Monotonicity, thinning and discrete versions of the Entropy Power Inequality

Joint work with Yaming Yu – see arXiv:0909.0641

Oliver Johnson
O.Johnson@bristol.ac.uk
http://www.stats.bris.ac.uk/∼maotj

Statistics Group, University of Bristol

24th June 2010

Abstract

Abstract

- Differential entropy $h = -\int f(x) \log f(x) dx$ has many nice properties.

Abstract

- Differential entropy $h = -\int f(x) \log f(x) dx$ has many nice properties.
- Often Gaussian provides case of equality.

Abstract

- Differential entropy $h = -\int f(x) \log f(x) dx$ has many nice properties.
- Often Gaussian provides case of equality.
- Focus on 3 such properties:

Abstract

- Differential entropy $h = -\int f(x) \log f(x) dx$ has many nice properties.
- Often Gaussian provides case of equality.
- Focus on 3 such properties:
  1. Maximum entropy

Abstract

- Differential entropy $h = -\int f(x) \log f(x) dx$ has many nice properties.
- Often Gaussian provides case of equality.
- Focus on 3 such properties:
  1. Maximum entropy
  2. Entropy power inequality

Abstract

- Differential entropy $h = -\int f(x) \log f(x) dx$ has many nice properties.
- Often Gaussian provides case of equality.
- Focus on 3 such properties:
  1. Maximum entropy
  2. Entropy power inequality
  3. Monotonicity

Abstract

- Differential entropy $h = -\int f(x) \log f(x) dx$ has many nice properties.
- Often Gaussian provides case of equality.
- Focus on 3 such properties:
  1. Maximum entropy
  2. Entropy power inequality
  3. Monotonicity
- Will discuss discrete analogues for discrete entropy $H = \sum_x p(x) \log p(x)$.

Abstract

- Differential entropy $h = -\int f(x) \log f(x) dx$ has many nice properties.
- Often Gaussian provides case of equality.
- Focus on 3 such properties:
  1. Maximum entropy
  2. Entropy power inequality
  3. Monotonicity
- Will discuss discrete analogues for discrete entropy $H = \sum_x p(x) \log p(x)$.
- Infinite divisibility suggests Poisson should be case of equality.

Property 1: Maximum entropy

Theorem (Shannon 1948)

If X has mean μ and variance σ and $Y \sim N(\mu, \sigma^2)$ then

$$h(X) \leq h(Y),$$

with equality if and only if $X \sim N(\mu, \sigma^2)$.

Property 2: Entropy Power Inequality

Property 2: Entropy Power Inequality

- Define $\mathcal{E}(t) = h(N(0,t)) = \frac{1}{2}\log_2(2\pi e t)$.

Property 2: Entropy Power Inequality

- Define $\mathcal{E}(t) = h(N(0,t)) = \frac{1}{2}\log_2(2\pi e t)$.
- Define entropy power $v(X) = \mathcal{E}^{-1}(h(X)) = 2^{2h(X)}/(2\pi e)$.

Property 2: Entropy Power Inequality

- Define $\mathcal{E}(t) = h(N(0,t)) = \frac{1}{2}\log_2(2\pi e t)$.
- Define entropy power $v(X) = \mathcal{E}^{-1}(h(X)) = 2^{2h(X)}/(2\pi e)$.

Theorem (EPI)

Consider independent continuous X and Y. Then

$$v(X+Y) \geq v(X) + v(Y),$$

with equality if and only if X and Y are Gaussian.

Property 2: Entropy Power Inequality

- Define $\mathcal{E}(t) = h(N(0,t)) = \frac{1}{2}\log_2(2\pi e t)$.
- Define entropy power $v(X) = \mathcal{E}^{-1}(h(X)) = 2^{2h(X)}/(2\pi e)$.

Theorem (EPI)

Consider independent continuous X and Y. Then

$$v(X+Y) \geq v(X) + v(Y),$$

with equality if and only if X and Y are Gaussian.

- First stated by Shannon.

