A General Approach to the Method of Cluster Expansions

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Program

1. General setting and results

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3. Combinatorics and the problem of convergence

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$\rightarrow$ ...using Ruelle's algebraic method

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$\longrightarrow$ ...using Ruelle's algebraic method

$\longrightarrow$ ...using Brydges & Federbush tree identities

Brief history

- Introduced in the 1930's in statistical mechanics in order to study gases of classical interacting particles.
- Made rigorous by mathematical-physicists in the 1960's, see Ruelle's book (1969). Method split afterwards.
- For continuous systems: applications to classical systems (Penrose '67, Минлос-Poghosyan '77, Brydges-Federbush '78), quantum systems (Ginibre '65, Poghosyan-Zessin '01), or quantum field theory (Glimm-Jaffe-Spencer '74, Малышев '80, Battle-Federbush '82, Brydges-Kennedy '87).
- For polymer systems: Gruber-Kunz '71, Kotecký-Preiss '86, Добрушин '96, Bovier-Zahradník '00, Miracle-Solé '00, Sokal '01, Fernández-Procacci '07, Procacci '08, Jackson-Procacci-Sokal '08.
- Attempts at unifying the methods : Ueltschi '04 and Faris '08
- Useful surveys by Brydges '86, Pfister '91, Abdesselam-Rivasseau '94.

General setting and results

State space: measure space $(\mathbb{X}, \mathcal{X}, \mu)$ with μ a complex measure ($|\mu|$ denotes its total variation).
Let u and ζ complex measurable symmetric functions on $\mathbb{X} \times \mathbb{X}$, that satisfy

$$\zeta(x,y) = \mathrm{e}^{-u(x,y)} - 1.$$

Partition function:

$$Z = \sum_{n \geqslant 0} \frac{1}{n!} \int \mathrm{d}\mu(x_1) \dots \int \mathrm{d}\mu(x_n) \exp\Bigl\{- \sum_{1 \leqslant i<j \leqslant n} u(x_i, x_j)\Bigr\}$$

or equivalently

$$Z = \sum_{n \geqslant 0} \frac{1}{n!} \int \mathrm{d}\mu(x_1) \dots \int \mathrm{d}\mu(x_n) \prod_{1 \leqslant i<j \leqslant n} \bigl(1 + \zeta(x_i, x_j)\bigr)$$

Stability: *There exists a nonnegative function* b *on* $\mathbb{X}$ *such that, for all* n *and almost all* $\{x_1, \dots, x_n \in \mathbb{X}\}$

$$\prod_{1 \leqslant i<j \leqslant n} \bigl|1+\zeta(x_i,x_j)\bigr| \leqslant \prod_{i=1}^n \mathrm{e}^{b(x_i)} \quad \text{or} \quad \sum_{1 \leqslant i<j \leqslant n} \operatorname{Re} u(x_i,x_j) \geqslant -\sum_{i=1}^n b(x_i)$$

Small interactions ("Kotecký-Preiss criterion"): We assume either of the following two conditions:

(a) There exists a nonnegative function a *on* $\mathbb{X}$ *such that*

$$\int \mathrm{d}|\mu|(y)\, |\zeta(x,y)|\, \mathrm{e}^{a(y)+2b(y)} \leqslant a(x)$$

(b) Let $\overline{u}(x,y) = 1$ *if* $\zeta(x,y) = -1$ *, and* $\overline{u}(x,y) = u(x,y)$ *otherwise. There exists a nonnegative function* a *on* $\mathbb{X}$ *such that*

$$\int \mathrm{d}|\mu|(y)\, |\overline{u}(x,y)|\, \mathrm{e}^{a(y)+b(y)} \leqslant a(x)$$

Remark: For positive u we can take $b(x) \equiv 0$; since $1 - \mathrm{e}^{-u} \leqslant u$, (a) is always better than (b).

