On random permutations with nonuniform distributions

Daniel Ueltschi

Department of Mathematics, University of Warwick

Loughborough University, 28 October 2009

Collaborators: V. Betz (Warwick), Y. Velenik (Genève)

Program

- Quantum Bose gas

Program

- Quantum Bose gas

→ Model of spatial permutations ("general")

Program

- Quantum Bose gas

→ Model of spatial permutations ("general")

→ Model of spatial permutations ("cycle weights")

Program

- Quantum Bose gas

→ Model of spatial permutations ("general")

→ Model of spatial permutations ("cycle weights")

→ (Nonspatial) random permutations with cycle weights

Program

- Quantum Bose gas

→ Model of spatial permutations ("general")

→ Model of spatial permutations ("cycle weights")

→ (Nonspatial) random permutations with cycle weights

- Results for random permutations with cycle weights

Program

- Quantum Bose gas
- → Model of spatial permutations ("general")
- → Model of spatial permutations ("cycle weights")
- → (Nonspatial) random permutations with cycle weights
- Results for random permutations with cycle weights
- Illustration: lattice permutations

Program

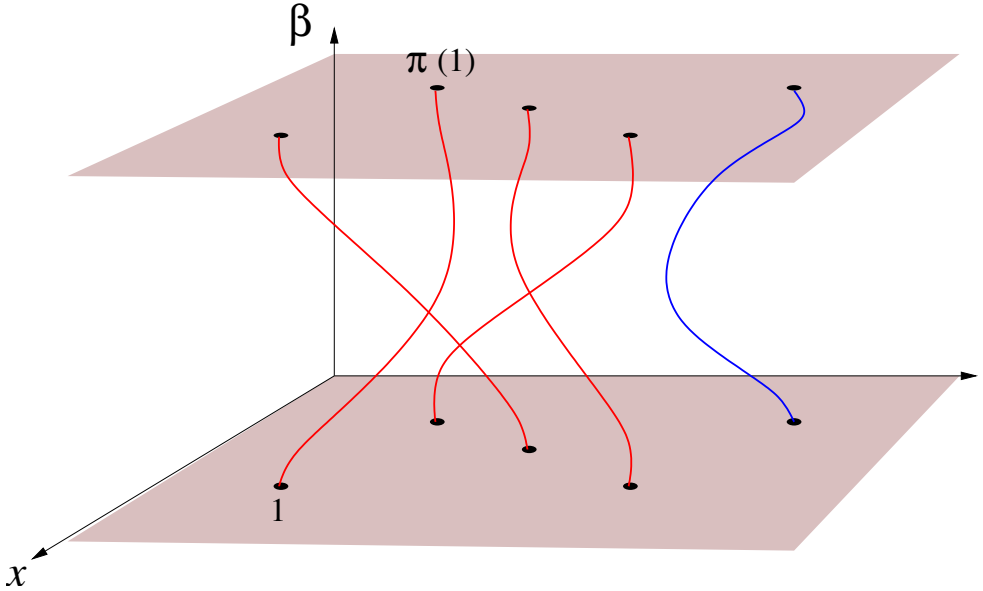
- Quantum Bose gas
- → Model of spatial permutations ("general")
- → Model of spatial permutations ("cycle weights")
- → (Nonspatial) random permutations with cycle weights
- Results for random permutations with cycle weights
- Illustration: lattice permutations
- Results for spatial permutations with small cycle weights

From the Bose gas to spatial permutations

n bosons in $\Lambda \subset \mathbb{R}^d$

State space $L^2_{\text{sym}}(\Lambda^n)$

Hamiltonian
$H = -\sum_{i=1}^n \Delta_i + \sum_{i<j} U(x_i - x_j)$



Feynman-Kac formula for
$Z = \text{Tr } \mathrm{e}^{-\beta H}$:
with $\boldsymbol{x} = (x_1, \ldots, x_n)$,

$$Z = \int_{\Lambda^n} \mathrm{d}\boldsymbol{x} \sum_{\pi \in \mathcal{S}_n} \mathrm{e}^{-H(\boldsymbol{x},\pi)} \quad \text{with "Hamiltonian"} \quad H : \Lambda^n \times \mathcal{S}_n \to \mathbb{R}$$

(Explicit expression for H in terms of Wiener trajectories)

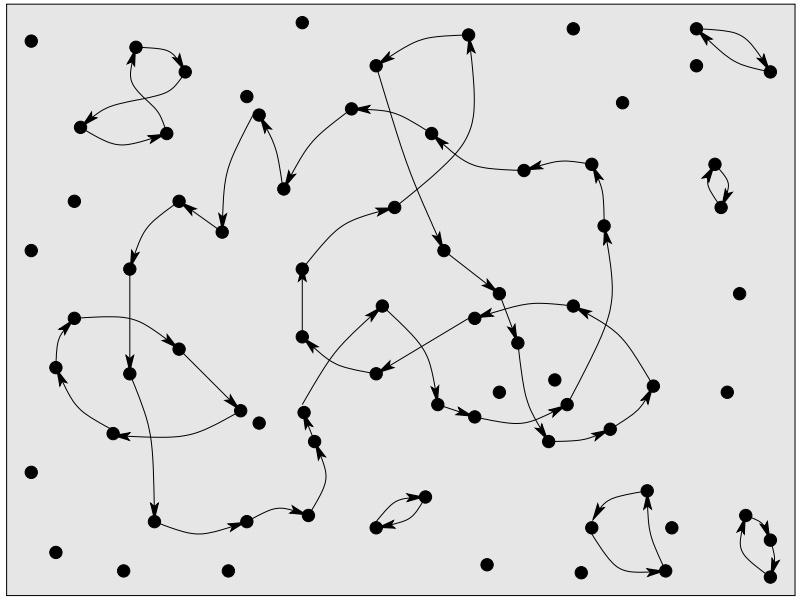
Model of spatial permutations (general)

State space: $\Omega_{\Lambda,n} = \Lambda^n \times \mathcal{S}_n$

Gibbs weight $\mathrm{e}^{-H(\boldsymbol{x},\pi)}$

Expectation of random variable Θ :

$$E_{\Lambda,n}(\Theta) = \frac{1}{Z} \int_{\Lambda^n} \mathrm{d}\boldsymbol{x} \sum_{\pi} \Theta(\boldsymbol{x},\pi)\, \mathrm{e}^{-H(\boldsymbol{x},\pi)}$$



General Hamiltonian:

$$H(\boldsymbol{x},\pi) = \sum_{i=1}^{n} \xi(x_i - x_{\pi(i)}) + \sum_{i<j} V(x_i, x_{\pi(i)}; x_j, x_{\pi(j)})$$

$V(x_i, x_{\pi(i)}; x_j, x_{\pi(j)})$ represents the pair interaction between the jumps $x_i \mapsto x_{\pi(i)}$ and $x_j \mapsto x_{\pi(j)}$

Model of spatial permutations (with cycle weights)

Bose gas: $\xi(x) = \frac{1}{4\beta}|x|^2$; interactions between $x \mapsto y$ and $x' \mapsto y'$ given by

$$V(x, y; x', y') = \int [1 - \mathrm{e}^{-\frac{1}{4}\int_0^\beta U(\omega(s))\mathrm{d}s}]\mathrm{d}W_{x-x',y-y'}(\omega)$$
$$= K(x - x', y - y')$$

with K the integral kernel of $\mathrm{e}^{2\beta\Delta} - \mathrm{e}^{2\beta(\Delta - U)}$

After additional calculations and some (exact?) approximations: Model with cycle weights

$$H(\boldsymbol{x}, \pi) = \sum_{i=1}^{n} \xi(x_i - x_{\pi(i)}) + \sum_{j \geqslant 1} \alpha_j r_j(\pi)$$

with α_j : parameters, and $r_j(\pi)$: number of cycles of length j

Preliminary study: *random permutations with cycle weights*

Random permutations with cycle weights

Probability of permutation $\pi \in \mathcal{S}_n$:

$$P(\pi) = \frac{1}{n! h_n} \exp\Big\{ - \sum_{j \geqslant 1} \alpha_j r_j(\pi) \Big\}$$

with α_j : parameters, and $r_j(\pi)$: number of cycles of length j
Uniform permutations: $\alpha_j \equiv 0$
Further motivation:

- *Ewens distribution* in mathematical biology: $\mathrm{e}^{-\alpha_j} \equiv \theta$. Model for genetic mutations in haploid populations; $\sum r_j$ describes the number of alleles; the Ewens distribution characterizes the stationary state
- This distribution also appears in Bayesian statistics, and in the Pólya urn model
- Links with random partitions

Random partitions

A partition $\boldsymbol{\lambda}$ of n : $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ with

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_k > 0 \quad \text{and} \quad \sum_{j=1}^{k} \lambda_j = n$$

The length of the partition, k , is not fixed
Occupation numbers: $r_j(\boldsymbol{\lambda})$ = number of indices i such that $\lambda_i = j$.
Occupation numbers determine the partition, and conversely. They satisfy

$$\sum_{j \geqslant 1} j r_j = n$$

We consider only random variables that depend on the cycle lengths (r_j) of permutations. Thus the model is really a model of random partitions. The Ewens distribution is often written

$$p(\boldsymbol{\lambda}) = \frac{n!}{(\theta)_n} \prod_{j \geqslant 1} \frac{\theta^{r_j}}{j^{r_j} r_j!}$$

Literature

Several studies for the number of cycles $K(\pi)$

- Gontcharov (1944): central limit theorem for K , for uniform random permutations:

$$\frac{K - \log n}{\sqrt{\log n}} \to N(0,1) \quad \text{in distribution}$$

- An amazing generalization of the central limit theorem above: Let $K(t,\pi) = \sum_{j=1}^{n^t} r_j$, and

$$Y(t,\pi) = \frac{K(t,\pi) - \theta t \log n}{\sqrt{\theta \log n}}$$

  Then $Y(t,\pi)$ converges weakly to the Wiener measure on the space of càdlàg functions on [0,1]. DeLaurentis & Bittel (1985) for uniform distributions, Hansen (1990) for Ewens

- Feng & Hoppe (1998): Large deviations for $\frac{K(t,\pi)}{\log n}$

Results for permutations with cycle weights

We are interested in the distribution of cycle lengths, rather than in the number of cycles. $\ell_1(\pi)$: length of cycle that contains 1 .
Weights $\theta_j = \mathrm{e}^{-\alpha_j}$ with $\alpha_j \asymp j^\gamma$

Theorem (with Volker Betz & Yvan Velenik, 2009)

- *The case* $\gamma < 0$ *is close to uniform distribution,* $P(\ell < sn) \to s$
- *In the case* $\gamma = 0$ *, i.e.* $\mathrm{e}^{-\alpha_j} \approx \theta$ *(Ewens),* $P(\ell_1 > sn) \to (1-s)^\theta$ *for any* $0 \leqslant s \leqslant 1$ *(cycles are macroscopic)*
- *The case* $0 < \gamma < 1$ *is surprising, with* $P(\ell_1 > n-k) \geqslant 1 - o(k)$ *as* $n \to \infty$ *. Almost all indices belong to a single giant cycle!*
  In addition: (i) $P(\ell_1 = n) > 0$ *; (ii) in some cases,* $P(\ell_1 = 1) > 0$
- *The case* $\gamma = 1$ *corresponds to uniform permutations*
- *When* $\gamma > 1$ *cycles become shorter, and* $\ell_1 \approx (\frac{1}{\gamma-1} \log n)^{1/\gamma}$ *with probability 1*

Sketch of proof for small weights

The case $\alpha_j \equiv 0$ is easy:

$$P\big(\ell_1(\pi) \in \{1, \dots, sn\}\big) = \sum_{j=1}^{sn} P\big(\ell_1(\pi) = j\big)$$

$$= \sum_{j=1}^{sn} \frac{1}{n!}(n-1)\dots(n-j+1)\cdot(n-j)! = \frac{sn}{n} = s$$