Property 2: Entropy Power Inequality

- Define $\mathcal{E}(t) = h(N(0,t)) = \frac{1}{2}\log_2(2\pi e t)$.
- Define entropy power $v(X) = \mathcal{E}^{-1}(h(X)) = 2^{2h(X)}/(2\pi e)$.

Theorem (EPI)

Consider independent continuous X and Y. Then

$$v(X+Y) \geq v(X) + v(Y),$$

with equality if and only if X and Y are Gaussian.

- First stated by Shannon.
- Lots of proofs (Stam/Blachman, Lieb, Dembo/Cover/Thomas, Tulino/Verdú/Guo).

Property 2: Entropy Power Inequality

- Define $\mathcal{E}(t) = h(N(0,t)) = \frac{1}{2}\log_2(2\pi e t)$.
- Define entropy power $v(X) = \mathcal{E}^{-1}(h(X)) = 2^{2h(X)}/(2\pi e)$.

Theorem (EPI)

Consider independent continuous X and Y. Then

$$v(X+Y) \geq v(X) + v(Y),$$

with equality if and only if X and Y are Gaussian.

- First stated by Shannon.
- Lots of proofs (Stam/Blachman, Lieb, Dembo/Cover/Thomas, Tulino/Verdú/Guo).
- Restricted versions easier to prove? (cf Costa).

Equivalent formulation

Theorem (ECI – not proved here!)

For independent X^*, Y^* with finite variance, for all $\alpha \in [0,1]$,

$$h(\sqrt{\alpha}X^* + \sqrt{1-\alpha}Y^*) \geq \alpha h(X^*) + (1-\alpha)h(Y^*).$$

Lemma

EPI is equivalent to ECI.

Equivalent formulation

Theorem (ECI – not proved here!)

For independent X^*, Y^* with finite variance, for all $\alpha \in [0,1]$,

$$h(\sqrt{\alpha}X^* + \sqrt{1-\alpha}Y^*) \geq \alpha h(X^*) + (1-\alpha)h(Y^*).$$

Lemma

EPI is equivalent to ECI.

- Key role played in Lemma by fact about scaling:

$$v(\sqrt{\alpha}X) = \alpha v(X). \tag{1}$$

Equivalent formulation

Theorem (ECI – not proved here!)

For independent X^*, Y^* with finite variance, for all $\alpha \in [0,1]$,

$$h(\sqrt{\alpha}X^* + \sqrt{1-\alpha}Y^*) \geq \alpha h(X^*) + (1-\alpha)h(Y^*).$$

Lemma

EPI is equivalent to ECI.

- Key role played in Lemma by fact about scaling:

$$v(\sqrt{\alpha}X) = \alpha v(X). \tag{1}$$

- This holds since $h(\sqrt{\alpha}X) = h(X) + \frac{1}{2}\log\alpha$, and $v(\sqrt{\alpha}X) = 2^{2h(\sqrt{\alpha}X)}/(2\pi e)$.

Proof of Lemma: EPI implies ECI

Proof of Lemma: EPI implies ECI

- By the EPI (where $X = \sqrt{\alpha} X^*$ and $Y = \sqrt{1-\alpha} Y^*$) and scaling relation (1),

$$
\begin{aligned}
v(\sqrt{\alpha} X^* + \sqrt{1-\alpha} Y^*) &\geq v(\sqrt{\alpha} X^*) + v(\sqrt{1-\alpha} Y^*) \\
&= \alpha v(X^*) + (1-\alpha) v(Y^*).
\end{aligned}
$$

Proof of Lemma: EPI implies ECI

- By the EPI (where $X = \sqrt{\alpha} X^*$ and $Y = \sqrt{1-\alpha} Y^*$) and scaling relation (1),

$$\begin{aligned} v(\sqrt{\alpha} X^* + \sqrt{1-\alpha} Y^*) &\geq v(\sqrt{\alpha} X^*) + v(\sqrt{1-\alpha} Y^*) \\ &= \alpha v(X^*) + (1-\alpha) v(Y^*). \end{aligned}$$

- Applying $\mathcal{E}$ to both sides and using Jensen (since $\mathcal{E} \sim \log$, so is concave):

$$\begin{aligned} h(\sqrt{\alpha} X^* + \sqrt{1-\alpha} Y^*) &\geq \mathcal{E}\Big(\alpha v(X^*) + (1-\alpha) v(Y^*) \Big) \\ &\geq \alpha \mathcal{E}(v(X^*)) + (1-\alpha) \mathcal{E}(v(Y^*)) \\ &= \alpha h(X^*) + (1-\alpha) h(Y^*) \end{aligned}$$

which is the ECI.

