The critical temperature of the dilute Bose gas: A tentative exact approach using spatial permutations

Daniel Ueltschi

Department of Mathematics, University of Warwick

January 9, 2010

Joint work with Volker Betz (University of Warwick)

Bose-Einstein condensation for ideal gas

Hamiltonian for N bosons in box $\Lambda \subset \mathbb{R}^d$:

$$\boldsymbol{H} = -\sum_{i=1}^{N} \Delta_i \qquad \text{in } L^2_{\text{sym}}(\Lambda^N)$$

In Fourier space: $\text{Tr } e^{-\beta \boldsymbol{H}} = \sum_{(n_k): \sum_k n_k = N} \prod_{k \in \Lambda^*} e^{-\beta k^2 n_k}$, $\Lambda^* = \frac{2\pi}{L}\mathbb{Z}^d$

Expectation of zero mode:

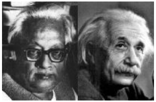
$$\frac{\langle n_0 \rangle}{V} = \frac{\sum_{(n_k)} n_0 \prod_{k \in \Lambda^*} e^{-\beta k^2 n_k}}{\text{Tr } e^{-\beta \boldsymbol{H}}} \longrightarrow \begin{cases} 0 & \text{if } \rho \leqslant \rho_c \\ \rho - \rho_c & \text{if } \rho \geqslant \rho_c \end{cases}$$

with critical density

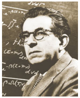
$$\rho_c = \frac{\zeta(\frac{d}{2})}{(4\pi\beta)^{d/2}}$$

Brief history

1924-25: Bose & Einstein understand that the ideal Bose gas displays "macroscopic occupation of zero mode"



1938: F. London suggests that superfluidity is related to BEC

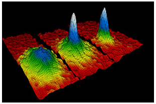


What does BEC mean in interacting systems?

1947: Боголюбов's microscopic theory of superfluidity
1951-53: Matsubara & Feynman consider the length of permutation cycles

1956: Penrose & Onsager introduce correct order parameter, the "off-diagonal long-range order"

1995: BEC finally observed (Cornell, Wieman, Ketterle)



Interacting Bose gas: Feynman-Kac representation

Hamiltonian: $H = -\sum_{i=1}^{N} \Delta_i + \sum_{1 \leqslant i<j \leqslant N} U(x_i - x_j)$ in $L^2_{\text{sym}}(\Lambda^N)$.

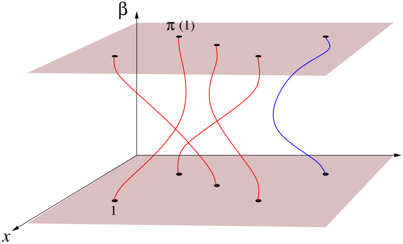
U : repulsive pair potential, $N = \rho|\Lambda|$

Feynman-Kac representation of the partition function:

$$\text{Tr } e^{-\beta H} = \frac{1}{N!} \sum_{\pi \in \mathcal{S}_N} \int dx_1 \dots dx_N$$

$$\int dW^{2\beta}_{x_1 x_{\pi(1)}}(\omega_1) \dots dW^{2\beta}_{x_N x_{\pi(N)}}(\omega_N)$$

$$\exp\Big\{ -\tfrac{1}{2} \sum_{i<j} \int_0^{2\beta} U\big(\omega_i(s) - \omega_j(s)\big) ds \Big\}$$



Conjecture 1

CONJECTURE 1. The critical temperature for Bose-Einstein condensation is identical to the critical temperature for the occurrence of infinite cycles.

This was proved by Sütő ('93, '02) for the ideal gas ($U = 0$).

Model of "spatial random permutations"

We write $\operatorname{Tr} \mathrm{e}^{-\beta H} = \frac{1}{Z} \sum_{\pi \in \mathcal{S}_N} \int \mathrm{d}x_1 \dots \mathrm{d}x_N \, \mathrm{e}^{-H_1(\boldsymbol{x},\pi)}$, with

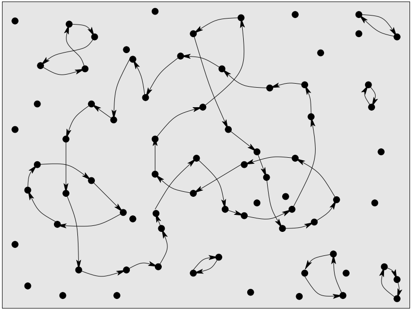
$$\mathrm{e}^{-H_1(\boldsymbol{x},\pi)} = \int \mathrm{d}W^{2\beta}_{x_1 x_{\pi(1)}}(\omega_1) \dots \mathrm{d}W^{2\beta}_{x_N x_{\pi(N)}}(\omega_N) \exp\Big\{ -\tfrac{1}{2} \sum_{i<j} \int_0^{2\beta} U\big(\omega_i(s) - \omega_j(s)\big) \mathrm{d}s \Big\}$$

