The dynamical Sine-Gordon equation

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Dynamical Sine-Gordon Equation

Space dimension $d = 2$. Equation depends on parameter $\beta > 0$.

$$\partial_t u = \frac{1}{2}\Delta u + \zeta \sin(\beta u) + \xi$$

ξ is space-time white noise.

Some parabolic stochastic PDEs

- Stochastic heat equation (ξ space-time white noise) (Walsh 1980s)

$$\partial_t u = \Delta u + \xi$$

- KPZ (Bertini-Giacomin 1997, Hairer 2011, Hairer-Quastel)

$$\partial_t h = \Delta h + (\nabla h)^2 + \xi$$

- Dynamical Φ^4 (Da Prato-Debussche 2003, Hairer 2013, Mourrat-Weber 2014, Hairer-Xu)

$$\partial_t \phi = \Delta \phi - \phi^3 + \xi$$

- Parabolic Anderson model ($\bar{\xi}$ space white noise) (Gubinelli-Imkeller-Perkovski 2012, Hairer 2013, Hairer-Labbe)

$$\partial_t u = \Delta u + f(u)\bar{\xi}$$

Difficulties of solving these equations

- Stochastic heat equation in d space dimension

$$\partial_t u = \Delta u + \xi$$

$$\text{Heuristically,} \quad \mathbb{E}[\xi(x,t)\xi(\bar{x},\bar{t})] = \delta^{(d)}(x-\bar{x})\delta(t-\bar{t})$$

$$\xi \in C^{-1-\frac{d}{2}-\varepsilon} \overset{\text{Schauder}}{\Longrightarrow} u \in C^{1-\frac{d}{2}-\varepsilon}$$

- KPZ ($d=1$)

$$\partial_t h = \Delta h + (\partial_x h)^2 + \xi$$

- Dynamical Φ^4 ($d = 2,3$)

$$\partial_t \phi = \Delta \phi - \phi^3 + \xi$$

- Parabolic Anderson model ($d=2$)

$$\partial_t u = \Delta u + f(u)\zeta \qquad (\zeta \in C^{-\frac{d}{2}-\varepsilon})$$

Dynamical Sine-Gordon Equation

Space dimension $d = 2$. Equation depends on parameter $\beta > 0$.

$$\partial_t u = \frac{1}{2}\Delta u + \zeta \sin(\beta u) + \xi$$

For the linear equation

$$\partial_t \Phi = \frac{1}{2}\Delta \Phi + \xi$$

$$\Phi \in C^{-\varepsilon}$$

Dynamical Sine-Gordon: motivations

Space dimension $d = 2$. Equation depends on parameter $\beta > 0$.

$$\partial_t u = \frac{1}{2}\Delta u + \zeta \sin(\beta u) + \xi$$

Formal invariant measure

$$\exp\left(-\frac{1}{2}\int |\partial u(x)|^2 dx + \zeta \int \cos(\beta u(x)) dx\right) \mathcal{D}u$$

- Dynamical Φ^4 equation

$$\partial_t \phi = \Delta \phi - \lambda \phi^3 + \xi$$

  has formal invariant measure

$$e^{-\frac{1}{2}\int |\partial \phi(x)|^2 dx - \frac{\lambda}{4}\int \phi(x)^4 dx} \mathcal{D}\phi$$

Physical motivation

- The Sine-Gordon field theory

$$\mathbf{P}\left(u\right) \propto e^{-\frac{1}{2}\int |\partial u(x)|^2 dx + \zeta \int \cos(\beta u(x)) dx}\, \mathcal{D}u$$

- 2D rotor model

$$\mathbf{P}\left(\{S_i\}_{i\in\mathbf{Z}^2}\right) \propto e^{\beta^2 \sum_{i\sim j} S_i\cdot S_j} \qquad (S_i \in \mathbf{S}^1)$$

- 2D Coulomb system: each charge $(x,\sigma) \in \mathbf{R}^2 \times \{\pm 1\}$,

$$\mathbf{P}\left(\{(x_1,\sigma_1),\ldots,(x_n,\sigma_n)\}\right) \propto \frac{\zeta^n}{n!} e^{-\beta^2 \sum_{i,j} \sigma_i\sigma_j V(x_i - x_j)}$$

$$V(x-y) \sim -\frac{1}{2\pi} \ln |x-y|$$

Kosterlitz-Thouless transition at $\beta^2 = 8\pi$.

- small β: Gaussian behavior at small scale
- large β: Gaussian behavior at large scale

Dynamical Sine-Gordon Equation

Stochastic PDE for $u(t,x)$ ($x \in \mathbb{T}^2$):

$$\partial_t u = \frac{1}{2}\Delta u + \zeta \sin(\beta u) + \xi$$

where ξ is the space-time white noise.

Is the initial value problem well-posed?