Proof of Lemma: ECI implies EPI

Proof of Lemma: ECI implies EPI

- For some α, define $X^* = X/\sqrt{\alpha}$ and $Y^* = Y/\sqrt{1-\alpha}$.

Proof of Lemma: ECI implies EPI

- For some α, define $X^* = X/\sqrt{\alpha}$ and $Y^* = Y/\sqrt{1-\alpha}$.
- Then the ECI and scaling (1) imply that

$$\begin{aligned} h(X+Y) &= h(\sqrt{\alpha}X^* + \sqrt{1-\alpha}Y^*) \\ &\geq \alpha h(X^*) + (1-\alpha)h(Y^*) \\ &= \alpha\mathcal{E}(v(X^*)) + (1-\alpha)\mathcal{E}(v(Y^*)) \\ &= \alpha\mathcal{E}\left(\frac{v(X)}{\alpha}\right) + (1-\alpha)\mathcal{E}\left(\frac{v(Y)}{1-\alpha}\right) \end{aligned}$$

Proof of Lemma: ECI implies EPI

- For some α, define $X^* = X/\sqrt{\alpha}$ and $Y^* = Y/\sqrt{1-\alpha}$.
- Then the ECI and scaling (1) imply that

$$\begin{aligned} h(X+Y) &= h(\sqrt{\alpha}X^* + \sqrt{1-\alpha}Y^*) \\ &\geq \alpha h(X^*) + (1-\alpha)h(Y^*) \\ &= \alpha\mathcal{E}(v(X^*)) + (1-\alpha)\mathcal{E}(v(Y^*)) \\ &= \alpha\mathcal{E}\left(\frac{v(X)}{\alpha}\right) + (1-\alpha)\mathcal{E}\left(\frac{v(Y)}{1-\alpha}\right) \end{aligned}$$

- Pick $\alpha = \frac{v(X)}{v(X)+v(Y)}$ and the above inequality becomes

$$h(X+Y) \geq \mathcal{E}(v(X)+v(Y)),$$

and applying $\mathcal{E}^{-1}$ to both sides gives the EPI.

Rephrased EPI

Rephrased EPI

- Note that this choice of α makes $v(X^*) = v(Y^*) = v(X) + v(Y)$.

Rephrased EPI

- Note that this choice of α makes $v(X^*) = v(Y^*) = v(X) + v(Y)$.
- This choice of scaling suggests the following rephrased EPI:

Corollary (Rephrased EPI)

Given independent X and Y with finite variance, there exist X^* and Y^* such that $X = \sqrt{\alpha} X^*$ and $Y = \sqrt{1-\alpha} Y^*$ for some α, and such that $h(X^*) = h(Y^*)$.
The EPI is equivalent to the fact that

$$h(X+Y) \geq h(X^*), \tag{2}$$

with equality if and only if X and Y are Gaussian.

Property 3: Monotonicity

Property 3: Monotonicity

- Exciting set of strong recent results, collectively referred to as 'monotonicity'.

Property 3: Monotonicity

- Exciting set of strong recent results, collectively referred to as 'monotonicity'.
- First proved by Artstein/Ball/Barthe/Naor, alternative proofs by Tulino/Verdú and Madiman/Barron.