Let $\mathcal{G}_n$ be the set of graphs with n vertices (unoriented, no loops) and $\mathcal{C}_n \subset \mathcal{G}_n$ the set of connected graphs with n vertices. Let

$$\varphi(x_1, \dots, x_n) = \begin{cases} 1 & \text{if } n = 1 \\ \frac{1}{n!} \sum_{G \in \mathcal{C}_n} \prod_{\{i,j\} \in G} \zeta(x_i, x_j) & \text{if } n \geqslant 2 \end{cases}$$

Theorem (Cluster expansions)

We have

$$Z = \exp\Big\{ \sum_{n \geqslant 1} \int \mathrm{d}\mu(x_1) \dots \mathrm{d}\mu(x_n) \, \varphi(x_1, \dots, x_n) \Big\}$$

Convergence is absolute. Furthermore, for almost all $x_1 \in \mathbb{X}$ *, we have*

$$\sum_{n \geqslant 2} n \int \mathrm{d}|\mu|(x_2) \dots \int \mathrm{d}|\mu|(x_n) \, |\varphi(x_1, \dots, x_n)| \leqslant (\mathrm{e}^{a(x_1)} - 1) \, \mathrm{e}^{2b(x_1)}$$

$$Z(x_1) = \sum_{n \geqslant 1} \frac{1}{(n-1)!} \int \mathrm{d}\mu(x_2) \dots \int \mathrm{d}\mu(x_n) \prod_{1 \leqslant i<j \leqslant n} \bigl(1 + \zeta(x_i, x_j)\bigr)$$

$$Z(x_1, x_2) = \sum_{n \geqslant 2} \frac{1}{(n-2)!} \int \mathrm{d}\mu(x_3) \dots \int \mathrm{d}\mu(x_n) \prod_{1 \leqslant i<j \leqslant n} \bigl(1+\zeta(x_i, x_j)\bigr)$$

$$\hat{Z}(x_1, x_2) = \sum_{n \geqslant 2} n(n-1) \int \mathrm{d}\mu(x_3) \dots \int \mathrm{d}\mu(x_n)\, \varphi(x_1, \dots, x_n)$$

Theorem (Correlation functions)

$$\frac{Z(x,y)}{Z} - \frac{Z(x)\,Z(y)}{Z^2} = \hat{Z}(x,y)$$

$$|\hat{Z}(x,y)| \leqslant \mathrm{e}^{a(y)+2b(y)} \Bigl[|\zeta(x,y)|\, \mathrm{e}^{a(x)+2b(x)} + + \sum_{m \geqslant 1} \int \mathrm{d}|\mu|(x_1) \dots \int \mathrm{d}|\mu|(x_m) \prod_{i=0}^{m} |\zeta(x_i, x_{i+1})|\, \mathrm{e}^{a(x_i)+2b(x_i)} \Bigr]$$

The classical interacting gas

Gas of point particles that interact with a pair potential; fugacity z and inverse temperature β

$\mathbb{X} \equiv \Lambda =$ open bounded subset of $\mathbb{R}^d$; $\mu(x) = z\mathrm{d}x$

Interaction $U : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$; $u(x,y) = \beta U(x-y)$

We suppose that U is stable: for any n and any $x_1, \dots, x_n \in \mathbb{R}^d$:

$$\sum_{1 \leqslant i<j \leqslant n} U(x_i - x_j) \geqslant -Bn$$

We can choose $b(x) \equiv \beta B$ and $a(x) \equiv a \equiv 1$. We get the condition

$$z\,\mathrm{e}^{2\beta B} \int_{\mathbb{R}^d} \big| \mathrm{e}^{-\beta U(y)} - 1 \big| \mathrm{d}y \leqslant \mathrm{e}^{-1}$$

This is the standard condition, see Ruelle '69. If U consists of a hard core of radius r and that it is otherwise integrable we get

$$z\,\mathrm{e}^{\beta B} \Big[|\mathbb{B}| r^d + \beta \int_{|y|>r} |U(y)| \mathrm{d}y \Big] \leqslant \mathrm{e}^{-1}$$

This new condition is often better.