U has scattering length a. A cluster expansion leads to $H_1 = H_2 + \mathcal{O}(a^2)$ with

$$H_2(\boldsymbol{x},\pi) = \frac{1}{4\beta} \sum_{i=1}^N |x_i - x_{\pi(i)}|^2 + \sum_{i<j} V(x_i, x_{\pi(i)}, x_j, x_{\pi(j)})$$

The 2-jump interaction $V(\cdot)$ is given by the integral kernel of $\mathrm{e}^{2\beta\Delta} - \mathrm{e}^{2\beta(\Delta - U)}$

Spatial random permutations



$$\mathbb{E}_N(\theta) = \frac{1}{Z_N} \sum_{\pi \in \mathcal{S}_N} \int_{\Lambda^n} \mathrm{d}x_1 \dots \mathrm{d}x_N \, \theta(x,\pi) \, \mathrm{e}^{-H_2(\boldsymbol{x},\pi)}$$

with

$$H_2(\boldsymbol{x},\pi) = \frac{1}{4\beta} \sum_{i=1}^{N} |x_i - x_{\pi(i)}|^2 + \sum_{i<j} V(x_i, x_{\pi(i)}, x_j, x_{\pi(j)}).$$

Conjecture 2

CONJECTURE 2. The critical temperature of the model H_2 is identical to that of the model H_1, up to a correction $o(a)$.

From 2-jump interactions to cycle weights

Goal: replace the 2-jump interactions $\sum_{i<j} V_{ij}(\boldsymbol{x},\pi)$ by cycle weights $\sum_{\ell \geqslant 1} \alpha_\ell r_\ell(\pi)$ in such a way that critical density remains identical (to lowest order)

Let θ be a random variable that depends only on the permutation, not on the positions. Its expectation can be written as

$$E(\theta) = \frac{1}{Z_2} \sum_\pi \theta(\pi) Z^{(\pi)} \int \mathrm{d}\mu^{(\pi)}(\boldsymbol{x})\, \mathrm{e}^{-\sum_{i<j} V_{ij}(\boldsymbol{x},\pi)}$$

where $\mu^{(\pi)}$ is a probability measure that depends on π, given by

$$\mathrm{d}\mu^{(\pi)}(\boldsymbol{x}) = \frac{1}{Z^{(\pi)}}\, \mathrm{e}^{-H_0(\boldsymbol{x},\pi)}\, \mathrm{d}\boldsymbol{x}, \qquad Z^{(\pi)} = \int \mathrm{e}^{-H_0(\boldsymbol{x},\pi)}\, \mathrm{d}\boldsymbol{x}$$

From 2-jump interactions to cycle weights

Introduce a "permutation free energy" by

$$\mathrm{e}^{-F(\pi)} = \int \mathrm{d}\mu^{(\pi)}(\boldsymbol{x})\, \mathrm{e}^{-\sum_{i<j} V_{ij}(\boldsymbol{x},\pi)}$$

Main idea: take the integral inside the exponential

$$\mathrm{e}^{-F(\pi)} \approx \exp\Big\{-\int \mathrm{d}\mu^{(\pi)}(\boldsymbol{x}) \sum_{i<j} V_{ij}(\boldsymbol{x},\pi)\Big\}$$

Justification: think in terms of typical positions. For given permutation, let $\boldsymbol{y}$ be typical realization of measure $\mu^{(\pi)}$, and let $\boldsymbol{z}$ be typical realization of measure

$$\frac{1}{\text{normalization}}\, \mathrm{e}^{-H_0(\boldsymbol{x},\pi)}\; \mathrm{e}^{-\sum_{i<j} V_{ij}(\boldsymbol{x},\pi)} .$$