Monotonicity theorem

Theorem

Given independent continuous X_i with finite variance, for any positive α_i such that $\sum_{i=1}^{n+1} \alpha_i = 1$, writing $\alpha^{(j)} = 1 - \alpha_j$, then

$$nh\left(\sum_{i=1}^{n+1} \sqrt{\alpha_i} X_i\right) \geq \sum_{j=1}^{n+1} \alpha^{(j)} h\left(\sum_{i \neq j} \sqrt{\alpha_i/\alpha^{(j)}} X_i\right).$$

Monotonicity theorem

Theorem

Given independent continuous X_i with finite variance, for any positive α_i such that $\sum_{i=1}^{n+1} \alpha_i = 1$, writing $\alpha^{(j)} = 1 - \alpha_j$, then

$$nh\left(\sum_{i=1}^{n+1} \sqrt{\alpha_i} X_i\right) \geq \sum_{j=1}^{n+1} \alpha^{(j)} h\left(\sum_{i \neq j} \sqrt{\alpha_i/\alpha^{(j)}} X_i\right).$$

- Choosing $\alpha_i = 1/(n+1)$ for IID X_i shows $h\left(\sum_{i=1}^{n} X_i/\sqrt{n}\right)$ is monotone increasing in n.

Monotonicity theorem

Theorem

Given independent continuous X_i with finite variance, for any positive α_i such that $\sum_{i=1}^{n+1} \alpha_i = 1$, writing $\alpha^{(j)} = 1 - \alpha_j$, then

$$nh\left(\sum_{i=1}^{n+1} \sqrt{\alpha_i} X_i\right) \geq \sum_{j=1}^{n+1} \alpha^{(j)} h\left(\sum_{i \neq j} \sqrt{\alpha_i/\alpha^{(j)}} X_i\right).$$

- Choosing $\alpha_i = 1/(n+1)$ for IID X_i shows $h\left(\sum_{i=1}^n X_i/\sqrt{n}\right)$ is monotone increasing in n.
- Equivalently relative entropy $D\left(\sum_{i=1}^n X_i/\sqrt{n} \middle\| Z\right)$ is monotone decreasing in n.

Monotonicity theorem

Theorem

Given independent continuous X_i with finite variance, for any positive α_i such that $\sum_{i=1}^{n+1} \alpha_i = 1$, writing $\alpha^{(j)} = 1 - \alpha_j$, then

$$nh\left(\sum_{i=1}^{n+1} \sqrt{\alpha_i} X_i\right) \geq \sum_{j=1}^{n+1} \alpha^{(j)} h\left(\sum_{i \neq j} \sqrt{\alpha_i/\alpha^{(j)}} X_i\right).$$

- Choosing $\alpha_i = 1/(n+1)$ for IID X_i shows $h\left(\sum_{i=1}^n X_i/\sqrt{n}\right)$ is monotone increasing in n.
- Equivalently relative entropy $D\left(\sum_{i=1}^n X_i/\sqrt{n} \middle\| Z\right)$ is monotone decreasing in n.
- Means CLT is equivalent of 2nd Law of Thermodynamics?

Monotonicity strengthens EPI

Monotonicity strengthens EPI

- By the right choice of α, monotonicity implies the following strengthened EPI.

Theorem (Strengthened EPI)

Given independent continuous Y_i with finite variance, the entropy powers satisfy

$$nv\left(\sum_{i=1}^{n+1} Y_i\right) \geq \sum_{j=1}^{n+1} v\left(\sum_{i \neq j} Y_i\right),$$

with equality if and only if all the Y_i are Gaussian.

Discrete Property 1: Poisson maximum entropy

Definition

For any λ, define class of ultra-log-concave V with mass function p_V satisfying

$$\mathbf{ULC}(\lambda) = \{V : \mathbb{E}V = \lambda \text{ and } p_V(i)/\Pi_\lambda(i) \text{ is log-concave}\}.$$

Discrete Property 1: Poisson maximum entropy

Definition

For any λ, define class of ultra-log-concave V with mass function p_V satisfying

$$\mathbf{ULC}(\lambda) = \{V : \mathbb{E}V = \lambda \text{ and } p_V(i)/\Pi_\lambda(i) \text{ is log-concave}\}.$$

That is

$$i p_V(i)^2 \geq (i+1) p_V(i+1) p_V(i-1), \text{ for all } i.$$

Discrete Property 1: Poisson maximum entropy

Definition

For any λ, define class of ultra-log-concave V with mass function p_V satisfying

$$\mathbf{ULC}(\lambda) = \{V : \mathbb{E}V = \lambda \text{ and } p_V(i)/\Pi_\lambda(i) \text{ is log-concave}\}.$$

That is

$$i p_V(i)^2 \geq (i+1) p_V(i+1) p_V(i-1), \text{ for all } i.$$

- Class includes Bernoulli sums and Poisson.