Thermodynamic pressure is defined as infinite volume limit of

$$p_\Lambda(\beta, z) = \frac{1}{|\Lambda|} \log Z$$

We have

$$p_\Lambda(\beta, z) = \frac{1}{|\Lambda|} \int_\Lambda \mathrm{d}x_1 \Big[\sum_{n \geqslant 1} z^n \int_\Lambda \mathrm{d}x_2 \dots \int_\Lambda \mathrm{d}x_n \varphi(x_1, \dots, x_n) \Big]$$

Increasing domains $\Lambda_1 \subset \Lambda_2 \subset \dots$ such that $\Lambda_n \to \mathbb{R}^d$. By translation invariance, we get

$$p(\beta, z) \equiv \lim_{n \to \infty} p_{\Lambda_n}(\beta, z) = \sum_{n \geqslant 1} z^n \int_{\mathbb{R}^d} \mathrm{d}x_2 \dots \int_{\mathbb{R}^d} \mathrm{d}x_n \, \varphi(0, x_2, \dots, x_n)$$

This expression is a convergent series of analytic functions of β, z . Then $p(\beta, z)$ is analytic in β and z by Vitali theorem

Models of polymers

$\mathbb{X}$: set of finite connected subsets of $\mathbb{Z}^d$. Measure μ is counting measure multiplied by activity $z(x) = \mathrm{e}^{-\gamma|x|}$. Interaction is hard core when polymers overlap and is attractive when they touch:

$$u(x,y) = \begin{cases} \infty & \text{if } x \cap y \neq \emptyset \\ -\eta\, c(x,y) & \text{if } x \cap y = \emptyset \end{cases}$$

Here, $c(x,y)$ is the number of “contacts” between x and y .
One can check that u is stable with function $b(x) = \eta d|x|$.
The function a must grow like $|x|$, so it is natural to choose $b(x) = a|x|$ for some constant a . A sufficient condition is that

$$\sum_{y, y\cap x \neq \emptyset} z(y)\, \mathrm{e}^{a|y| + \eta d|y|} + \sum_{y, \mathrm{dist}\,(x,y)=1} \eta z(y) c(x,y)\, \mathrm{e}^{a|y|+\eta d|y|} \leqslant a|x|$$

We can bound $\eta c(x,y)$ by $2\eta d|y|$.

Models of polymers

Summing over the sites of x , and requiring that y contains the given site or comes at distance 1, we get

$$\sum_{y \ni 0} (1 + 2d\eta|y|) z(y)\, \mathrm{e}^{a|y| + \eta d|y|} \leqslant a$$

The number of connected sets of cardinality n that contain the origin is smaller than the number of walks of length $2n-3$ starting at the origin, which is equal to $(2d)^{2n-3}$. Then it suffices that

$$\sum_{n \geqslant 1} \mathrm{e}^{-n(\gamma - a - 3d\eta - 2\log 2d)} \leqslant (2d)^3 a$$

This is equivalent to

$$\gamma \geqslant a + \log\bigl(1 + \tfrac{1}{(2d)^3 a}\bigr) + 3d\eta + 2\log 2d$$

Using $\log(1+t) \leqslant t$ and optimizing on a , we find the sufficient condition

$$\gamma \geqslant 2(2d)^{-3/2} + 3d\eta + 2\log 2d$$

The interacting quantum gas

Thanks to the Feynman-Kac representation, it can be represented as a gas of interacting Brownian bridges and we can use cluster expansions to control the dilute regime at sufficiently high temperature. See (Ginibre 1971) or our article for details.