Main ideas for the case $\alpha_j \neq 0$, small: $(P(\pi) = \frac{1}{n!h_n}\prod_j \mathrm{e}^{-\alpha_j r_j(\pi)}\ )$
Observe that

$$P(\ell_1 \in [a,b]) = \frac{1}{n}\sum_{j=a}^{b} \underbrace{\mathrm{e}^{-\alpha_j}}_{\approx 1} \underbrace{\frac{h_{n-j}}{h_n}}_{\text{show that } \approx 1}$$

Expression for h_n using generating functions

$$h_n = \frac{1}{n}\sum_{j=1}^{n} \mathrm{e}^{-\alpha_j}\, h_{n-j} \qquad (\text{because } P(\ell_1 \in [1,n]) = 1)$$

Then

$$(n+1)h_{n+1} = \sum_{j=0}^{n} b_j\, h_{n-j} \qquad (b_j = \mathrm{e}^{-\alpha_{j+1}})$$

$$\big((n+1)h_{n+1}\big) = b * h$$

Introduce generating functions:

$$G_h(s) = \sum_{n \geqslant 0} h_n s^n, \qquad G_b(s) = \sum_{n \geqslant 0} b_n s^n$$

Then

$$G_h'(s) = G_b(s)G_h(s), \qquad G_h(0) = 1$$

$$\rightarrow \quad G_h(s) = \exp\int_0^s G_b(t)\mathrm{d}t = \exp\sum_{n \geqslant 1} \frac{b_{n-1}}{n} s^n$$

Expanding the exponential, and matching the terms with the generating function, we get

$$h_n = \sum_{k=1}^{n} \frac{1}{k!} \sum_{\substack{j_1,\dots,j_k \\ j_1+\cdots+j_k=n}} \prod_{i=1}^{k} \frac{\mathrm{e}^{-\alpha_{j_i}}}{j_i}$$

Remark: in the case $\alpha_j \equiv 0$, we get a puzzling identity:

$$\sum_{k=1}^{n} \frac{1}{k!} \sum_{\substack{j_1,\dots,j_k \\ j_1+\cdots+j_k=n}} \frac{1}{j_1 \cdots j_k} = 1$$

for all n !

One can also obtain

$$h_n = \sum_{k=0}^{n} \frac{1}{k!} \sum_{\substack{j_1,\dots,j_k \geqslant 1 \\ j_1+\cdots+j_k \leqslant n}} \prod_{i=1}^{k} \frac{\mathrm{e}^{-\alpha_{j_i}} - 1}{j_i}$$

Then if $\sum \frac{1}{j} |1 - \mathrm{e}^{-\alpha_j}| < \infty$, we have

$$h_\infty = \lim_{n\to\infty} h_n = \exp \sum_{j \geqslant 1} \frac{\mathrm{e}^{-\alpha_j} - 1}{j}$$

and $\frac{h_n - j}{h_n} \approx 1$ when $n - j \gg 1$ and $n \gg 1$. The claim of the theorem easily follows.

Lattice permutations

State space: set of permutations of $\mathbb{Z}^d$ (i.e. bijections $\mathbb{Z}^d \to \mathbb{Z}^d$)
σ-algebra generated by "cylinder sets" $B_{x,y} = \{\pi : \pi(x) = y\}$

$$P(B_{x,y}) = \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{Z(\Lambda)} \sum_{\substack{\pi:\pi(x)=y \\ \pi(z)=z\ \forall z \notin \Lambda}} \mathrm{e}^{-H(\pi)}$$

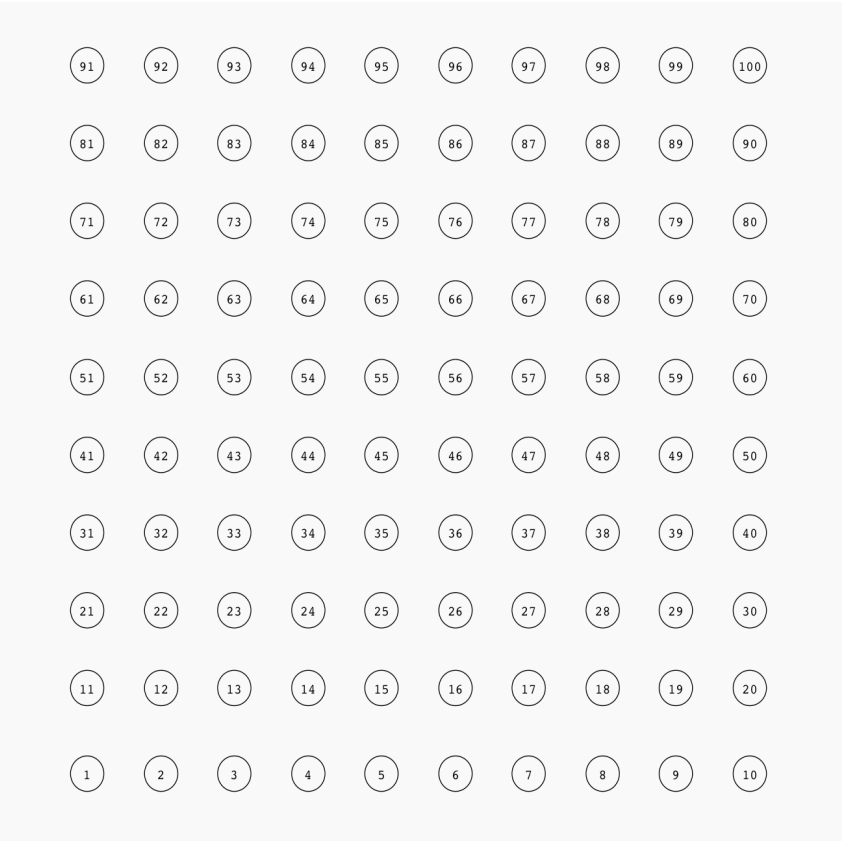
with "Hamiltonian"

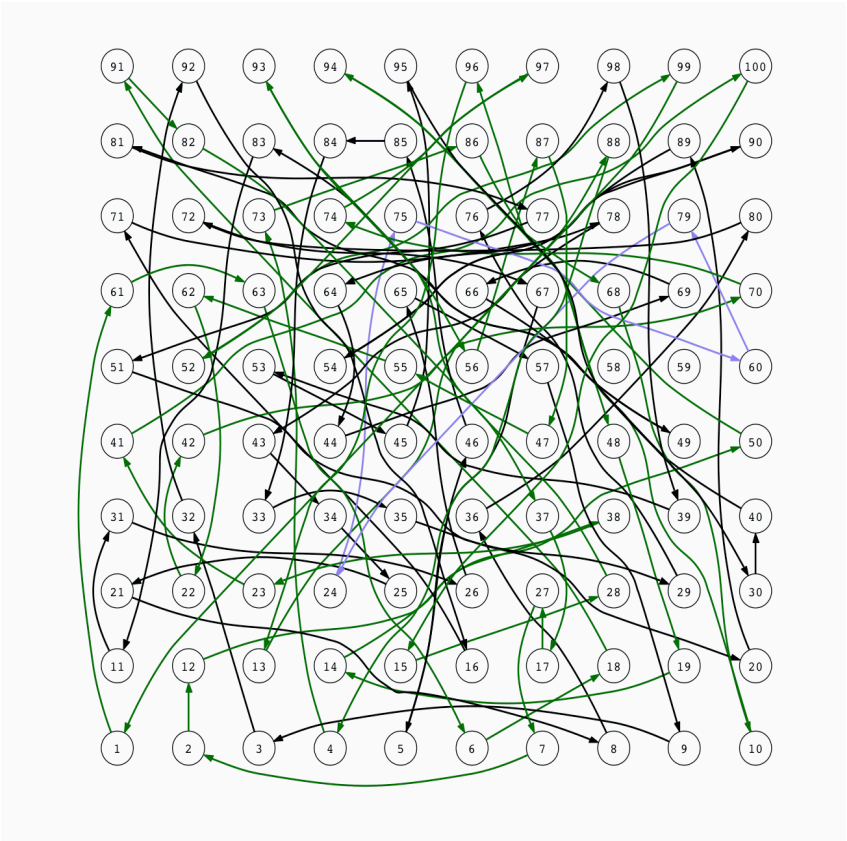
$$H(\pi) = \frac{1}{4\beta} \sum_{x \in \Lambda} |x - \pi(x)|^2$$

Existence of limit $\Lambda \nearrow \mathbb{Z}^d$: not clear, requires compactness argument.
Extension to a measure? Yes, provided $\sum_y P(B_{x,y}) = 1$ for all x , but this property has not been proved yet.

Monte Carlo simulations (dimension $d = 2$)

Step 0

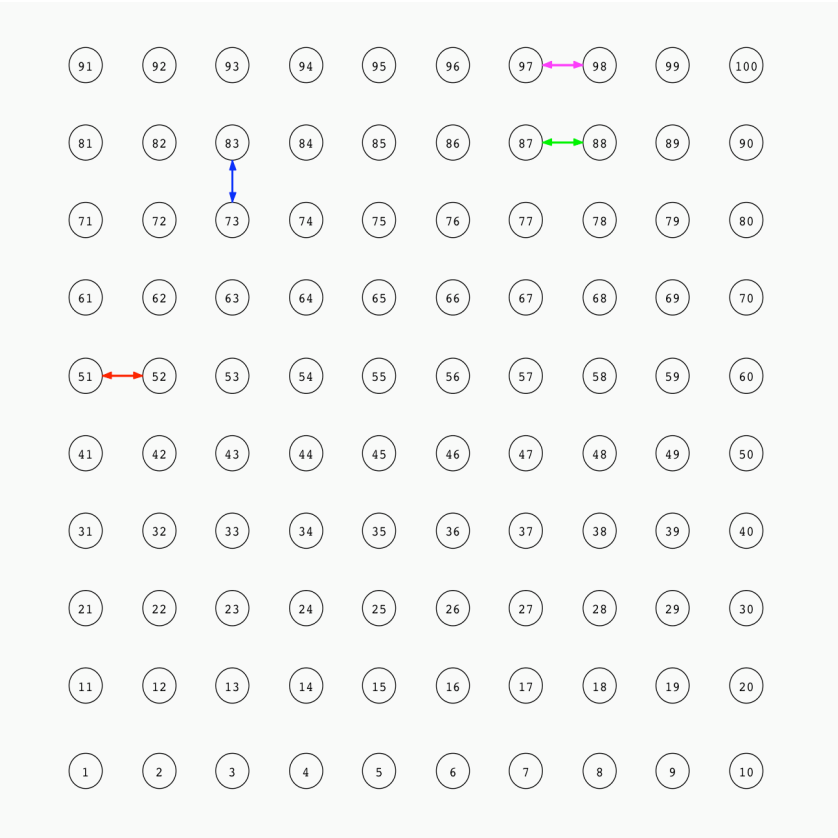


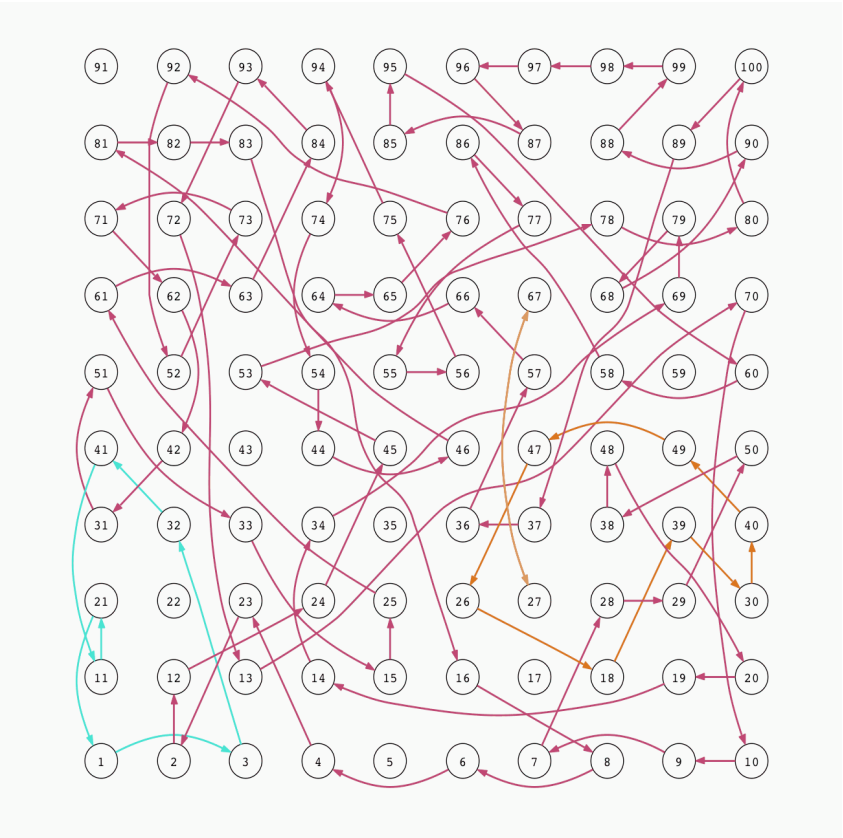


(Courtesy of Daniel Gandolfo and Jean Ruiz, CNRS, Marseille)