From 2-jump interactions to cycle weights

For all macroscopic observables A we expect that

$$A(\boldsymbol{y}) = A(\boldsymbol{z})\,(1 + O(a))$$

This holds in particular when macroscopic observable is $\sum_{i<j} V_{ij}(\boldsymbol{x}, \pi)$
We have $\mathrm{e}^{-F(\pi)} \approx \mathrm{e}^{-\sum_{i<j} V_{ij}(\boldsymbol{z},\pi)}$.
We replace it by $\mathrm{e}^{-F(\pi)} \approx \mathrm{e}^{-\sum_{i<j} V_{ij}(\boldsymbol{y},\pi)}$, the difference of free energies should be of order $O(a^2)$.
We get the new Hamiltonian

$$H_3(\boldsymbol{x}, \pi) = H_0(\boldsymbol{x}, \pi) + \sum_{i<j} \int V_{ij}(\boldsymbol{y}, \pi)\,\mathrm{d}\mu^{(\pi)}(\boldsymbol{y}).$$

CONJECTURE 3. The critical temperature of the model H_3 is identical to that of the model H_2, up to a correction $o(a)$.

From 2-jump interactions to cycle weights

Define the weights

$$\alpha_\ell = \sum_{i,j\in\gamma,i<j} \int V_{ij}(\boldsymbol{x},\gamma)\mathrm{d}\mu^{(\gamma)}(\boldsymbol{x}), \qquad \gamma = (2,\dots,\ell,1)$$

$$\alpha_{\ell,\ell'} = \tfrac{1}{2} \sum_{i\in\gamma,j\in\gamma'} \int V_{ij}(\boldsymbol{x},\gamma\cup\gamma')\mathrm{d}\mu^{(\gamma\cup\gamma')}(\boldsymbol{x}), \quad \gamma\cup\gamma' = (2,\dots,\ell,1)(\ell+2,\dots,\ell+\ell',\ell+1)$$

Then

$$H_3(\boldsymbol{x},\pi) = \frac{1}{4\beta}\sum_i |x_i - x_{\pi(i)}|^2 + \sum_{\ell\geqslant 1}(\alpha_\ell - \alpha_{\ell,\ell})r_\ell(\pi) + \sum_{\ell,\ell'\geqslant 1}\alpha_{\ell,\ell'}r_\ell(\pi)r_{\ell'}(\pi)$$

where $r_\ell(\pi)$ is the number of ℓ-cycles in the permutation π

Intermission: effects of interactions on crit. temperature

$\boldsymbol{H} = -\sum_{i=1}^{N} \Delta_i + \sum_{i<j} U(x_i - x_j)$, $U(x) \geqslant 0$ with scattering length a

Intermission: effects of interactions on crit. temperature

$$\boldsymbol{H} = -\sum_{i=1}^{N} \Delta_i + \sum_{i<j} U(x_i - x_j)$$, $U(x) \geqslant 0$ with scattering length a

1953 Feynman: $\Delta T_c < 0$

Intermission: effects of interactions on crit. temperature

$$\boldsymbol{H} = -\sum_{i=1}^{N} \Delta_i + \sum_{i<j} U(x_i - x_j)$$, $U(x) \geqslant 0$ with scattering length a

1953 Feynman: $\Delta T_\mathrm{c} < 0$

1958 Lee & Yang: $\frac{\Delta T_\mathrm{c}}{T_\mathrm{c}} \sim a\rho^{1/3}$

Intermission: effects of interactions on crit. temperature

$$\boldsymbol{H} = -\sum_{i=1}^{N} \Delta_i + \sum_{i<j} U(x_i - x_j)$$, $U(x) \geqslant 0$ with scattering length a

1953 Feynman: $\Delta T_\mathrm{c} < 0$

1958 Lee & Yang: $\frac{\Delta T_\mathrm{c}}{T_\mathrm{c}} \sim a\rho^{1/3}$

1960 Glassgold, Kaufman, Watson: $\frac{\Delta T_\mathrm{c}}{T_\mathrm{c}} \sim (a\rho^{1/3})^{1/2}$, increases

Intermission: effects of interactions on crit. temperature

$$\boldsymbol{H} = -\sum_{i=1}^{N} \Delta_i + \sum_{i<j} U(x_i - x_j)$$, $U(x) \geqslant 0$ with scattering length a

1953 Feynman: $\Delta T_c < 0$

1958 Lee & Yang: $\frac{\Delta T_c}{T_c} \sim a\rho^{1/3}$

1960 Glassgold, Kaufman, Watson: $\frac{\Delta T_c}{T_c} \sim (a\rho^{1/3})^{1/2}$, increases

1964 Huang: $\frac{\Delta T_c}{T_c} \sim (a\rho^{1/3})^{3/2}$, increases

Intermission: effects of interactions on crit. temperature

$$\boldsymbol{H} = -\sum_{i=1}^{N} \Delta_i + \sum_{i<j} U(x_i - x_j)$$, $U(x) \geqslant 0$ with scattering length a

