Scaling limit of the two-dimensional dynamic Ising-Kac model

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A famous result

Theorem (Bertini-Giacomin '97)

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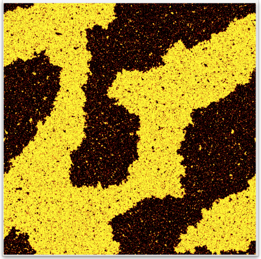
- Scaling leaves stochastic heat equation invariant.
- Gives interpretation for infinite renormalisation in KPZ.
- Particle model changes with the scaling (asymmetry).

Kac-Ising model I



Grid: $\Lambda = \mathbb{Z}^d/(2N+1)\mathbb{Z}^d \approx \{-N,\ldots,N\}^d$,

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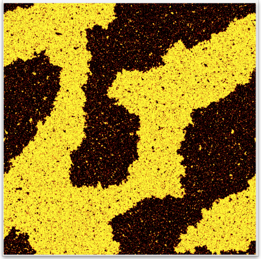

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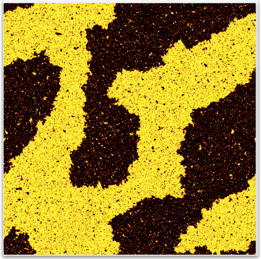

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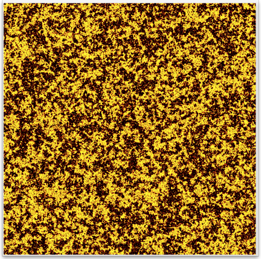

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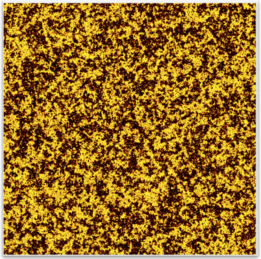

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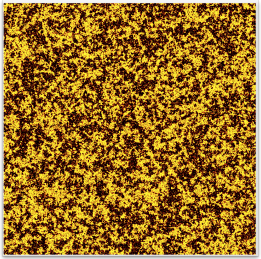

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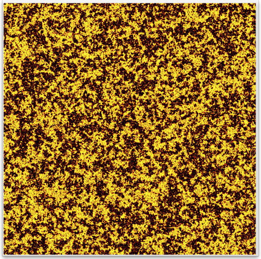

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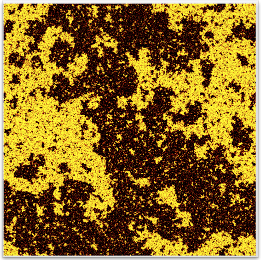
Defines reversible Markov process.

Kac -Ising III



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- Very detailed study by Presutti et al. in 90s. Study hydrodynamic limit, linear fluctuations, ...

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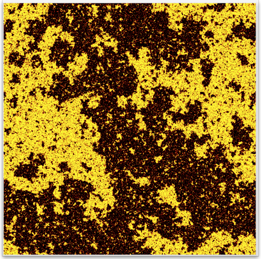

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Aim: Describe non-linear fluctuations dynamical near criticality in dimension $d = 2$! Local averages described by SPDE

$$\partial_t X = \Delta X - X^3 + AX + \xi.$$

Coarse-grained field and rescaling I

Local averages: $h_\gamma(\sigma(t),k) = \sum_{\ell\in\Lambda} \kappa_\gamma(\ell-k)\,\sigma(t,\ell)$.

Dynamic can be described in terms of h_γ:

$$\mathscr{H}_\gamma(\sigma) = -\frac{1}{2}\sum_{k\in\Lambda_N} \sigma(k)\,h_\gamma(\sigma,k)\ ,$$

$$c_\gamma(\sigma,j) = \frac{1}{2}\Big(1-\sigma(j)\ \underbrace{\tanh\left(\beta h_\gamma(\sigma,j)\right)}_{\approx\beta h_\gamma(\sigma,j)-\frac{1}{3}\beta^3 h_\gamma(\sigma,j)^3}\ \Big)\ .$$

Coarse-grained field and rescaling II

Evolution equation:

$$h_\gamma(t,k) = h_\gamma(0,k) + \int_0^t \mathscr{L}_\gamma \, h_\gamma(s,k) \, ds + m_\gamma(s,k),$$

where $m_\gamma(\cdot, k)$ martingale.