Monte Carlo simulations (dimension $d = 2$)

Step 1

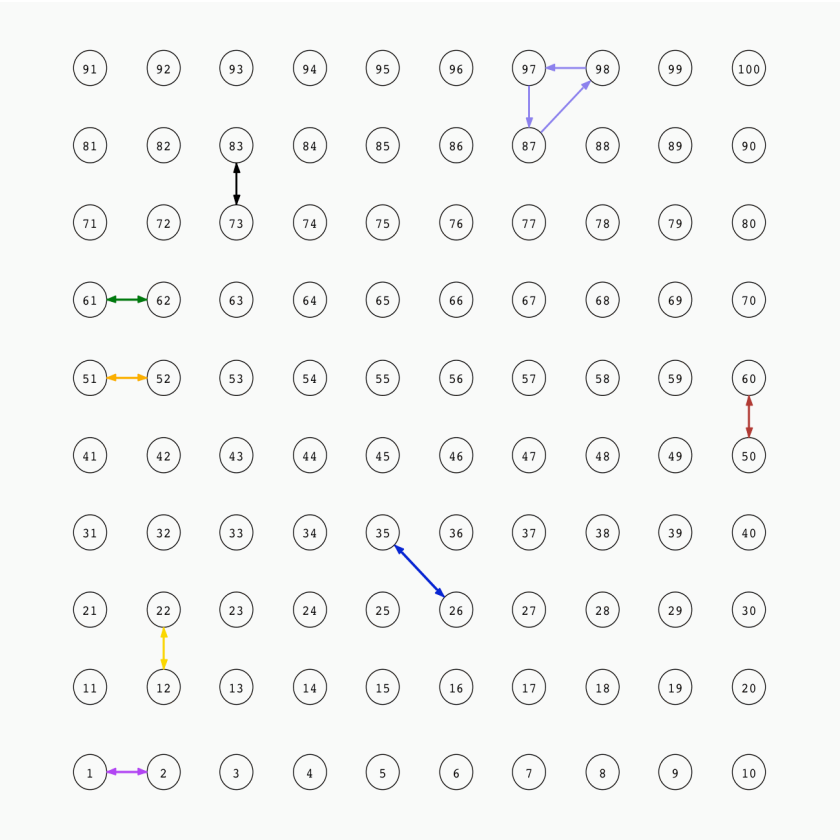


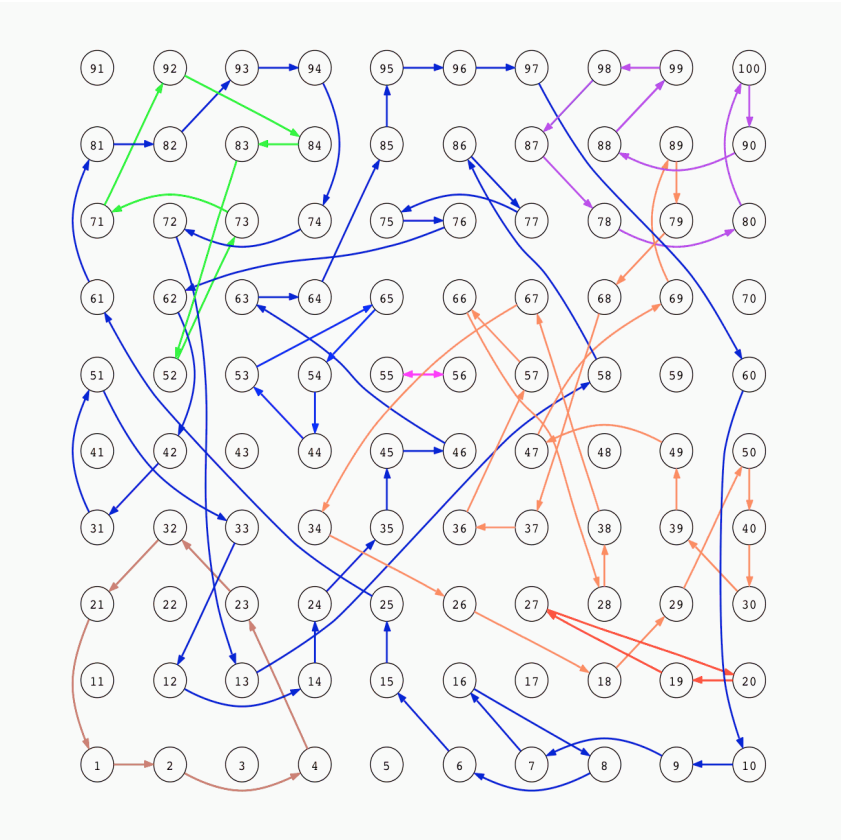


(Courtesy of Daniel Gandolfo and Jean Ruiz, CNRS, Marseille)

Monte Carlo simulations (dimension $d = 2$)

Step 2

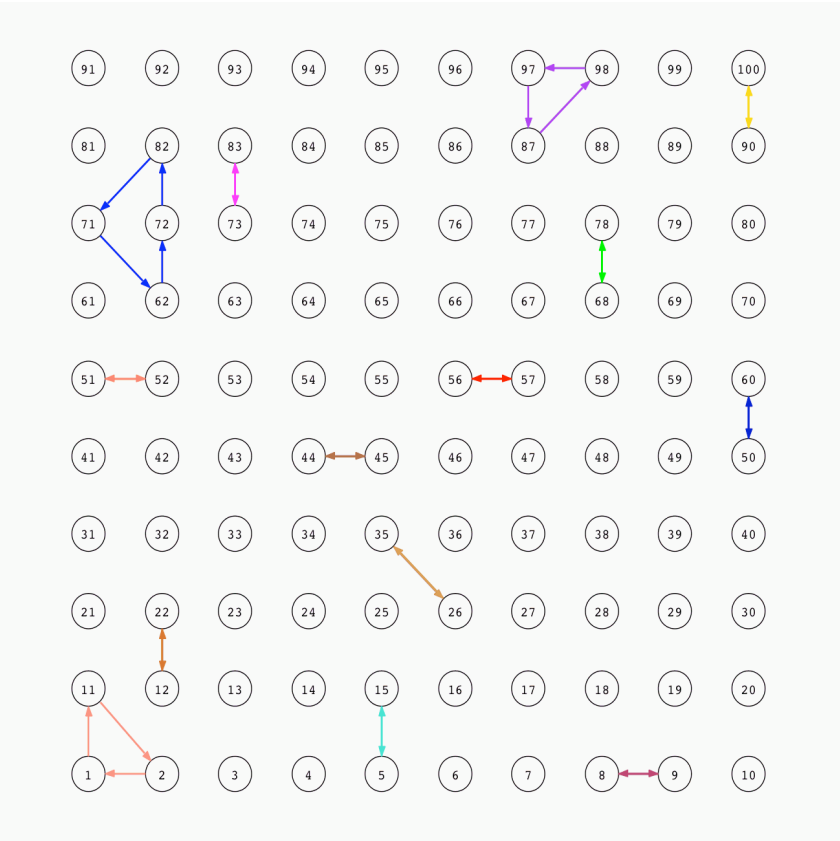


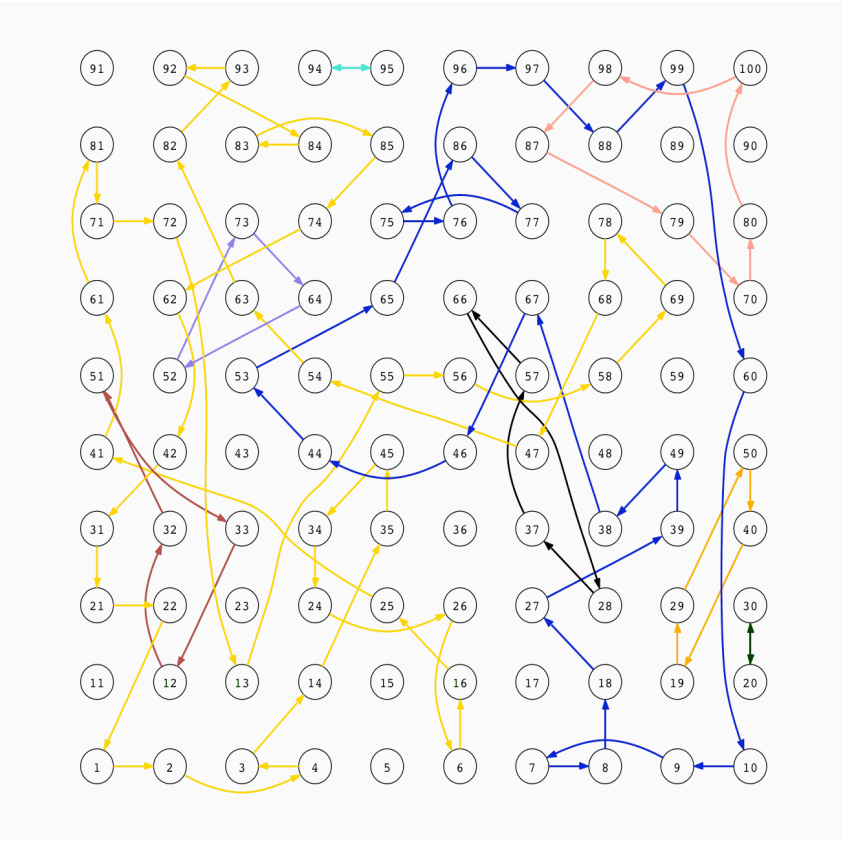


(Courtesy of Daniel Gandolfo and Jean Ruiz, CNRS, Marseille)

Monte Carlo simulations (dimension $d = 2$)