1953 Feynman: $\Delta T_c < 0$

1958 Lee & Yang: $\frac{\Delta T_c}{T_c} \sim a\rho^{1/3}$

1960 Glassgold, Kaufman, Watson: $\frac{\Delta T_c}{T_c} \sim (a\rho^{1/3})^{1/2}$, increases

1964 Huang: $\frac{\Delta T_c}{T_c} \sim (a\rho^{1/3})^{3/2}$, increases

1971 Fetter & Walecka: $\frac{\Delta T_c}{T_c}$ decreases

Intermission: effects of interactions on crit. temperature

$$\boldsymbol{H} = -\sum_{i=1}^{N} \Delta_i + \sum_{i<j} U(x_i - x_j)$$, $U(x) \geqslant 0$ with scattering length a

1953 Feynman: $\Delta T_c < 0$

1958 Lee & Yang: $\frac{\Delta T_c}{T_c} \sim a\rho^{1/3}$

1960 Glassgold, Kaufman, Watson: $\frac{\Delta T_c}{T_c} \sim (a\rho^{1/3})^{1/2}$, increases

1964 Huang: $\frac{\Delta T_c}{T_c} \sim (a\rho^{1/3})^{3/2}$, increases

1971 Fetter & Walecka: $\frac{\Delta T_c}{T_c}$ decreases

1982 Toyoda: $\frac{\Delta T_c}{T_c} \approx -3.5(a\rho^{1/3})^{1/2}$

Intermission: effects of interactions on crit. temperature

$$\boldsymbol{H} = -\sum_{i=1}^{N} \Delta_i + \sum_{i<j} U(x_i - x_j)$$, $U(x) \geqslant 0$ with scattering length a

1953 Feynman: $\Delta T_{\mathrm{c}} < 0$

1958 Lee & Yang: $\frac{\Delta T_{\mathrm{c}}}{T_{\mathrm{c}}} \sim a\rho^{1/3}$

1960 Glassgold, Kaufman, Watson: $\frac{\Delta T_{\mathrm{c}}}{T_{\mathrm{c}}} \sim (a\rho^{1/3})^{1/2}$, increases

1964 Huang: $\frac{\Delta T_{\mathrm{c}}}{T_{\mathrm{c}}} \sim (a\rho^{1/3})^{3/2}$, increases

1971 Fetter & Walecka: $\frac{\Delta T_{\mathrm{c}}}{T_{\mathrm{c}}}$ decreases

1982 Toyoda: $\frac{\Delta T_{\mathrm{c}}}{T_{\mathrm{c}}} \approx -3.5(a\rho^{1/3})^{1/2}$

1992 Stoof: $\frac{\Delta T_{\mathrm{c}}}{T_{\mathrm{c}}} = c\, a\rho^{1/3} + o(a\rho^{1/3}), \quad c > 0$

Intermission: effects of interactions on crit. temperature

$$\boldsymbol{H} = -\sum_{i=1}^{N} \Delta_i + \sum_{i<j} U(x_i - x_j) \ , \ U(x) \geqslant 0 \text{ with scattering length } a$$

1953 Feynman: $\Delta T_c < 0$

1958 Lee & Yang: $\frac{\Delta T_c}{T_c} \sim a\rho^{1/3}$

1960 Glassgold, Kaufman, Watson: $\frac{\Delta T_c}{T_c} \sim (a\rho^{1/3})^{1/2}$, increases

1964 Huang: $\frac{\Delta T_c}{T_c} \sim (a\rho^{1/3})^{3/2}$, increases

1971 Fetter & Walecka: $\frac{\Delta T_c}{T_c}$ decreases

1982 Toyoda: $\frac{\Delta T_c}{T_c} \approx -3.5(a\rho^{1/3})^{1/2}$

1992 Stoof: $\frac{\Delta T_c}{T_c} = c\, a\rho^{1/3} + o(a\rho^{1/3}), \quad c > 0$

1996 Bijlsma & Stoof: $c = 4.66$

Intermission: effects of interactions on crit. temperature

$\boldsymbol{H} = -\sum_{i=1}^{N} \Delta_i + \sum_{i<j} U(x_i - x_j)$, $U(x) \geqslant 0$ with scattering length a