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$$\mathscr{L}_\gamma h_\gamma(\sigma,\cdot) = \Big(\kappa_\gamma * h_\gamma(\sigma,\cdot) - h_\gamma(\sigma,\cdot)\Big) + (\beta-1)\kappa_\gamma * h_\gamma(\sigma,\cdot)$$
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Predictable quadratic variation:

$$\langle m_\gamma(\cdot,k), m_\gamma(\cdot,j)\rangle_t = 4\int_0^t \sum_{\ell\in\Lambda_N} \kappa_\gamma(k-\ell)\,\kappa_\gamma(j-\ell)\, \underbrace{c_\gamma(\sigma(s),\ell)}_{=\frac{1}{2}+\ldots}\, ds\,.$$

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Rescaled dynamics: $X_\gamma(t,x) = \frac{1}{\delta} h_\gamma\left(\frac{t}{\alpha}, \frac{x}{\varepsilon}\right).$

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Non-trivial scaling:

$$\varepsilon = \gamma^{\frac{4}{4-d}} \qquad \delta = \gamma^{\frac{d}{4-d}} \qquad \alpha = \gamma^{\frac{2d}{4-d}}$$

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Scaling limit

For $d = 1$ result known:

Theorem (Bertini-Presutti-Rüdiger-Saada '93, Fritz-Rüdiger '95)

If $d = 1$ and

$$\varepsilon = \gamma^{\frac{4}{3}} \qquad \delta = \gamma^{\frac{1}{3}} \qquad \alpha = \gamma^{\frac{2}{3}} \qquad (\beta - 1) = C\alpha,$$

then X_γ converges in law to the solution of the stochastic PDE

$$\partial_t X = \Delta X - \frac{1}{3}X^3 + CX + \sqrt{2}\xi,$$

where $\xi =$ space-time white noise.

Main result

Theorem (Mourrat, W. 2014+)

If $d = 2$ *and*

$$\varepsilon = \gamma^2 \qquad \delta = \gamma \qquad \alpha = \gamma^2 \qquad \frac{(\beta - 1)}{\alpha} = \mathfrak{c}_\gamma + A$$

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$$\mathbf{X}_\gamma(t,\varphi) = \sum_{x \in \Lambda_\varepsilon} \varepsilon^2 \varphi(x)\, \delta^{-1} \sigma(t/\alpha, x/\varepsilon),$$

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Higher dimensions:

- In principle scaling works also for $d = 3$. Hairer / Catellier-Chouk have stable notion of solutions for SPDE.
- For $d = 4$ scaling and SPDE theory break down.

Discussion II

Literature:

- Conjectured by Giacomin-Lebowitz-Presutti ’99.
- In static case: trivial limit without renormalisation Cassandro-Marra-Presutti ’95, ’97.
- Glimm-Jaffe-Spencer: Phase transition for static Φ_2^4 ’75. Mass parameter A plays role of inverse temperature.
- Construction of da Prato-Debussche ’03 crucial for our argument.

Renormalisation of Φ^4_2

$$\partial_t X_\delta = \Delta X_\delta - (X_\delta^3 - C_\delta X_\delta) + \xi_\delta.$$

$\xi_\delta(t,x) := \sum_{|\mathbf{k}|\leq \frac{1}{\delta}} e^{i\mathbf{k}\cdot x}\dot{\beta}(t,\mathbf{k})$. Noise white in time, spatial corellations $\sim \delta$.