Step 3

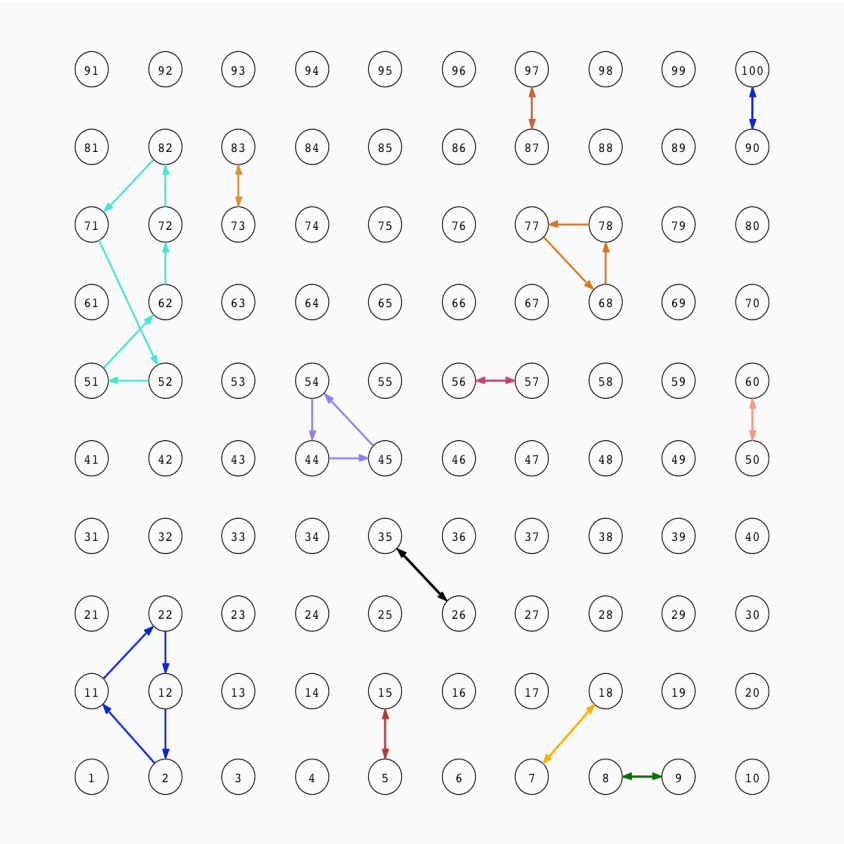


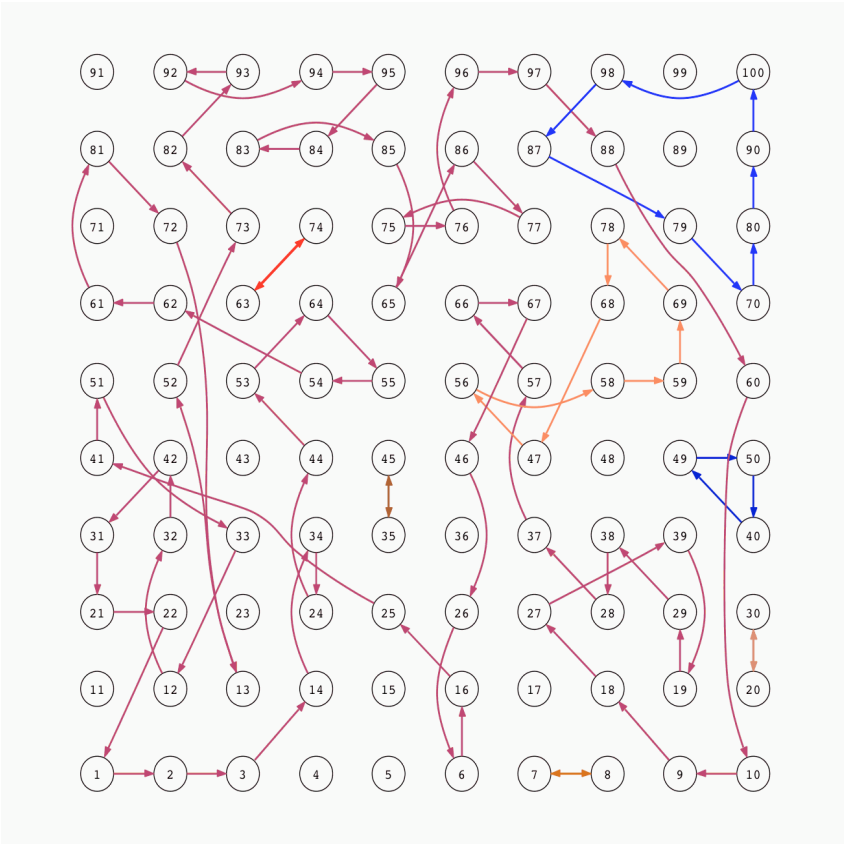


(Courtesy of Daniel Gandolfo and Jean Ruiz, CNRS, Marseille)

Monte Carlo simulations (dimension $d = 2$)

Step 4

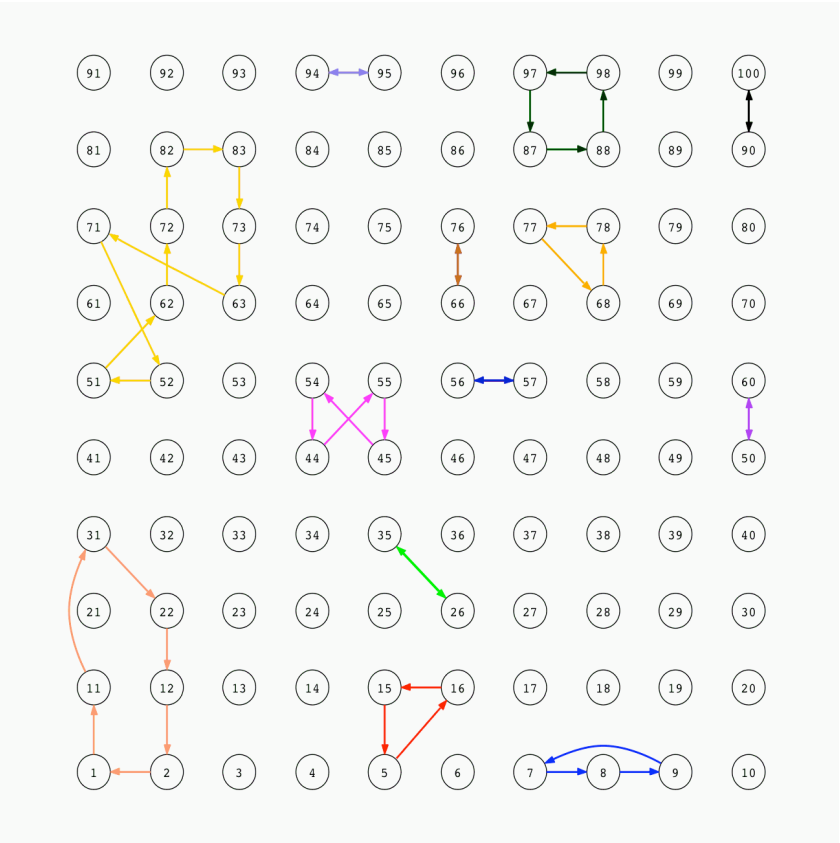


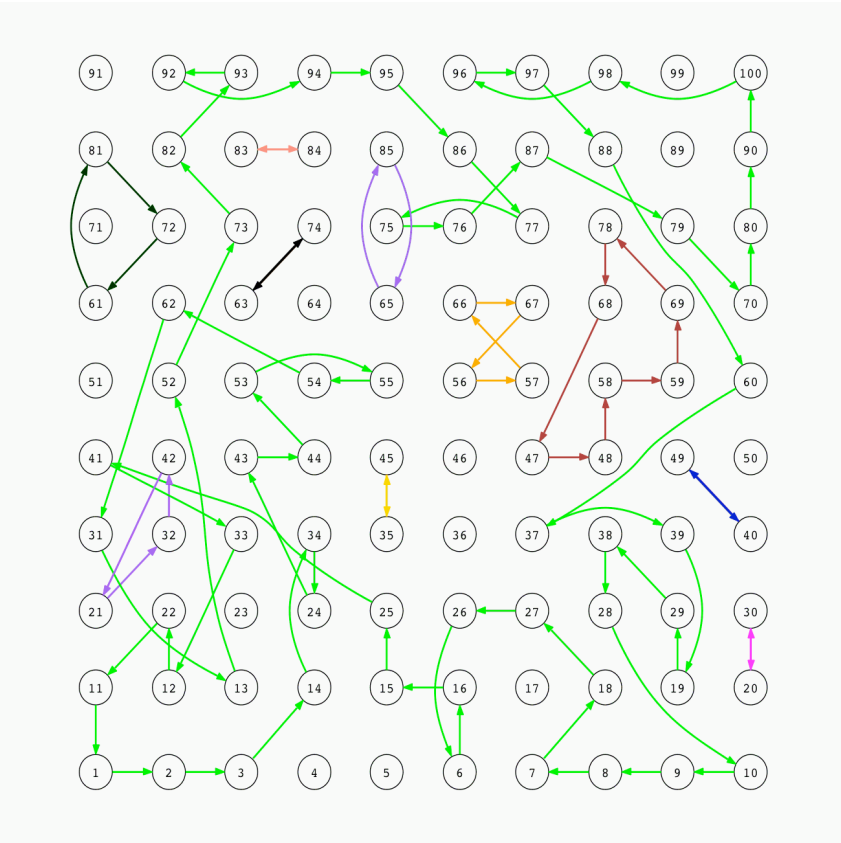


(Courtesy of Daniel Gandolfo and Jean Ruiz, CNRS, Marseille)

Monte Carlo simulations (dimension $d = 2$)

Step 5

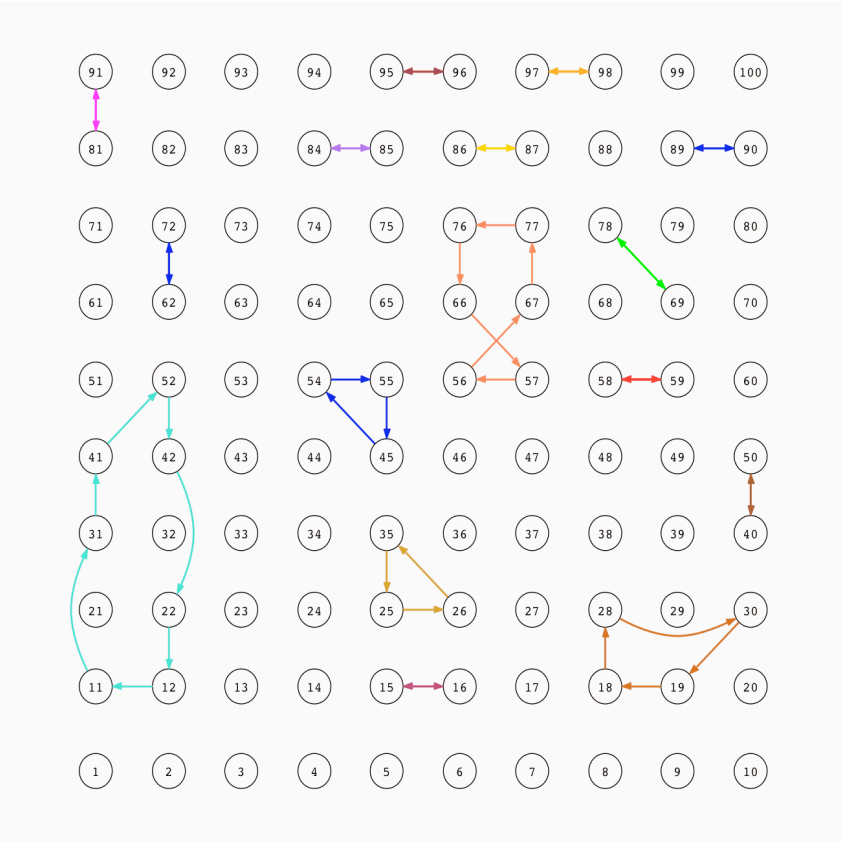


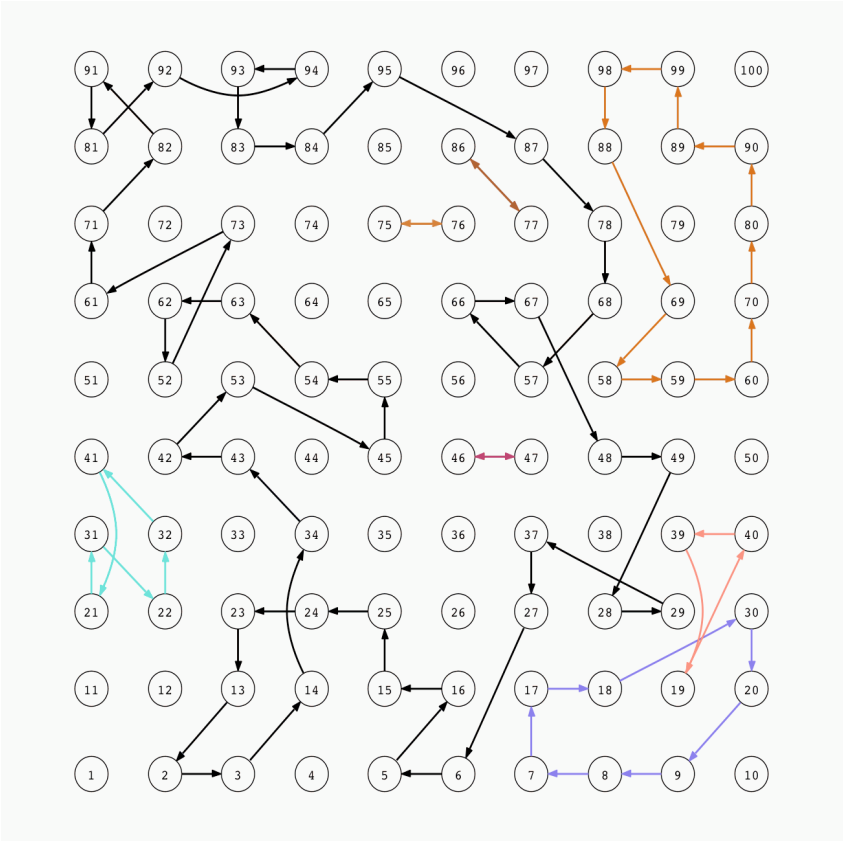


(Courtesy of Daniel Gandolfo and Jean Ruiz, CNRS, Marseille)