1953 Feynman: $\Delta T_\mathrm{c} < 0$

1958 Lee & Yang: $\frac{\Delta T_\mathrm{c}}{T_\mathrm{c}} \sim a\rho^{1/3}$

1960 Glassgold, Kaufman, Watson: $\frac{\Delta T_\mathrm{c}}{T_\mathrm{c}} \sim (a\rho^{1/3})^{1/2}$, increases

1964 Huang: $\frac{\Delta T_\mathrm{c}}{T_\mathrm{c}} \sim (a\rho^{1/3})^{3/2}$, increases

1971 Fetter & Walecka: $\frac{\Delta T_\mathrm{c}}{T_\mathrm{c}}$ decreases

1982 Toyoda: $\frac{\Delta T_\mathrm{c}}{T_\mathrm{c}} \approx -3.5(a\rho^{1/3})^{1/2}$

1992 Stoof: $\frac{\Delta T_\mathrm{c}}{T_\mathrm{c}} = c\, a\rho^{1/3} + o(a\rho^{1/3}), \quad c > 0$

1996 Bijlsma & Stoof: $c = 4.66$

1997 Grüter, Ceperley, Laloë: $c = 0.34$

Intermission: effects of interactions on crit. temperature

$\boldsymbol{H} = -\sum_{i=1}^{N} \Delta_i + \sum_{i<j} U(x_i - x_j)$, $U(x) \geqslant 0$ with scattering length a

1953 Feynman: $\Delta T_{\rm c} < 0$

1958 Lee & Yang: $\frac{\Delta T_{\rm c}}{T_{\rm c}} \sim a\rho^{1/3}$

1960 Glassgold, Kaufman, Watson: $\frac{\Delta T_{\rm c}}{T_{\rm c}} \sim (a\rho^{1/3})^{1/2}$, increases

1964 Huang: $\frac{\Delta T_{\rm c}}{T_{\rm c}} \sim (a\rho^{1/3})^{3/2}$, increases

1971 Fetter & Walecka: $\frac{\Delta T_{\rm c}}{T_{\rm c}}$ decreases

1982 Toyoda: $\frac{\Delta T_{\rm c}}{T_{\rm c}} \approx -3.5(a\rho^{1/3})^{1/2}$

1992 Stoof: $\frac{\Delta T_{\rm c}}{T_{\rm c}} = c\, a\rho^{1/3} + o(a\rho^{1/3}), \quad c > 0$

1996 Bijlsma & Stoof: $c = 4.66$

1997 Grüter, Ceperley, Laloë: $c = 0.34$

1999 Holzmann, Grüter, Laloë: $c = 0.7$

Intermission: effects of interactions on crit. temperature

$\boldsymbol{H} = -\sum_{i=1}^{N} \Delta_i + \sum_{i<j} U(x_i - x_j)$, $U(x) \geqslant 0$ with scattering length a

1953 Feynman: $\Delta T_c < 0$

1958 Lee & Yang: $\frac{\Delta T_c}{T_c} \sim a\rho^{1/3}$

1960 Glassgold, Kaufman, Watson: $\frac{\Delta T_c}{T_c} \sim (a\rho^{1/3})^{1/2}$, increases

1964 Huang: $\frac{\Delta T_c}{T_c} \sim (a\rho^{1/3})^{3/2}$, increases

1971 Fetter & Walecka: $\frac{\Delta T_c}{T_c}$ decreases

1982 Toyoda: $\frac{\Delta T_c}{T_c} \approx -3.5(a\rho^{1/3})^{1/2}$

1992 Stoof: $\frac{\Delta T_c}{T_c} = c\, a\rho^{1/3} + o(a\rho^{1/3}), \quad c > 0$

1996 Bijlsma & Stoof: $c = 4.66$

1997 Grüter, Ceperley, Laloë: $c = 0.34$

1999 Holzmann, Grüter, Laloë: $c = 0.7$; Holzmann, Krauth: $c = 2.3$;

Intermission: effects of interactions on crit. temperature

$\boldsymbol{H} = -\sum_{i=1}^{N} \Delta_i + \sum_{i<j} U(x_i - x_j)$, $U(x) \geqslant 0$ with scattering length a