Maximum entropy and **ULC**(λ)

Theorem (Johnson, Stoch. Proc. Appl. 2007)

If $X \in \mathbf{ULC}(\lambda)$ *and* $Y \sim \Pi_\lambda$ *then*

$$H(X) \leq H(Y),$$

with equality if and only if $X \sim \Pi_\lambda$*.*

(see also Harremoës, 2001)

Key operation: thinning

Definition

Given Y, define the α-thinned version of Y by

$$T_\alpha Y = \sum_{i=1}^{Y} B_i,$$

where $B_1, B_2 \ldots$ i.i.d. Bernoulli(α), independent of Y.

Key operation: thinning

Definition

Given Y, define the α-thinned version of Y by

$$T_\alpha Y = \sum_{i=1}^{Y} B_i,$$

where $B_1, B_2 \dots$ i.i.d. Bernoulli(α), independent of Y.

- Thinning has many interesting properties.

Key operation: thinning

Definition

Given Y, define the α-thinned version of Y by

$$T_\alpha Y = \sum_{i=1}^{Y} B_i,$$

where $B_1, B_2 \ldots$ i.i.d. Bernoulli(α), independent of Y.

- Thinning has many interesting properties.
- We believe T_α is the discrete equivalent of scaling by $\sqrt{\alpha}$.

Key operation: thinning

Definition

Given Y, define the α-thinned version of Y by

$$T_\alpha Y = \sum_{i=1}^{Y} B_i,$$

where $B_1, B_2 \ldots$ i.i.d. Bernoulli(α), independent of Y.

- Thinning has many interesting properties.
- We believe T_α is the discrete equivalent of scaling by $\sqrt{\alpha}$.
- Preserves several parametric families.

Key operation: thinning

Definition

Given Y, define the α-thinned version of Y by

$$T_\alpha Y = \sum_{i=1}^{Y} B_i,$$

where $B_1, B_2 \ldots$ i.i.d. Bernoulli(α), independent of Y.

- Thinning has many interesting properties.
- We believe T_α is the discrete equivalent of scaling by $\sqrt{\alpha}$.
- Preserves several parametric families.
- 'Mean-preserving transform' $T_\alpha X + T_{1-\alpha} Y$ equivalent to 'variance-preserving transform' $\sqrt{\alpha} X + \sqrt{1-\alpha} Y$ in continuous case? (Matches max. ent. condition).

Discrete Property 2: EPI

Discrete Property 2: EPI

- Define $\mathcal{E}(t) = H(\Pi_t)$, an increasing, concave function.

Discrete Property 2: EPI

- Define $\mathcal{E}(t) = H(\Pi_t)$, an increasing, concave function.
- Define $V(X) = \mathcal{E}^{-1}(H(X))$.

Discrete Property 2: EPI

- Define $\mathcal{E}(t) = H(\Pi_t)$, an increasing, concave function.
- Define $V(X) = \mathcal{E}^{-1}(H(X))$.

Conjecture

Consider independent discrete X and Y. Then

$$V(X+Y) \geq V(X) + V(Y),$$

with equality if and only if X and Y are Poisson.

Discrete Property 2: EPI

- Define $\mathcal{E}(t) = H(\Pi_t)$, an increasing, concave function.
- Define $V(X) = \mathcal{E}^{-1}(H(X))$.

Conjecture

Consider independent discrete X and Y. Then

$$V(X+Y) \geq V(X) + V(Y),$$

with equality if and only if X and Y are Poisson.

- Turns out not to be true!