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In $d = 3$ recently by Hairer. Then $C_\delta = \frac{C_1}{\delta} + C_2 \log(\delta^{-1})$.

Regularity of the linear part

Stochastic heat equation:

$$\partial_t Z_\delta = \Delta Z_\delta + \sqrt{2}\xi_\delta \qquad \text{on } [0,\infty) \times \mathbb{T}^d.$$

Solution $Z_\delta(t,x) = \int_0^t \int_{\mathbb{T}^d} K_\delta(t-s, x-y)\, \xi(ds, dy)$.

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$$\mathbb{E}\langle Z_\delta(t,\cdot), \mathcal{S}_x^\lambda \eta\rangle^2 = \int_0^t \int_{\mathbb{T}^d} \Big(\int_0^t \int_{\mathbb{T}^d} K_\delta(t-s, x-y) \mathcal{S}_{\bar{x}}^\lambda \eta(x) \Big)^2 \, ds\, dy$$

$$\lesssim_{\text{unif. in } \delta} \begin{cases} \log(\lambda^{-1}) & \text{if } d = 2, \\ \lambda^{-d-2} & \text{if } d \geq 3. \end{cases}$$

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Gaussian $\Rightarrow$ equivalent bounds in L^p $\Rightarrow$ regularity estimates.

Powers of the solutions

$$Z_\delta(t,x) = \int_0^t \int_{\mathbb{T}^d} K_\delta(t-s, x-y)\, \xi(ds, dy).$$

Itô formula:

$$\begin{aligned} Z_\delta(t,x)^3 = & \int_{([0,t]\times\mathbb{T})^3} \mathcal{W}_\delta^{(3)}(t,x;z_1,z_2,z_3)\, \xi(dz_1)\, \xi(dz_2)\, \xi(dz_3) \\ & + 3Z_\delta(t,x) \underbrace{\int_{([0,t]\times\mathbb{T})} K_\delta(t-s,x-y)^2 ds\, dy}_{C_\delta} . \end{aligned}$$

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- Expectations of first term tested against scaled test functions can be bounded uniformly in δ for $d = 1, 2, (3)$.
- Second term diverges for $d = 2, 3$.

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Nelson estimate $\Rightarrow$ equivalent bounds in L^p $\Rightarrow$ regularity estimates.

Construction of the “model":

For $d = 2$

Lemma

For every $p \geq 1$, $\nu > 0$

- $Z_\delta \to Z$,
- $Z_\delta^2 - C_\delta \to Z^{:2:}$,
- $Z_\delta^3 - 3C_\delta Z_\delta \to Z^{:3:}$,

in $L^p(C([0,T], \mathcal{B}^{-\nu}_{\infty,\infty}(\mathbb{T}^2)))$.

- $Z^{:n:}$ are called Wick powers.
- Key ingredient: Equivalence of moments in fixed Wiener chaos - Nelson estimate.
- Regularity measured in Besov spaces.

Analysis of linearised particle system I

Evolution equation:

$$X_\gamma(t,x) \approx X_\gamma(0,x) + M_\gamma(t,x) + \int_0^t \Big[\Delta_\gamma X_\gamma(s,x) - \frac{1}{3} K_\gamma *_\varepsilon \Big(X_\gamma^3(s,x) - C_1 \log(\gamma^{-1}) X_\gamma(s,x) + \mathrm{Err}\Big)\Big] ds\,.$$

Linearised system:

$R_{\gamma,t}(s,x) := \int_{r=0}^{s} P_{t-r}^\gamma \, dM_\gamma(r,x)$ where $P_t^\gamma = e^{t\Delta_\gamma}$.

Approximate Wick powers:

$R_{\gamma,t}^{:n:}(s,x) = n \int_{r=0}^{s} R_{\gamma,t}^{:n-1:}(r^-,x) \, dR_{\gamma,t}(r,x)$.