1953 Feynman: $\Delta T_c < 0$

1958 Lee & Yang: $\frac{\Delta T_c}{T_c} \sim a\rho^{1/3}$

1960 Glassgold, Kaufman, Watson: $\frac{\Delta T_c}{T_c} \sim (a\rho^{1/3})^{1/2}$, increases

1964 Huang: $\frac{\Delta T_c}{T_c} \sim (a\rho^{1/3})^{3/2}$, increases

1971 Fetter & Walecka: $\frac{\Delta T_c}{T_c}$ decreases

1982 Toyoda: $\frac{\Delta T_c}{T_c} \approx -3.5(a\rho^{1/3})^{1/2}$

1992 Stoof: $\frac{\Delta T_c}{T_c} = c\, a\rho^{1/3} + o(a\rho^{1/3}), \quad c > 0$

1996 Bijlsma & Stoof: $c = 4.66$

1997 Grüter, Ceperley, Laloë: $c = 0.34$

1999 Holzmann, Grüter, Laloë: $c = 0.7$; Holzmann, Krauth: $c = 2.3$;
Baym, B.H.L.V.: $c = 2.9$

Intermission: effects of interactions on crit. temperature

$\boldsymbol{H} = -\sum_{i=1}^{N} \Delta_i + \sum_{i<j} U(x_i - x_j)$, $U(x) \geqslant 0$ with scattering length a

1953 Feynman: $\Delta T_c < 0$

1958 Lee & Yang: $\frac{\Delta T_c}{T_c} \sim a\rho^{1/3}$

1960 Glassgold, Kaufman, Watson: $\frac{\Delta T_c}{T_c} \sim (a\rho^{1/3})^{1/2}$, increases

1964 Huang: $\frac{\Delta T_c}{T_c} \sim (a\rho^{1/3})^{3/2}$, increases

1971 Fetter & Walecka: $\frac{\Delta T_c}{T_c}$ decreases

1982 Toyoda: $\frac{\Delta T_c}{T_c} \approx -3.5(a\rho^{1/3})^{1/2}$

1992 Stoof: $\frac{\Delta T_c}{T_c} = c\,a\rho^{1/3} + o(a\rho^{1/3}), \quad c > 0$

1996 Bijlsma & Stoof: $c = 4.66$

1997 Grüter, Ceperley, Laloë: $c = 0.34$

1999 Holzmann, Grüter, Laloë: $c = 0.7$; Holzmann, Krauth: $c = 2.3$; Baym, B.H.L.V.: $c = 2.9$

1999 Huang: $\frac{\Delta T_c}{T_c} \approx 3.5(a\rho^{1/3})^{1/2}$

Intermission: effects of interactions on crit. temperature

$\boldsymbol{H} = -\sum_{i=1}^{N} \Delta_i + \sum_{i<j} U(x_i - x_j)$, $U(x) \geqslant 0$ with scattering length a

1953 Feynman: $\Delta T_c < 0$

1958 Lee & Yang: $\frac{\Delta T_c}{T_c} \sim a\rho^{1/3}$

1960 Glassgold, Kaufman, Watson: $\frac{\Delta T_c}{T_c} \sim (a\rho^{1/3})^{1/2}$, increases

1964 Huang: $\frac{\Delta T_c}{T_c} \sim (a\rho^{1/3})^{3/2}$, increases

1971 Fetter & Walecka: $\frac{\Delta T_c}{T_c}$ decreases

1982 Toyoda: $\frac{\Delta T_c}{T_c} \approx -3.5(a\rho^{1/3})^{1/2}$

1992 Stoof: $\frac{\Delta T_c}{T_c} = c\,a\rho^{1/3} + o(a\rho^{1/3}), \quad c > 0$

1996 Bijlsma & Stoof: $c = 4.66$

1997 Grüter, Ceperley, Laloë: $c = 0.34$

1999 Holzmann, Grüter, Laloë: $c = 0.7$; Holzmann, Krauth: $c = 2.3$; Baym, B.H.L.V.: $c = 2.9$

1999 Huang: $\frac{\Delta T_c}{T_c} \approx 3.5(a\rho^{1/3})^{1/2}$

2000 Reppy et. al.: $c = 5.1$

Effects of interactions on critical temperature

$$\boldsymbol{H} = -\sum_{i=1}^{N} \Delta_i + \sum_{i<j} U(x_i - x_j) \;,\; U(x) \geqslant 0 \text{ with scattering length } a$$

2001 Arnold, Moore: $c = 1.32$

Effects of interactions on critical temperature

$$\boldsymbol{H} = -\sum_{i=1}^{N} \Delta_i + \sum_{i<j} U(x_i - x_j) \text{ , } U(x) \geqslant 0 \text{ with scattering length } a$$