Discrete Property 2: EPI

- Define $\mathcal{E}(t) = H(\Pi_t)$, an increasing, concave function.
- Define $V(X) = \mathcal{E}^{-1}(H(X))$.

Conjecture

Consider independent discrete X and Y. Then

$$V(X+Y) \geq V(X) + V(Y),$$

with equality if and only if X and Y are Poisson.

- Turns out not to be true!
- Even natural restrictions e.g. ULC, Bernoulli sums don't help

Discrete Property 2: EPI

- Define $\mathcal{E}(t) = H(\Pi_t)$, an increasing, concave function.
- Define $V(X) = \mathcal{E}^{-1}(H(X))$.

Conjecture

Consider independent discrete X and Y. Then

$$V(X+Y) \geq V(X) + V(Y),$$

with equality if and only if X and Y are Poisson.

- Turns out not to be true!
- Even natural restrictions e.g. ULC, Bernoulli sums don't help
- Counterexample (not mine!): $X \sim Y$, $P_X(0) = 1/6$, $P_X(1) = 2/3$, $P_X(2) = 1/6$.

Thinned Entropy Power Inequality

Conjecture (TEPI)

Consider independent discrete ULC X and Y. For any α, conjecture that

$$V(T_\alpha X + T_{1-\alpha} Y) \geq \alpha V(X) + (1-\alpha) V(Y),$$

with equality if and only if X and Y are Poisson.

Thinned Entropy Power Inequality

Conjecture (TEPI)

Consider independent discrete ULC X and Y. For any α, conjecture that

$$V(T_\alpha X + T_{1-\alpha} Y) \geq \alpha V(X) + (1-\alpha) V(Y),$$

with equality if and only if X and Y are Poisson.

- Again, not true in general!

Thinned Entropy Power Inequality

Conjecture (TEPI)

Consider independent discrete ULC X and Y. For any α, conjecture that

$$V(T_\alpha X + T_{1-\alpha} Y) \geq \alpha V(X) + (1-\alpha) V(Y),$$

with equality if and only if X and Y are Poisson.

- Again, not true in general!
- Perhaps not all α?

Thinned Entropy Power Inequality

Conjecture (TEPI)

Consider independent discrete ULC X and Y. For any α, conjecture that

$$V(T_\alpha X + T_{1-\alpha} Y) \geq \alpha V(X) + (1-\alpha) V(Y),$$

with equality if and only if X and Y are Poisson.

- Again, not true in general!
- Perhaps not all α?
- Have partial results, but not full description of which α.

Thinned Entropy Power Inequality

Conjecture (TEPI)

Consider independent discrete ULC X and Y. For any α, conjecture that

$$V(T_\alpha X + T_{1-\alpha} Y) \geq \alpha V(X) + (1-\alpha) V(Y),$$

with equality if and only if X and Y are Poisson.

- Again, not true in general!
- Perhaps not all α?
- Have partial results, but not full description of which α.
- For example, true for Poisson Y with $H(Y) \leq H(X)$.

Two weaker results

Two weaker results

- Analogues of the continuous concavity and scaling results do hold. (Again, proofs not given here!)

Theorem (TECI, Johnson/Yu, ISIT '09)

Consider independent ULC X and Y. For any α,

$$H(T_\alpha X + T_{1-\alpha} Y) \geq \alpha H(X) + (1-\alpha) H(Y).$$

Theorem (RTEPI, Johnson/Yu, arXiv:0909.0641)

Consider ULC X. For any α,

$$V(T_\alpha X) \geq \alpha V(X).$$

Discrete EPI?

Discrete EPI?

- Duplicating steps from the continuous case above, we deduce an analogue of rephrased EPI

Theorem (Johnson/Yu, arXiv:0909.0641)

Given independent ULC X and Y, suppose there exist X^* and Y^* such that $X = T_\alpha X^*$ and $Y = T_{1-\alpha} Y^*$ for some α, and such that $H(X^*) = H(Y^*)$. Then

$$H(X+Y) \geq H(X^*), \tag{3}$$

with equality if and only if X and Y are Poisson.