Combinatorics and convergence

The main idea is to decompose graphs in terms of connected graphs. In combinatorics, this is known as “exponential structures”.

$$
\begin{aligned}
Z &= 1 + \sum_{n \geqslant 1} \frac{1}{n!} \int \mathrm{d}\mu(x_1) \dots \int \mathrm{d}\mu(x_n) \sum_{G \in \mathcal{G}_n} \prod_{\{i,j\} \in G} \zeta(x_i, x_j) \\
&= 1 + \sum_{n \geqslant 1} \sum_{k \geqslant 1} \frac{1}{k!} \sum_{\substack{m_1, \dots, m_k \geqslant 1 \\ m_1 + \dots + m_k = n}} \frac{1}{m_1! \dots m_k!} \\
&\qquad \prod_{\ell=1}^{k} \Big\{ \int \mathrm{d}\mu(x_1) \dots \int \mathrm{d}\mu(x_{m_\ell}) \sum_{G \in \mathcal{C}_{m_\ell}} \prod_{\{i,j\} \in G} \zeta(x_i, x_j) \Big\} \\
&= 1 + \sum_{n \geqslant 1} \sum_{k \geqslant 1} \frac{1}{k!} \sum_{\substack{m_1, \dots, m_k \geqslant 1 \\ m_1 + \dots + m_k = n}} \prod_{\ell=1}^{k} \Big\{ \int \mathrm{d}\mu(x_1) \dots \int \mathrm{d}\mu(x_{m_\ell}) \varphi(x_1, \dots, x_{m_\ell}) \Big\}
\end{aligned}
$$

Combinatorics and convergence

The triple sum is absolutely convergent thanks to the estimate in the theorem. One can then interchange the sums by the dominated convergence theorem, yielding

$$Z = \exp\Big\{ \sum_{n \geqslant 1} \int \mathrm{d}\mu(x_1) \dots \mathrm{d}\mu(x_n)\, \varphi(x_1, \dots, x_n) \Big\}$$

The main problem is to prove an estimate for the cluster terms

Tree estimates

Let $\mathcal{T}_n \subset \mathcal{C}_n$ denote the set of trees with n vertices. Let n be an integer, $b_1, \dots, b_n$ be real nonnegative numbers, and $\zeta_{ij} = \zeta_{ji}$, $1 \leqslant i, j \leqslant n$, be complex numbers. We assume that the following bound holds for any subset $I \subset \{1, \dots, n\}$:

$$\prod_{i,j \in I, i<j} |1 + \zeta_{ij}| \leqslant \prod_{i \in I} \mathrm{e}^{b_i}$$

Let u_{ij} be such that $\zeta_{ij} = \mathrm{e}^{-u_{ij}} - 1$; let $\overline{u}_{ij} = 1$ if $\zeta_{ij} = -1$, and $\overline{u}_{ij} = u_{ij}$ otherwise.

Theorem

(a) $$\Big| \sum_{G \in \mathcal{C}_n} \prod_{\{i,j\} \in G} \zeta_{ij} \Big| \leqslant \Big(\prod_{i=1}^n \mathrm{e}^{2b_i} \Big) \sum_{G \in \mathcal{T}_n} \prod_{\{i,j\} \in G} |\zeta_{ij}|$$

(b) $$\Big| \sum_{G \in \mathcal{C}_n} \prod_{\{i,j\} \in G} \zeta_{ij} \Big| \leqslant \Big(\prod_{i=1}^n \mathrm{e}^{b_i} \Big) \sum_{G \in \mathcal{T}_n} \prod_{\{i,j\} \in G} |\overline{u}_{ij}|$$

Ruelle's algebraic method

Sketch proof of tree estimate (a).
We consider complex functions on the power set $\mathcal{P}(\{1,\dots,n\})$.
Multiplication operation: $f * g(I) = \sum_{J \subset I} f(J) g(I \setminus J)$
Define

$$\Phi(I) = \sum_{G \in \mathcal{C}(I)} \prod_{\{i,j\} \in G} \zeta_{ij},$$

$$\Psi(I) = \prod_{i,j \in I, i<j} (1 + \zeta_{ij}) = \sum_{G \in \mathcal{G}(I)} \prod_{\{i,j\} \in G} \zeta_{ij}.$$