2001 Arnold, Moore: $c = 1.32$; Kashurnikov, Prokof'ev, Svistunov: $c = 1.29$

Effects of interactions on critical temperature

$$\boldsymbol{H} = -\sum_{i=1}^{N} \Delta_i + \sum_{i<j} U(x_i - x_j) \text{ , } U(x) \geqslant 0 \text{ with scattering length } a$$

2001 Arnold, Moore: $c = 1.32$; Kashurnikov, Prokof'ev, Svistunov: $c = 1.29$

2003 Kleinert: $c = 1.14$

Effects of interactions on critical temperature

$$\boldsymbol{H} = -\sum_{i=1}^{N} \Delta_i + \sum_{i<j} U(x_i - x_j) \;,\; U(x) \geqslant 0 \text{ with scattering length } a$$

2001 Arnold, Moore: $c = 1.32$; Kashurnikov, Prokof'ev, Svistunov: $c = 1.29$

2003 Kleinert: $c = 1.14$

2004 Kastening: $c = 1.27$

Effects of interactions on critical temperature

$$\boldsymbol{H} = -\sum_{i=1}^{N} \Delta_i + \sum_{i<j} U(x_i - x_j) \ , \ U(x) \geqslant 0 \text{ with scattering length } a$$

2001 Arnold, Moore: $c = 1.32$; Kashurnikov, Prokof'ev, Svistunov: $c = 1.29$

2003 Kleinert: $c = 1.14$

2004 Kastening: $c = 1.27$; Nho, Landau: $c = 1.32$

Effects of interactions on critical temperature

$$\boldsymbol{H} = -\sum_{i=1}^{N} \Delta_i + \sum_{i<j} U(x_i - x_j)$$, $U(x) \geqslant 0$ with scattering length a

2001 Arnold, Moore: $c = 1.32$; Kashurnikov, Prokof'ev, Svistunov: $c = 1.29$

2003 Kleinert: $c = 1.14$

2004 Kastening: $c = 1.27$; Nho, Landau: $c = 1.32$

A partial but rigorous result:

Theorem (with Robert Seiringer, 2009)

There is no BEC when $\dfrac{\Delta T_{\rm c}}{T_{\rm c}} > 5.09\sqrt{a\rho^{1/3}}$

Back to the cycle model for the Bose gas

$$H_3(\boldsymbol{x},\pi) = \frac{1}{4\beta} \sum_i |x_i - x_{\pi(i)}|^2 + \sum_{\ell \geqslant 1} (\alpha_\ell - \alpha_{\ell,\ell}) r_\ell(\pi) + \sum_{\ell,\ell' \geqslant 1} \alpha_{\ell,\ell'} r_\ell(\pi) r_{\ell'}(\pi)$$

A computation gives

$$\alpha_{\ell,\ell'} = \frac{4\pi\beta\ell\ell' a}{|\Lambda|}$$

(to first order). Then

$$\sum_{\ell,\ell' \geqslant 1} \alpha_{\ell,\ell'} r_\ell(\pi) r_{\ell'}(\pi) = \frac{4\pi\beta a N^2}{|\Lambda|}$$

is constant in the canonical ensemble. This term can be dismissed!
We also get

$$\alpha_\ell = \frac{\ell a}{(4\pi\beta)^{1/2}} \sum_{j=1}^{\ell-1} \Big(\frac{\ell}{j(\ell-j)} \Big)^{3/2}$$

(to first order)

We have $\lim_{\ell\to\infty} \frac{\alpha_\ell}{\ell} = \frac{2\zeta(\frac32)a}{(4\pi\beta)^{1/2}}$. Define $\alpha'_\ell = \alpha_\ell - \frac{2\zeta(\frac32)a\ell}{(4\pi\beta)^{1/2}}$. Then

$$\sum_{\ell \geqslant 1} \alpha'_\ell r_\ell(\pi) = \sum_{\ell \geqslant 1} \alpha_\ell r_\ell(\pi) - \frac{2\zeta(\frac32)aN}{(4\pi\beta)^{1/2}}$$

so that both weights yield the same Gibbs distribution.

$$H_4 = \frac{1}{4\beta} \sum_i |x_i - x_{\pi(i)}|^2 + \sum_{\ell \geqslant 1} \alpha'_\ell r_\ell(\pi)$$

CONJECTURE 4. The critical temperature of the model H_4 is identical to that of the model H_3, up to a correction $o(a)$.