Monte Carlo simulations (dimension $d = 2$)

Step 10

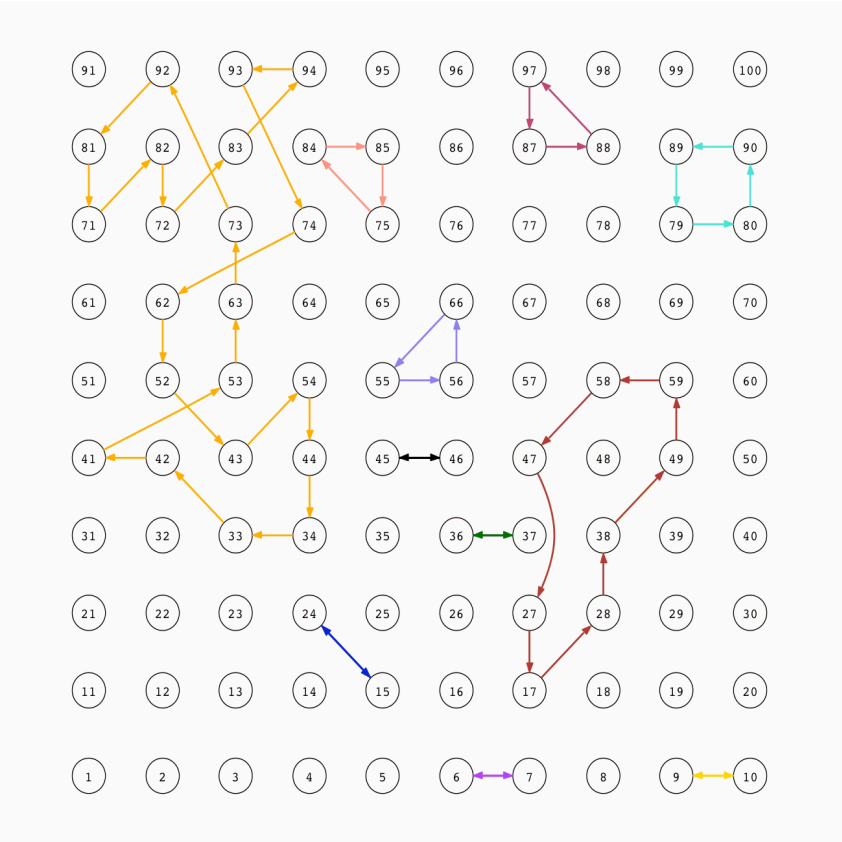


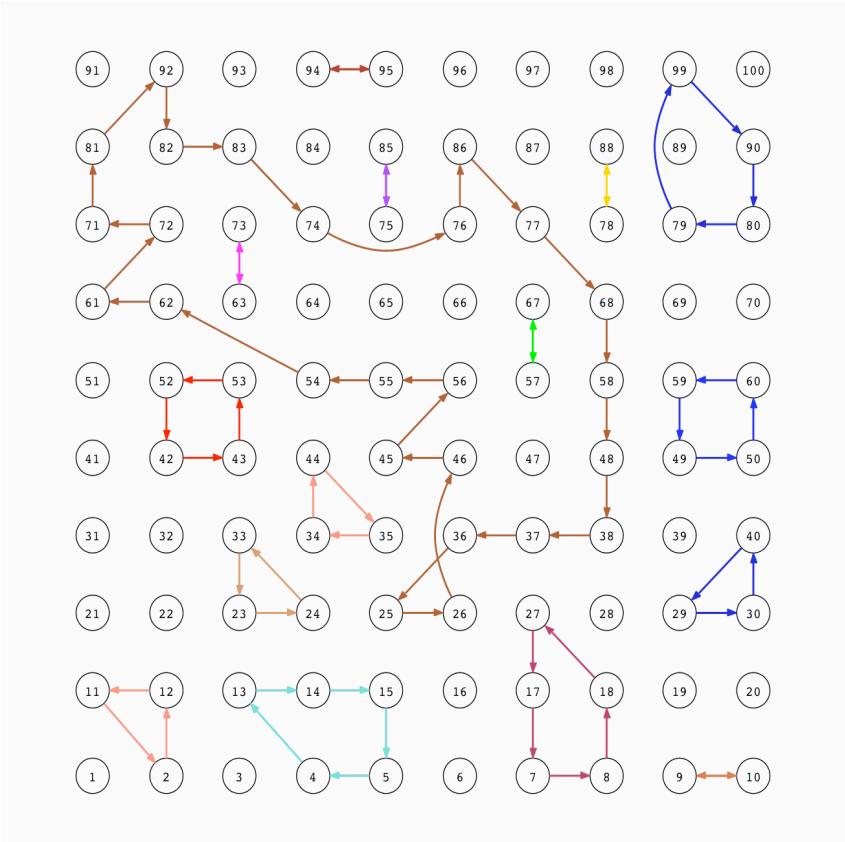


(Courtesy of Daniel Gandolfo and Jean Ruiz, CNRS, Marseille)

Monte Carlo simulations (dimension $d = 2$)

Step 30

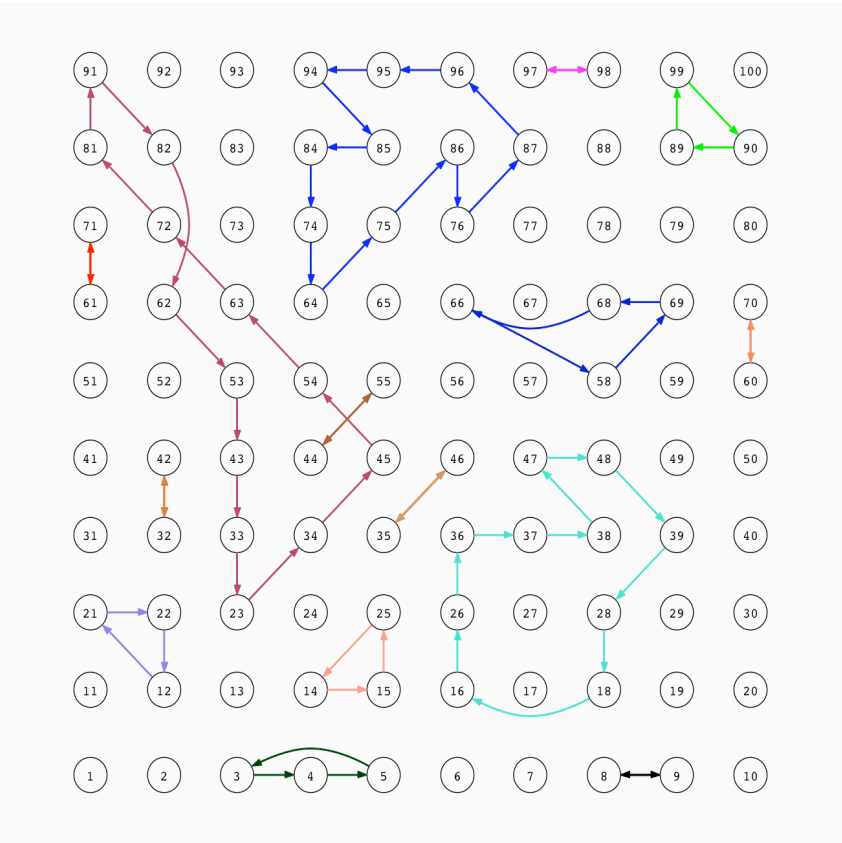


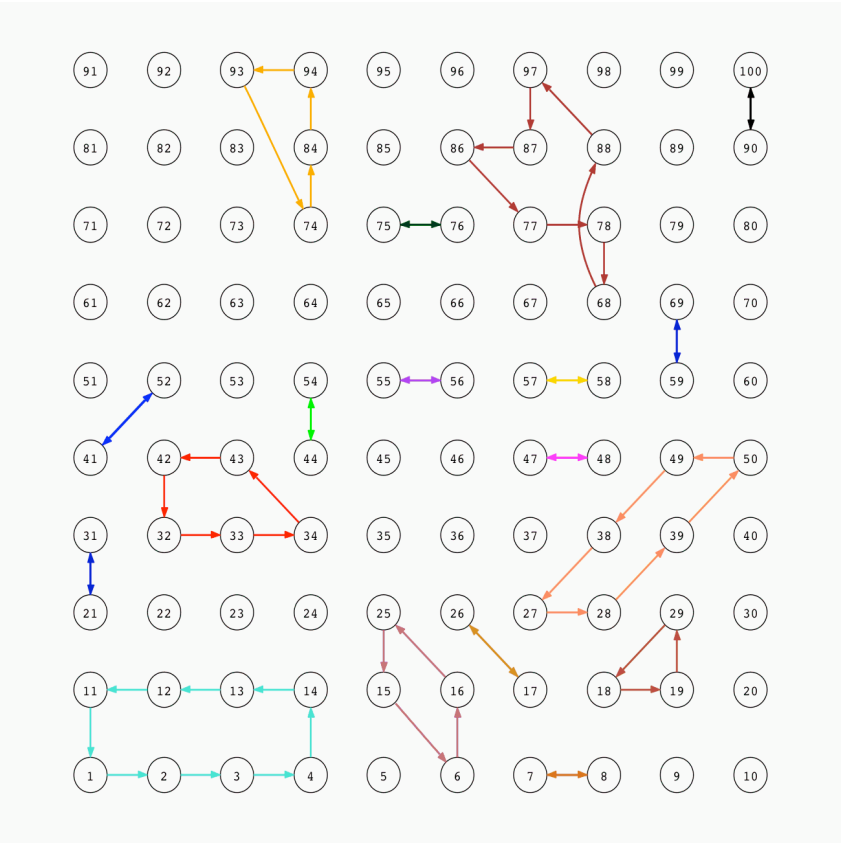


(Courtesy of Daniel Gandolfo and Jean Ruiz, CNRS, Marseille)

Monte Carlo simulations (dimension $d = 2$)