Discrete Property 3: Monotonicity

Discrete Property 3: Monotonicity

- Write $D(X)$ for $D(X \| \Pi_{\mathbb{E}X})$.

Discrete Property 3: Monotonicity

- Write $D(X)$ for $D(X \| \Pi_{\mathbb{E}X})$.
- By convex ordering arguments, Yu showed that for IID X_i:
  1. relative entropy $D\left(\sum_{i=1}^n T_{1/n} X_i\right)$ is monotone decreasing in n,
  2. for ULC X_i the entropy $H\left(\sum_{i=1}^n T_{1/n} X_i\right)$ is monotone increasing in n.

Discrete Property 3: Monotonicity

- Write $D(X)$ for $D(X \| \Pi_{\mathbb{E}X})$.
- By convex ordering arguments, Yu showed that for IID X_i:
  1. relative entropy $D\left(\sum_{i=1}^n T_{1/n} X_i\right)$ is monotone decreasing in n,
  2. for ULC X_i the entropy $H\left(\sum_{i=1}^n T_{1/n} X_i\right)$ is monotone increasing in n.
- In fact, implicit in work of Yu is following stronger theorem:

Theorem

Given positive α_i such that $\sum_{i=1}^{n+1} \alpha_i = 1$, and writing $\alpha^{(j)} = 1 - \alpha_j$, then for any independent ULC X_i,

$$nD\left(\sum_{i=1}^{n+1} T_{\alpha_i} X_i\right) \leq \sum_{j=1}^{n+1} \alpha^{(j)} D\left(\sum_{i \neq j} T_{\alpha_i/\alpha^{(j)}} X_i\right).$$

Generalization of monotonicity

Theorem (Johnson/Yu, arXiv:0909.0641)

Given positive α_i *such that* $\sum_{i=1}^{n+1} \alpha_i = 1$*, and writing* $\alpha^{(j)} = 1 - \alpha_j$*, then for any independent ULC* X_i*,*

$$nH\left(\sum_{i=1}^{n+1} T_{\alpha_i} X_i\right) \geq \sum_{j=1}^{n+1} \alpha^{(j)} H\left(\sum_{i \neq j} T_{\alpha_i/\alpha^{(j)}} X_i\right).$$

Generalization of monotonicity

Theorem (Johnson/Yu, arXiv:0909.0641)

Given positive α_i *such that* $\sum_{i=1}^{n+1} \alpha_i = 1$, *and writing* $\alpha^{(j)} = 1 - \alpha_j$, *then for any independent ULC* X_i,

$$nH\left(\sum_{i=1}^{n+1} T_{\alpha_i} X_i\right) \geq \sum_{j=1}^{n+1} \alpha^{(j)} H\left(\sum_{i \neq j} T_{\alpha_i/\alpha^{(j)}} X_i\right).$$

- Exact analogue of Artstein/Ball/Barthe/Naor result,

$$nh\left(\sum_{i=1}^{n+1} \sqrt{\alpha_i} X_i\right) \geq \sum_{j=1}^{n+1} \alpha^{(j)} h\left(\sum_{i \neq j} \sqrt{\alpha_i/\alpha^{(j)}} X_i\right),$$

  replacing scalings by thinnings.

Future work

Future work

- Resolve for which α, the

$$V(T_\alpha X + T_{1-\alpha} Y) \geq \alpha V(X) + (1-\alpha) V(Y).$$

Future work

- Resolve for which α, the

$$V(T_\alpha X + T_{1-\alpha} Y) \geq \alpha V(X) + (1-\alpha) V(Y).$$

- Relation to Shepp-Olkin conjecture

Future work

- Resolve for which α, the

$$V(T_\alpha X + T_{1-\alpha} Y) \geq \alpha V(X) + (1-\alpha) V(Y).$$

- Relation to Shepp-Olkin conjecture
- **Conjecture**: if there exist X^* and Y^* such that $X = T_\alpha X^*$ and $Y = T_{1-\alpha} Y^*$, where $\alpha = V(X)/(V(X) + V(Y))$, then

$$V(X+Y) \geq V(X) + V(Y).$$