Analysis of linearised particle system II

$R_{\gamma,t}(s,x) := \int_{r=0}^{s} P_{t-r}^{\gamma}\, dM_\gamma(r,x)$

$R_{\gamma,t}^{:n:}(s,x) = n \int_{r=0}^{s} R_{\gamma,t}^{:n-1:}(r^-,x)\, dR_{\gamma,t}(r,x)$.

Lemma

For every n, the process $Z_\gamma^{:n:}(t) := R_{\gamma,t}^{:n:}(t,\cdot)$ converges in distribution to $Z^{:\,n:}$ with respect to $\mathcal{D}(\mathbb{R}_+,\mathcal{B}_{\infty,\infty}^{-\nu}(\mathbb{T}^2))$ topology .

- In tightness proof Nelson estimate replaced by martingale inequality (BDG).
- $\mathcal{D}(\dots) =$ space of càdlàg functions.

A Lemma for iterated integrals

$$\left(\mathbb{E}\sup_{0\le r\le t}\left|(R^{:n:}_{\gamma,t}(r,x),f(x))\right|^p\right)^{\frac{2}{p}}$$
$$\lesssim\int_{r_1=0}^{t}\dots\int_{r_n=0}^{r_{n-1}}\sum_{\mathbf{z}\in\Lambda_\varepsilon^n}\varepsilon^{2n}\,(\mathcal{W}^{(n)}(\cdot,\mathbf{r},\mathbf{z}),f(\cdot))^2\,d\mathbf{r}+\mathrm{Err}\,,$$

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where

$$\mathcal{W}^{(n)}(\cdot,\mathbf{r},\mathbf{z}) = \prod_{j=1}^{n} P^{\gamma}_{t-r_j} \star K_\gamma(\cdot - z_j) \; ,$$

and

$$\mathrm{Err} = C\, \varepsilon^4 \delta^{-2} \sum_{\ell=1}^{n} \int_{r_1=0}^{t} \dots \int_{r_{\ell-1}=0}^{r_{\ell-2}} \sum_{z_1,\dots,z_{\ell-1}\in\Lambda_\varepsilon} \varepsilon^{2(\ell-1)}$$

$$\left(\mathbb{E} \sup_{\substack{0\le r_\ell\le r_{\ell-1}\\ z_\ell\in\Lambda_\varepsilon}} \left|\left(f(\cdot), R^{:n-\ell:}_{\gamma,t}(r_\ell,\cdot)\prod_{j=1}^{\ell} P^{\gamma}_{t-r_j} \star K_\gamma(\cdot - z_j)\right)\right|^p\right)^{\frac{2}{p}} d\mathbf{r}_{\ell-1} \; .$$

Da Prato-Debussche Strategy II

2.) Non-linear evolution as continuous function of lifted Gaussian process

Regularisation trick: $v_\delta = X_\delta - Z_\delta$.

$$\begin{aligned}\partial_t v &= \Delta v_\delta - \left((Z_\delta + v_\delta)^3 - 3\, C_\delta (Z_\delta + v_\delta)\right) \\ &= \Delta v_\delta - \left(Z_\delta^{:3:} + 3 Z_\delta^{:2:}\, v_\delta + 3 Z_\delta\, v_\delta^2 + v_\delta^3\right).\end{aligned}$$

Multiplicative inequality: If $\alpha < 0 < \beta$ with $\alpha + \beta > 0$

$$\|u\, v\|_{\mathcal{B}^\alpha_{\infty,\infty}} \lesssim \|u\|_{\mathcal{B}^\alpha_{\infty,\infty}} \|v\|_{\mathcal{B}^\beta_{\infty,\infty}}.$$

Used to deal with nonlinearity.

Can be repeated on level of X_γ.

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Conclusion

- Prove conjecture about two dimensional Kac-Ising model near criticality.
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    1.) Treat a linearised version with probabilistic techniques.
    2.) Treat the non-linear system as a perturbation.
- Core part of proof: Convergence of a linearised particle system and nonlinear functions thereof.