Step 100

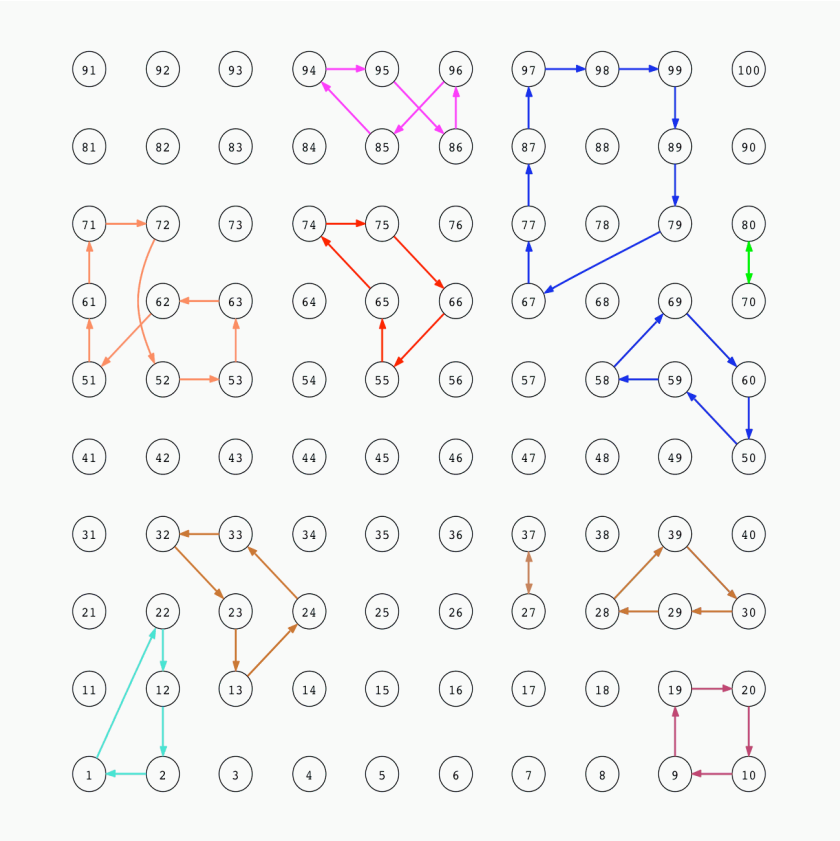


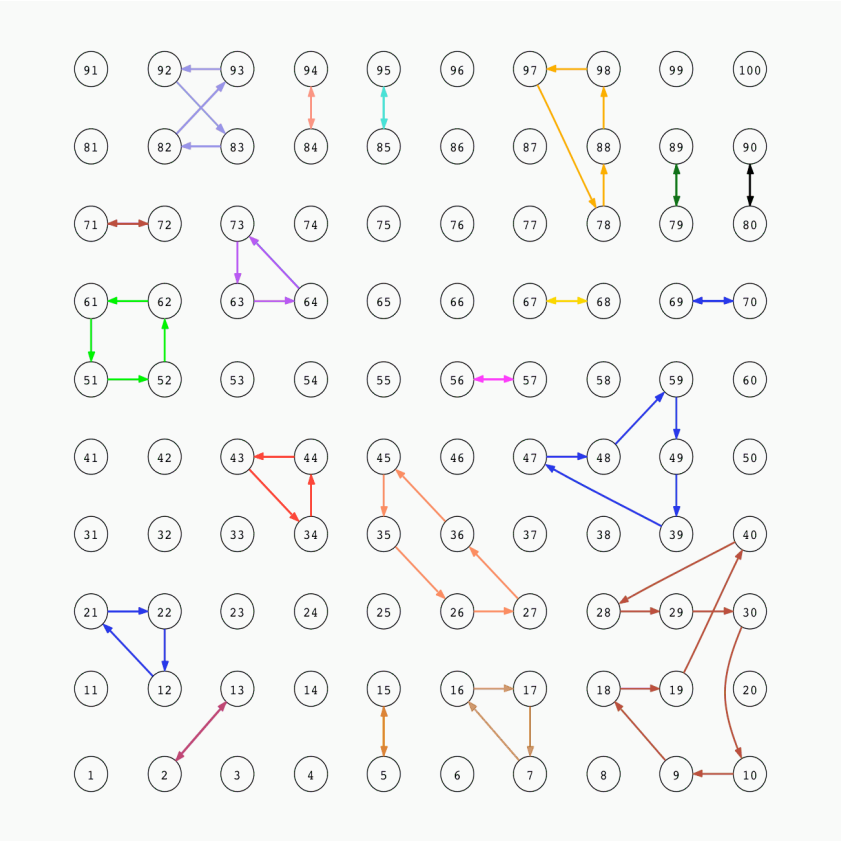


(Courtesy of Daniel Gandolfo and Jean Ruiz, CNRS, Marseille)

Monte Carlo simulations (dimension $d = 2$)

Step ∞





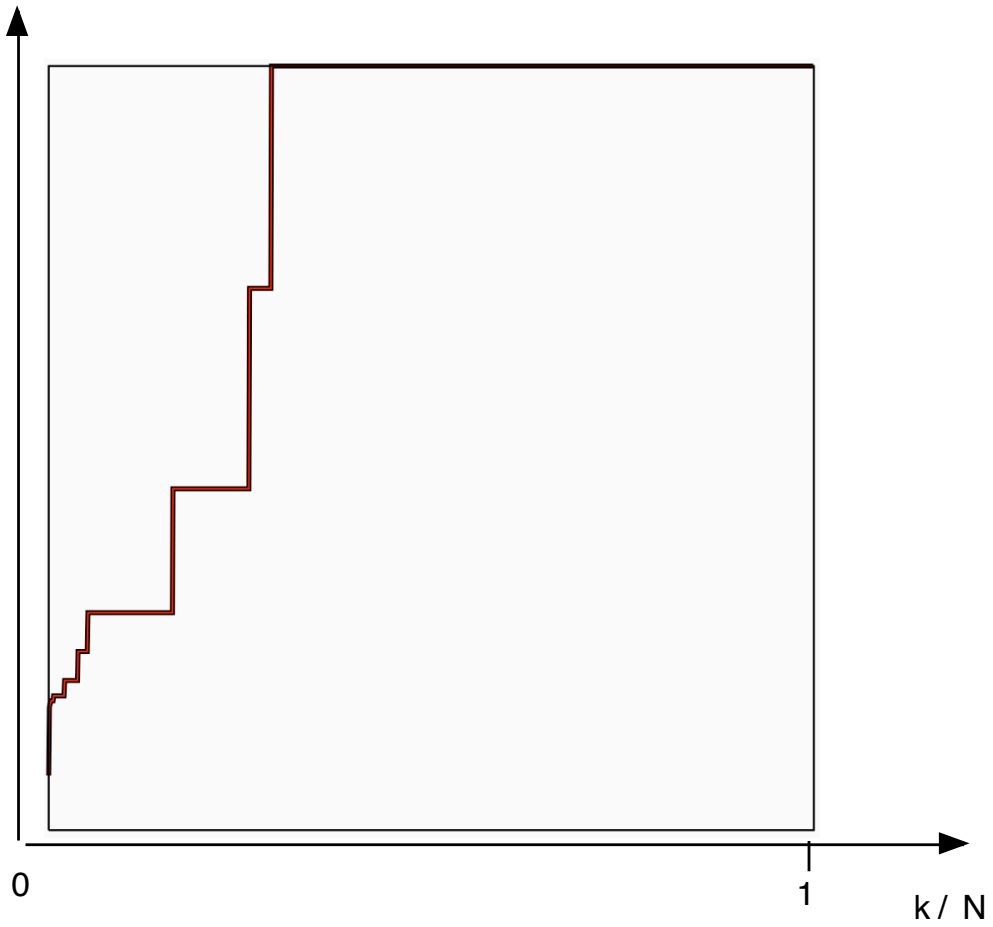
(Courtesy of Daniel Gandolfo and Jean Ruiz, CNRS, Marseille)

Monte Carlo simulations: cycle distribution

Simulation of a large $3d$ system in equilibrium.

Cubic lattice with $L = 30$, $\beta = 0.5$.

The graph shows the number of sites in cycles of length $\leqslant k$, as a function of k/N .
red = current
black = average



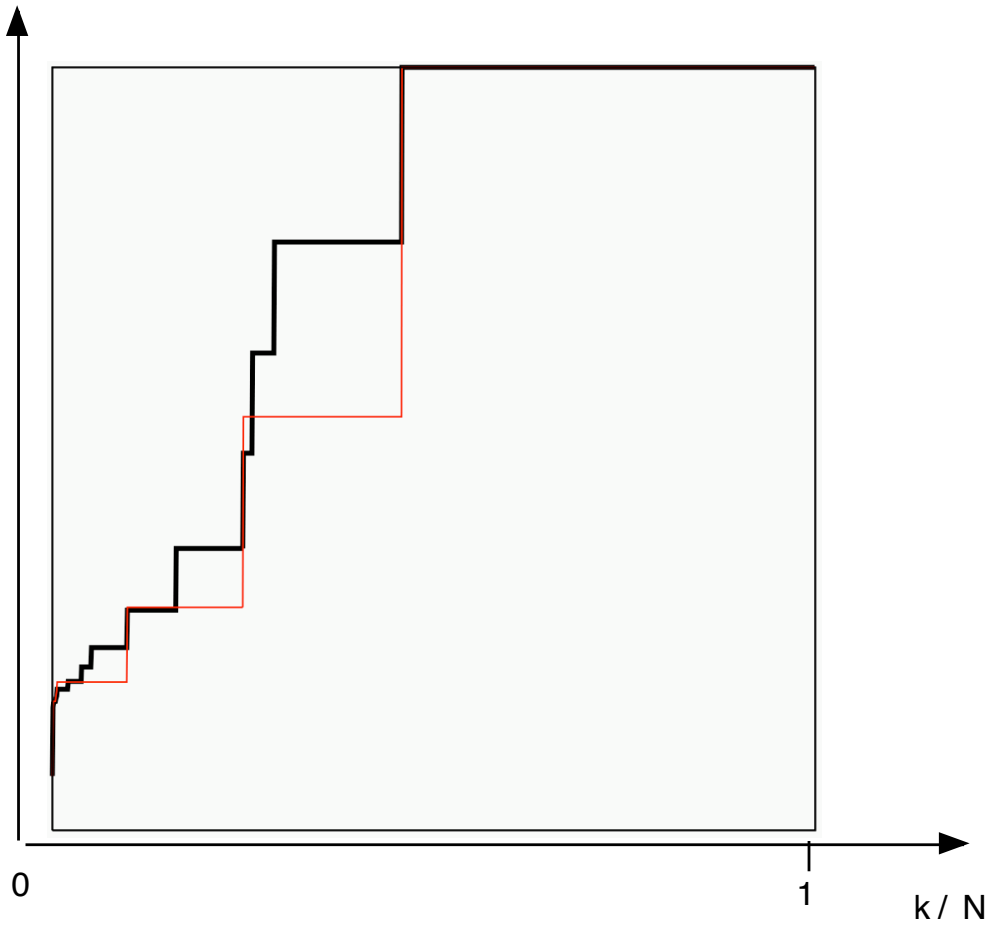
Gandolfo, Ruiz, U (JSP '07)

Monte Carlo simulations: cycle distribution

Simulation of a large $3d$ system in equilibrium.

Cubic lattice with $L = 30$, $\beta = 0.5$.