Critical density

Since $\frac{\alpha'_\ell}{\ell} \to 0$, we have the formula for the critical density:

$$\rho_{\mathrm{c}}^{(a)} = \sum_{\ell \geqslant 1} \frac{\mathrm{e}^{-\alpha'_\ell}}{(4\pi\beta\ell)^{3/2}}$$

This is computed by analogy to the ideal Bose gas. Is it relevant for the occurrence of infinite cycles?

- Sütő proved it for $\alpha'_\ell \equiv 0$ ('93, '02)
- With Betz, we proved it when $\alpha'_\ell \to 0$ faster than $1/\log \ell$ ('10)

Here, one can check that $\alpha'_\ell = -\frac{6-3\gamma_{1/2}}{(4\pi\beta)^{1/2}}\, a \left(1 + O(\ell^{-1/5})\right)$

The weights are negative, and $\alpha'_\ell \not\to 0$, but we conjecture that the formula for the critical density applies nonetheless

CONJECTURE 5. The critical density of the model H_5 is given by the formula above.

Change in critical density

We get

$$\frac{\rho_{\rm c}^{(a)} - \rho_{\rm c}^{(0)}}{\rho_{\rm c}^{(0)}} = -\frac{2a}{(4\pi\beta)^2} \sum_{\ell \geqslant 1} \frac{1}{\ell^{1/2}} \Big[\frac{1}{2} \sum_{j=1}^{\ell-1} \Big(\frac{\ell}{j(\ell-j)} \Big)^{3/2} - \zeta(3/2) \Big]$$

$$= +\frac{2\sqrt{\pi}}{\zeta(3/2)} a\beta^{-1/2}$$

This implies that

$$\frac{T_{\rm c}^{(a)} - T_{\rm c}^{(0)}}{T_{\rm c}^{(0)}} = -\frac{8\pi}{3\zeta(\frac{3}{2})^{4/3}}\, a\rho^{1/3} = -2.33\, a\rho^{1/3}$$

This contradicts the consensus of the physics literature!!!
(Constant is $+1.3$)

A verification: the free energy

In Oberwolfach, an excellent suggestion of ROBERT SEIRINGER: calculate the free energy of the simplified model, and compare it with the one of the dilute Bose gas, which is known explicitly (Seiringer '08, Yin '09)

We have that

$$f(\beta,\rho) = 4\pi a\rho(\rho + 2\rho_{\rm c}) + \hat{f}(\beta,\rho),$$

with $\hat{f}$ the free energy of the model with weights $\{\alpha'_\ell\}$. We find

$$\hat{f}(\beta,\rho) = f^{(0)}(\beta,\rho) + \begin{cases} 4\pi a\rho(\rho - 2\rho_{\rm c}) & \text{if } \rho \leqslant \rho_{\rm c} \\ -4\pi a\rho_{\rm c}^2 & \text{if } \rho \geqslant \rho_{\rm c} \end{cases}$$

Then one recovers the free energy of the dilute Bose gas:

$$f(\beta,\rho) = f^{(0)}(\beta,\rho) + 4\pi a[2\rho^2 - (\rho - \rho_{\rm c})_+^2]$$

(All identities are valid to first order in $a\rho^{1/3}$ only)

Conclusion

We computed the change of critical density of BEC using the following (exact?) simplifications:

- BEC is equivalent to occurrence of infinite cycles in FK representation

Conclusion

We computed the change of critical density of BEC using the following (exact?) simplifications:

- BEC is equivalent to occurrence of infinite cycles in FK representation
- The multibody interactions from FK are replaced by 2-jump interactions

Conclusion

We computed the change of critical density of BEC using the following (exact?) simplifications:

- BEC is equivalent to occurrence of infinite cycles in FK representation
- The multibody interactions from FK are replaced by 2-jump interactions
- We neglect the fluctuations of a certain macroscopic observable

Conclusion

We computed the change of critical density of BEC using the following (exact?) simplifications:

- BEC is equivalent to occurrence of infinite cycles in FK representation
- The multibody interactions from FK are replaced by 2-jump interactions
- We neglect the fluctuations of a certain macroscopic observable
- The formula for the critical density has not been rigorously established for our weights

Conclusion

We computed the change of critical density of BEC using the following (exact?) simplifications:

- BEC is equivalent to occurrence of infinite cycles in FK representation
- The multibody interactions from FK are replaced by 2-jump interactions
- We neglect the fluctuations of a certain macroscopic observable
- The formula for the critical density has not been rigorously established for our weights
- Despite these caveats, our method should give the right critical temperature. The discrepancy with the physics literature is puzzling.

THANK YOU!