Background:

- Formally, the Sine-Gordon measure is an invariant measure of the above dynamics.
- Dynamic of solid-vapour interfaces at the roughening transition (Chui-Weeks PRL'78, Neudecker Zeit.Phys'83)
- Crystal surface fluctuations in equilibrium (Kahng-Park Phys.Rev.B'93-'94)

Dynamical Sine-Gordon Equation

Theorem

If $\beta^2 < 16\pi/3$, then "a renormalized version" of the equation

$$\partial_t u = \frac{1}{2}\Delta u + \zeta \sin(\beta u) + \xi$$

is locally well-posed for any initial data $u^{(0)} \in C^\eta(\mathbb{T}^2)$ with $\eta > -\frac{1}{3}$.

- Well-posedness is expected for all $\beta^2 < 8\pi$, but we have not proved it.
- The same result holds with some generalizations:

$$\partial_t u = \frac{1}{2}\Delta u + \sum_{k=1}^{M} \zeta_k \sin(k\beta u + \theta_k) + \xi$$

Methods of the proof

- Da Prato - Debussche method applies after some extra work for $\beta^2 < 4\pi$.
  - Also applies to: Dynamical Φ^4 in 2D (Da Prato-Debussche), Dynamical Φ^3 in 3D (E-Jentzen-S)
- Hairer's theory of regularity structures applies for $4\pi \leq \beta^2 < \frac{16\pi}{3}$ (in principle should work for $\beta^2 < 8\pi$ but has not been done)
  - Also applies to: Dynamical Φ^4 in 3D, KPZ in 1D, Parabolic Anderson model in 2D, and many other subcritical (super-renormalizable) equations (Hairer)

The main difficulty

Stochastic PDE for $u(t,x)$ ($x \in \mathbb{T}^2$):

$$\partial_t u = \frac{1}{2}\Delta u + \zeta \sin(\beta u) + \xi$$

- The solution to the linear equation

  $$\partial_t u = \frac{1}{2}\Delta u + \xi$$

  is a.s. a distribution – $\sin(\beta u)$ is meaningless!
- Replace ξ by smooth noise ξ_ϵ

  $$\partial_t u_\epsilon = \frac{1}{2}\Delta u_\epsilon + \zeta \sin(\beta u_\epsilon) + \xi_\epsilon$$

  where $\xi_\epsilon \to \xi$ as $\epsilon \to 0$. Then u_ϵ does not converge to any nontrivial limit as $\epsilon \to 0$.

Da Prato - Debussche method

Let ξ_ϵ be smooth noise and $\xi_\epsilon \to \xi$. Write $u_\epsilon = \Phi_\epsilon + v_\epsilon$ where

$$\partial_t u_\epsilon = \frac{1}{2}\Delta u_\epsilon + \zeta \sin(\beta u_\epsilon) + \xi_\epsilon$$

$$\partial_t \Phi_\epsilon = \frac{1}{2}\Delta \Phi_\epsilon + \xi_\epsilon$$

Then v_ϵ satisfies

$$\partial_t v_\epsilon = \frac{1}{2}\Delta v_\epsilon + \zeta\Big(\sin(\beta\Phi_\epsilon)\cos(\beta v_\epsilon) + \cos(\beta\Phi_\epsilon)\sin(\beta v_\epsilon)\Big)$$

New random input: $\exp(i\beta\Phi_\epsilon) = \cos(\beta\Phi_\epsilon) + i\,\sin(\beta\Phi_\epsilon)$.

- Parabolic Anderson $\partial_t u = \Delta u +$ "Gaussian noise"·f(u)

A general PDE argument

Let f be a smooth function, and $\Psi \in C^\gamma$ with $\gamma > -1$,

$$\partial_t v = \frac{1}{2}\Delta v + \Psi f(v)$$

Let $K = (\partial_t - \frac{1}{2}\Delta)^{-1}$ be the heat kernel. Then:

$$\mathcal{M} : \; v \mapsto K * (\Psi f(v))$$

defines a map from C^1 to C^1 itself:

- Young's Thm: $g \in C^\alpha, h \in C^\beta$, $\alpha + \beta > 0 \Rightarrow gh \in C^{\min(\alpha,\beta)}$

$$\Psi f(v) \in C^\gamma \qquad (\gamma > -1)$$

- Schauder's estimate: "heat kernel gives two more regularities"

$$\mathcal{M}v \in C^{\gamma+2} \subseteq C^1$$

Da Prato - Debussche method

- Back to our equation

$$\partial_t v_\epsilon = \frac{1}{2}\Delta v_\epsilon + \zeta\Big(\sin(\beta\Phi_\epsilon)\cos(\beta v_\epsilon) + \cos(\beta\Phi_\epsilon)\sin(\beta v_\epsilon)\Big)$$

Q: Does $\exp(i\beta\Phi_\epsilon)$ converge to a limit in C^γ with $\gamma > -1$?