The graph shows the number of sites in cycles of length $\leqslant k$, as a function of k/N .
red = current
black = average



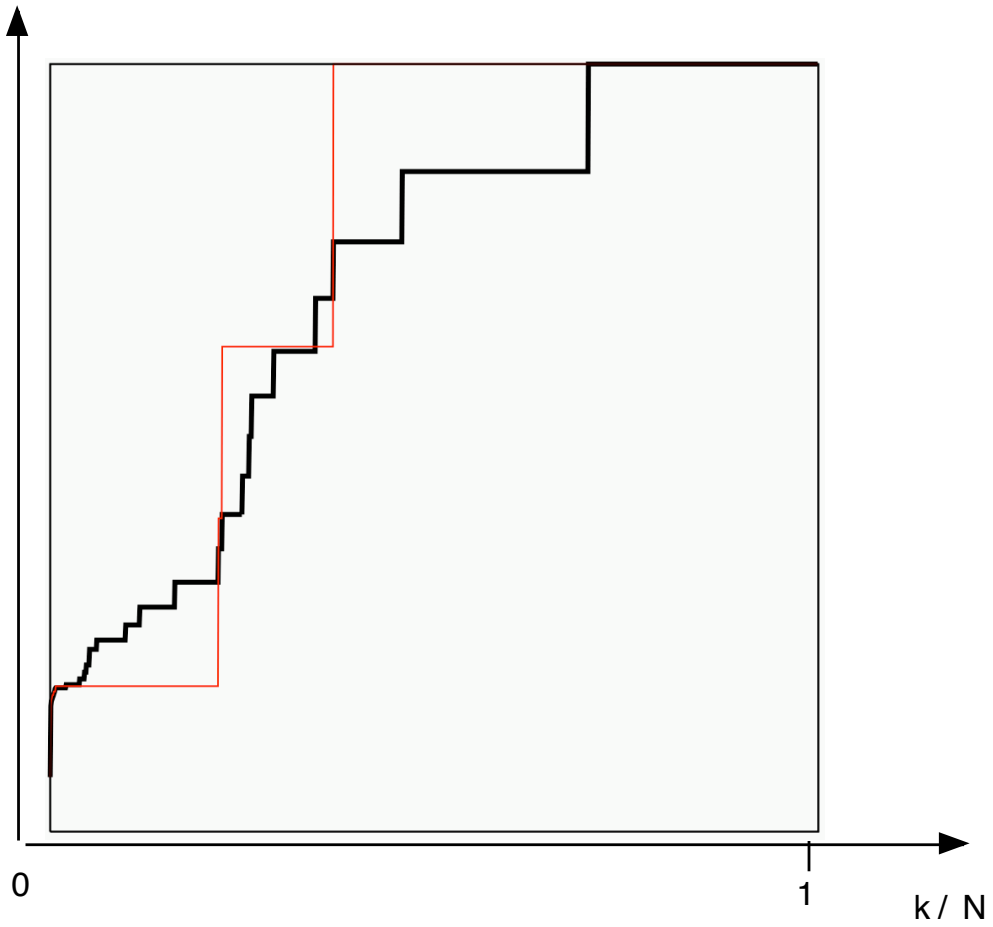
Gandolfo, Ruiz, U (JSP '07)

Monte Carlo simulations: cycle distribution

Simulation of a large $3d$ system in equilibrium.

Cubic lattice with $L = 30$, $\beta = 0.5$.

The graph shows the number of sites in cycles of length $\leqslant k$, as a function of k/N .
red = current
black = average



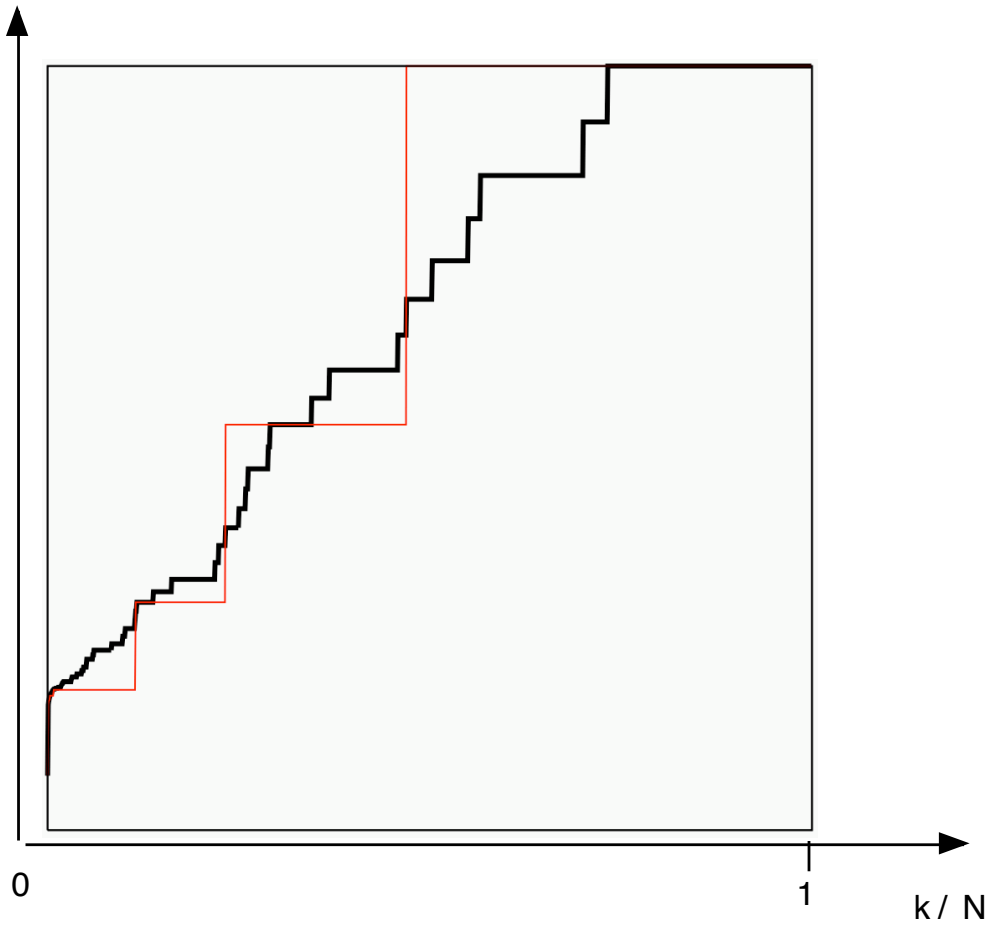
Gandolfo, Ruiz, U (JSP '07)

Monte Carlo simulations: cycle distribution

Simulation of a large $3d$ system in equilibrium.

Cubic lattice with $L = 30$, $\beta = 0.5$.

The graph shows the number of sites in cycles of length $\leqslant k$, as a function of k/N .
red = current
black = average



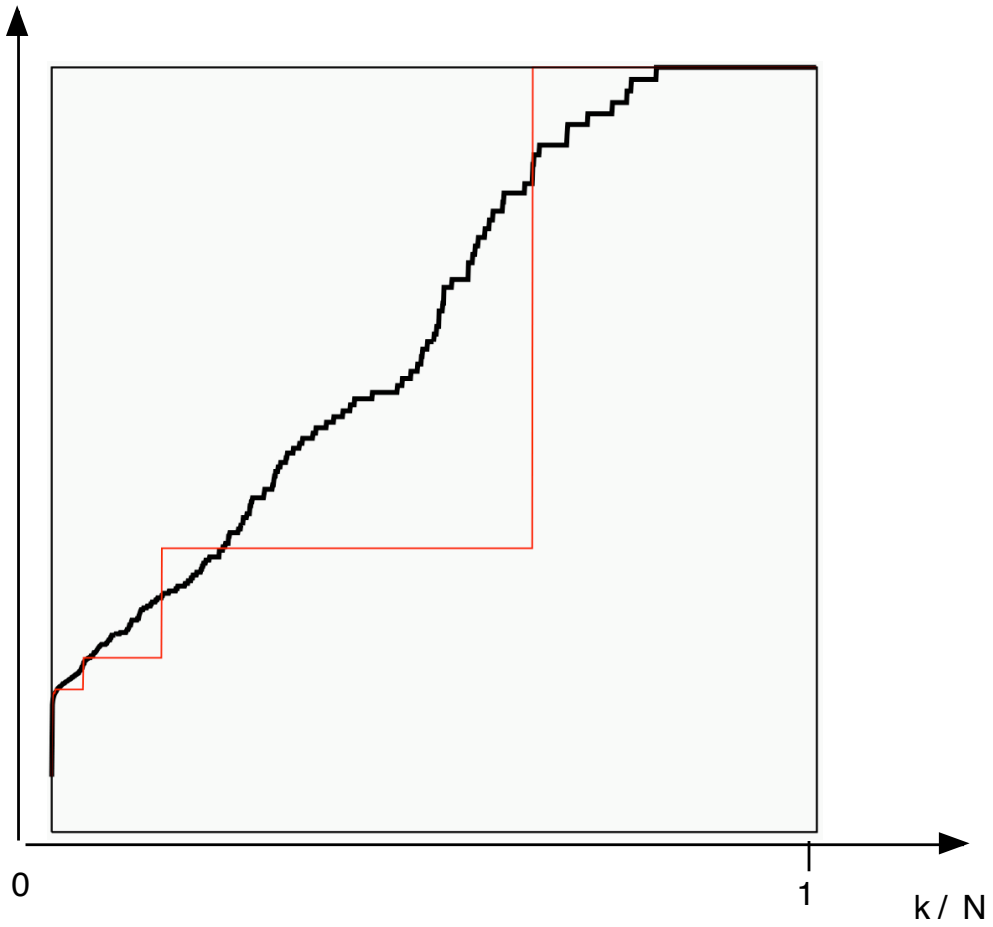
Gandolfo, Ruiz, U (JSP '07)

Monte Carlo simulations: cycle distribution

Simulation of a large $3d$ system in equilibrium.

Cubic lattice with $L = 30$, $\beta = 0.5$.

The graph shows the number of sites in cycles of length $\leqslant k$, as a function of k/N .
red = current
black = average



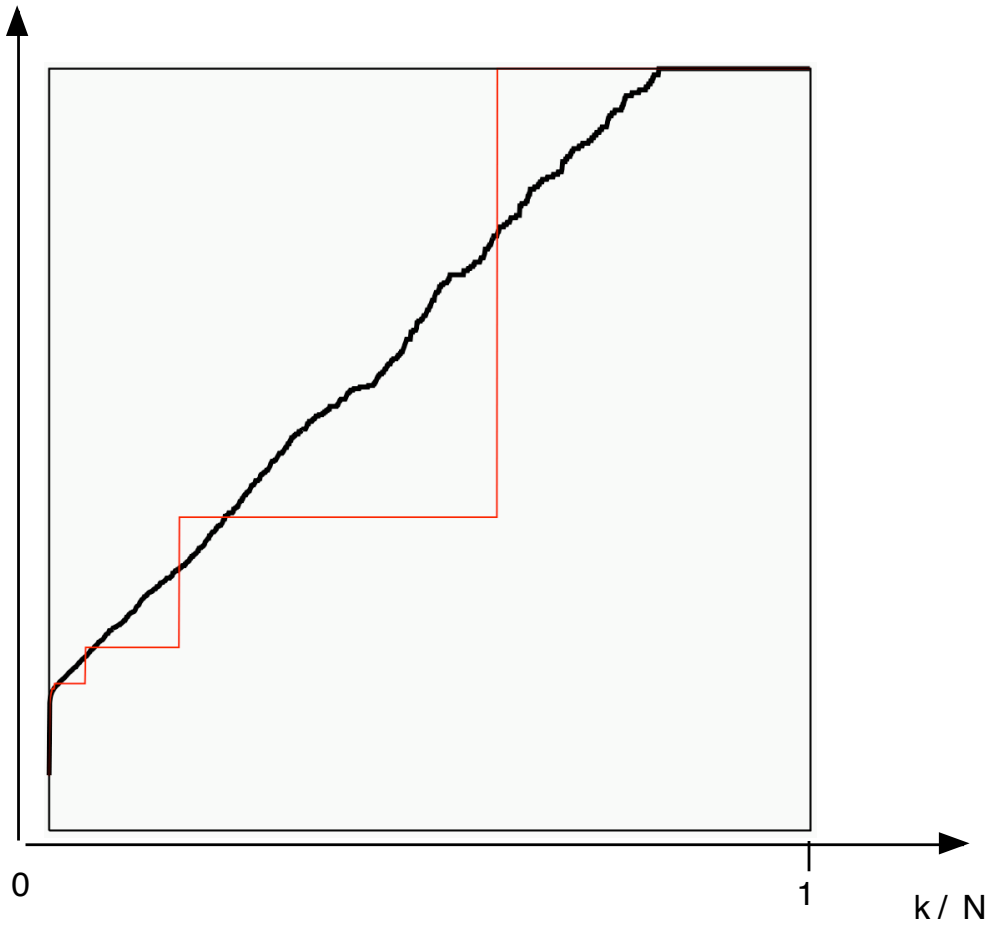
Gandolfo, Ruiz, U (JSP '07)

Monte Carlo simulations: cycle distribution

Simulation of a large $3d$ system in equilibrium.

