimentation." One idea, for instance, was a unit on professional development. "It was harder than we thought,"