Notice that $\Psi = \exp_{\mathcal{A}} \Phi$.
"Differentiation":

$$D_J f(I) = \begin{cases} f(I \cup J) & \text{if } I \cap J = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

For disjoint $I, J \subset \{1, \dots, n\}$, define

$$g(I,J) = \Big(\Psi^{*(-1)} * D_I \Psi\Big)(J)$$

Because of stability, there exists $i \in I$ such that $\prod_{j \in I \setminus \{i\}} |1 + \zeta_{ij}| \leqslant \mathrm{e}^{2b_i}$
Let $\iota(I)$ the function that assigns this index, and $I' = I \setminus \{\iota(I)\}$.

Lemma

The function $g(I,J)$ is solution of the following equation.

$$\begin{cases} g(\emptyset, J) = \delta_{\emptyset, J}, \\ g(I,J) = \Big(\prod_{i \in I'} (1 + \zeta_{i,\iota(I)})\Big) \sum_{K \subset J} \Big(\prod_{i \in K} \zeta_{i,\iota(I)}\Big) g(I' \cup K, J \setminus K) \quad \text{if } I \neq \emptyset. \end{cases}$$

Since the equation gives $g(I,J)$ in terms of $g(K,L)$ with $|K| + |L| = |I| + |J| - 1$, it is well defined inductively and it has a unique solution.

Let h satisfy

$$\begin{cases} h(\emptyset, J) = \delta_{\emptyset,J}, \\ h(I,J) = \mathrm{e}^{2b_{\iota(I)}} \displaystyle\sum_{K \subset J} \Big(\prod_{i \in K} |\zeta_{i,\iota(I)}| \Big) h(I' \cup K, J \setminus K) \quad \text{if } I \neq \emptyset. \end{cases}$$

Since $\prod_{i \in I'} |1 + \zeta_{i,\iota(I)}| \leqslant \mathrm{e}^{2b_{\iota(I)}}$, we can check inductively that

$$|g(I,J)| \leqslant h(I,J)$$

Let $\mathcal{F}_I(J)$ be the set of forests on $I \cup J$ rooted in I.

Lemma

The solution of the equation above is

$$h(I,J) = \Big(\prod_{i \in I \cup J} \mathrm{e}^{2b_i} \Big) \sum_{G \in \mathcal{F}_I(J)} \prod_{\{i,j\} \in G} |\zeta_{ij}|$$

This proves the tree estimate (a), since it is the inequality $g(\{1\},\{2,\dots,n\}) \leqslant h(\{1\},\{2,\dots,n\})$, which is now clear.

Brydges and Federbush tree identities

Sketch proof of the tree estimate (b).

Tree identity of Brydges, Battle, and Federbush:

$$\sum_{G\in\mathcal{C}_n}\prod_{\{i,j\}\in G}\big(\mathrm{e}^{-u_{ij}}-1\big)=\sum_{G\in\mathcal{T}_n}\prod_{\{i,j\}\in G}(-u_{ij})\int\mathrm{d}\lambda_G(\{s_{ij}\})\,\mathrm{e}^{-\sum_{i<j}s_{ij}u_{ij}}$$

with λ_G a probability measure, and $s_{ij}v_{ij}$ satisfies the stability property.

The tree estimate (b) is immediate consequence.

In the presence of hard cores, the tree identity can be proved using a trick due to Procacci.

Conclusion

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- General theorem about cluster expansions. Applies to both discrete and continuous systems
- Two criteria for the convergence, proved in two different manners (algebraic approach & tree identity)
- The result suggests that there should be a better criterion for the convergence
- Interesting applications to classical & quantum gases, polymer models, etc...

THANK YOU!