Cubic lattice with $L = 30$, $\beta = 0.5$.

The graph shows the number of sites in cycles of length $\leqslant k$, as a function of k/N .
red = current
black = average



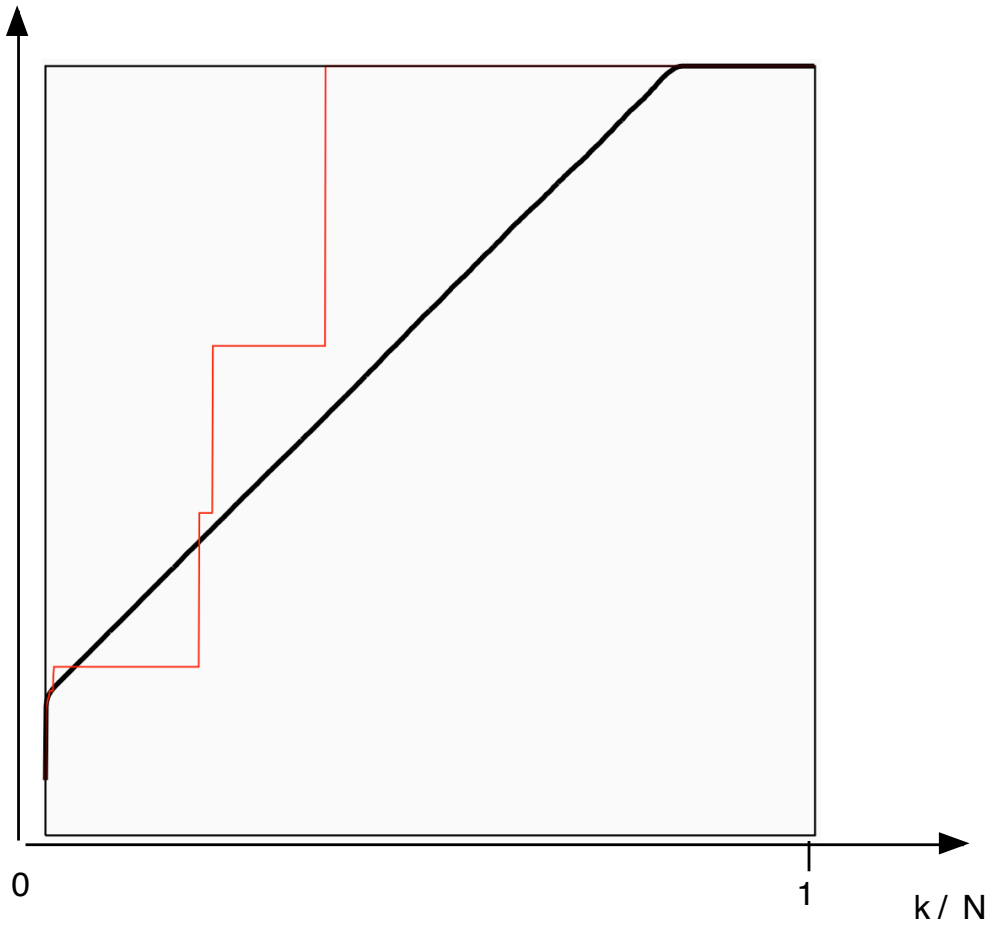
Gandolfo, Ruiz, U (JSP '07)

Monte Carlo simulations: cycle distribution

Simulation of a large $3d$ system in equilibrium.

Cubic lattice with $L = 30$, $\beta = 0.5$.

The graph shows the number of sites in cycles of length $\leqslant k$, as a function of k/N .
red = current
black = average



Gandolfo, Ruiz, U (JSP '07)

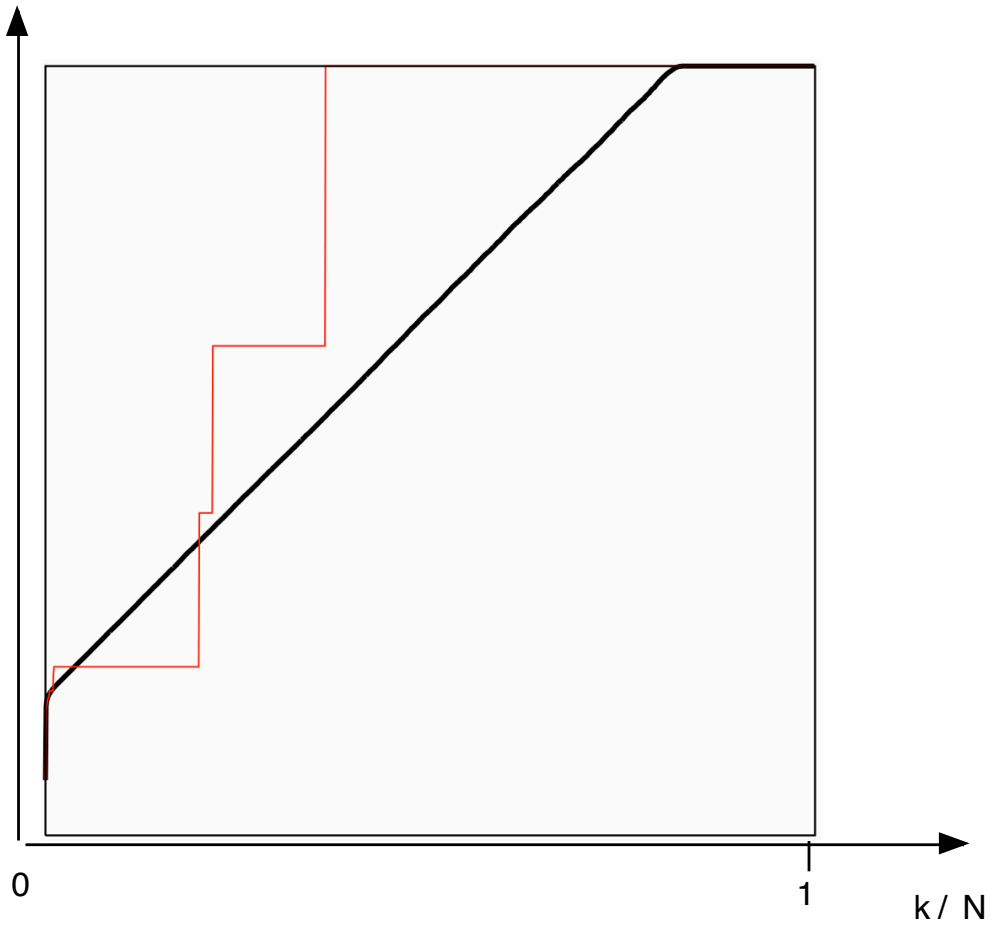
Monte Carlo simulations: cycle distribution

Simulation of a large $3d$ system in equilibrium.

Cubic lattice with $L = 30$, $\beta = 0.5$.

The graph shows the number of sites in cycles of length $\leqslant k$, as a function of k/N .
red = current
black = average

Conclusion: "infinite" cycles appear!



Gandolfo, Ruiz, U (JSP '07)

Results for spatial permutations with cycle weights

"Annealed" model, with $E_{\Lambda,n}(\Theta) = \frac{1}{Z}\int_{\Lambda^n} \mathrm{d}\boldsymbol{x} \sum_\pi \Theta(\boldsymbol{x},\pi)\,\mathrm{e}^{-H(\boldsymbol{x},\pi)}$ and $H(\boldsymbol{x},\pi) = \sum_{i=1}^n \xi(x_i - x_{\pi(i)}) + \sum_{j\geqslant 1} \alpha_j r_j(\pi)$

We suppose that $\xi(x) = \xi(|x|)$ is increasing, $\int \mathrm{e}^{-\xi} = 1$, $\widehat{\mathrm{e}^{-\xi}} \geqslant 0$, and that $\alpha_j \to 0$ faster than $1/\log j$

Results for spatial permutations with cycle weights

"Annealed" model, with $E_{\Lambda,n}(\Theta) = \frac{1}{Z}\int_{\Lambda^n} \mathrm{d}\boldsymbol{x} \sum_\pi \Theta(\boldsymbol{x},\pi)\,\mathrm{e}^{-H(\boldsymbol{x},\pi)}$ and $H(\boldsymbol{x},\pi) = \sum_{i=1}^n \xi(x_i - x_{\pi(i)}) + \sum_{j\geqslant 1} \alpha_j r_j(\pi)$

We suppose that $\xi(x) = \xi(|x|)$ is increasing, $\int \mathrm{e}^{-\xi} = 1$, $\widehat{\mathrm{e}^{-\xi}} \geqslant 0$, and that $\alpha_j \to 0$ faster than $1/\log j$

Theorem (with Volker Betz, 2008)

Assume that $\eta(V) \to \infty$ *with* $\eta(V)/V \to 0$ *, and that* $s \geqslant 0$ *. Then*

(a) $$\lim_{V\to\infty} P_{\Lambda,\rho V}(\ell_1 \in [1,\eta(V)]) = \begin{cases} 1 & \text{if } \rho \leqslant \rho_{\mathrm{c}} \\ \rho_{\mathrm{c}}/\rho & \text{if } \rho \geqslant \rho_{\mathrm{c}} \end{cases}$$

(b) $$\lim_{V\to\infty} P_{\Lambda,\rho V}(\ell_1 \in [\eta(V), \tfrac{V}{\eta(V)}]) = 0$$

(c) $$\lim_{V\to\infty} P_{\Lambda,\rho V}(\ell_1 \in [\tfrac{V}{\eta(V)}, sV]) = \begin{cases} 0 & \text{if } \rho \leqslant \rho_{\mathrm{c}} \\ s/\rho & \text{if } 0 \leqslant s \leqslant \rho - \rho_{\mathrm{c}} \\ 1 - \rho_{\mathrm{c}}/\rho & \text{if } 0 \leqslant \rho - \rho_{\mathrm{c}} \leqslant s \end{cases}$$

Critical density

The critical density is exactly known!
Define $\varepsilon(k)$ by

$$\mathrm{e}^{-\varepsilon(k)} = \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi \mathrm{i} k x}\, \mathrm{e}^{-\xi(x)}\, \mathrm{d}x$$

Notice that $\varepsilon(0) = 0$ and $\varepsilon(k) > a|k|^2$ near $k = 0$. Then

$$\rho_{\mathrm{c}} = \sum_{j \geqslant 1} \mathrm{e}^{-\alpha_j} \int_{\mathbb{R}^d} \mathrm{e}^{-j\varepsilon(k)}\, \mathrm{d}k$$

Notice that infinite cycles also occur in $d = 1$ with an integrable jump function, e.g.

$$\mathrm{e}^{-\xi(x)} = (|x| + 1)^{-\gamma}$$

with $1 < \gamma < 2$. Then $\varepsilon(k) \sim |k|^{\gamma - 1}$ near 0 , and

$$\rho_{\mathrm{c}} = \int_{\mathbb{R}^d} \frac{\mathrm{d}k}{\mathrm{e}^{\varepsilon(k)} - 1} < \infty$$

Conclusion

- Interesting probabilistic models: **random permutations** with **cycle weights** and **spatial structure**

Conclusion

- Interesting probabilistic models: **random permutations** with **cycle weights** and **spatial structure**
- Occurrence of **infinite, macroscopic** cycles; proved in "annealed" model with small cycle weights

Conclusion

- Interesting probabilistic models: **random permutations** with **cycle weights** and **spatial structure**
- Occurrence of **infinite, macroscopic** cycles; proved in "annealed" model with small cycle weights
- Links with **Bose-Einstein condensation**

Conclusion

- Interesting probabilistic models: **random permutations** with **cycle weights** and **spatial structure**
- Occurrence of **infinite, macroscopic** cycles; proved in "annealed" model with small cycle weights
- Links with **Bose-Einstein condensation**
- **Many open questions!** Suitable for analytical or numerical studies

THANK YOU!