- $\varphi_{z_0}^\lambda$: rescaled test function centered at z_0 $\left(\varphi_{z_0}^\lambda(z) = \lambda^{-4}\varphi(\frac{z-z_0}{\lambda})\right)$
Kolmogorov: For random process f_ϵ, suppose $\forall z_0 \in \mathbb{R}^{2+1}$

$$\mathbb{E}\left|\langle f_\epsilon, \varphi_{z_0}^\lambda\rangle\right|^p \lesssim \lambda^{\gamma p} \qquad \lambda^{-\gamma p}\,\mathbb{E}\left|\langle f_\epsilon - f, \varphi_{z_0}^\lambda\rangle\right|^p \to 0$$

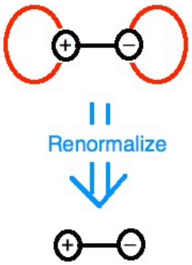
for $\forall p \geq 1$, uniformly in λ, φ. Then, $f_\epsilon \to f \in C^\gamma$.

Da Prato - Debussche method: bound on second moment

Back to the question $e^{i\beta\Phi_\epsilon} \to ?$ in C^γ with $\gamma > -1$

- Want: $\mathbb{E}\left[\left|\int \varphi_0^\lambda(z) e^{i\beta\Phi_\epsilon(z)} dz\right|^2\right] \lesssim \lambda^{2\gamma}$
- By Fourier transform

$$\mathbb{E}\left[e^{i\beta\Phi_\epsilon(z)} e^{-i\beta\Phi_\epsilon(z')}\right]$$
$$= \exp\left(-\frac{\beta^2}{2}\mathbb{E}\left[(\Phi_\epsilon(z) - \Phi_\epsilon(z'))^2\right]\right)$$



- $\mathbb{E}\left[\Phi_\epsilon(z)\Phi_\epsilon(z')\right] \sim -\frac{1}{2\pi}\log(|z - z'| + \epsilon)$
- $\exp\left(-\frac{\beta^2}{2}\mathbb{E}\left[\Phi_\epsilon(z)^2\right]\right) \sim \epsilon^{\beta^2/(4\pi)} \to 0 \qquad (\epsilon \to 0)$

To obtain a nontrival limit, consider the renormalized object

$$\Psi_\epsilon = \epsilon^{-\beta^2/(4\pi)}\, e^{i\beta\Phi_\epsilon}$$

Da Prato - Debussche method: bound on second moment

- Second moment bound

$$\mathbb{E}\Big[\Big|\int \varphi_0^\lambda(z)\Psi_\epsilon(z)\,dz\Big|^2\Big] \lesssim \iint |z-z'|^{-\beta^2/(2\pi)}\,\varphi_0^\lambda(z)\varphi_0^\lambda(z')\,dzdz'$$
$$\lesssim \lambda^{-\beta^2/(2\pi)}$$

(integrable when $\beta^2 < 8\pi$)

- Indicates $\Psi_\epsilon(z) \to \Psi(z) \in C^{-\beta^2/(4\pi)}$.
  Therefore, when $\beta^2 < 4\pi$, we have $\Psi(z) \in C^\gamma$ with $\gamma > -1$.
- Replace $e^{i\beta\Phi_\epsilon}$ by $\Psi_\epsilon \iff$ renormalize the original equation

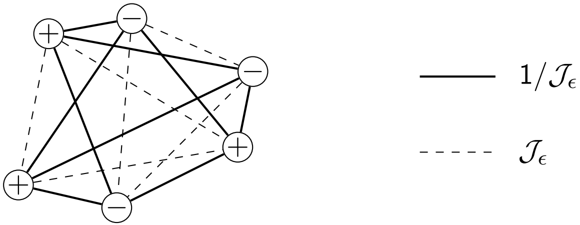
$$\partial_t u_\epsilon = \frac{1}{2}\Delta u_\epsilon + \zeta\,\epsilon^{-\beta^2/(4\pi)}\,\sin(\beta u_\epsilon) + \xi_\epsilon$$

Da Prato - Debussche method: bound on higher moments

However, second moment bound is not sufficient!
Higher order correlations look like:

$$\mathbb{E}\left[\Psi_\epsilon(z_1^+)\cdots\Psi_\epsilon(z_m^+)\,\bar\Psi_\epsilon(z_1^-)\cdots\bar\Psi_\epsilon(z_m^-)\right]$$
$$= \frac{\prod_{i\neq j}\mathcal{J}_\epsilon(z_i^+ - z_j^+)\,\prod_{i\neq j}\mathcal{J}_\epsilon(z_i^- - z_j^-)}{\prod_{i,j}\mathcal{J}_\epsilon(z_i^+ - z_j^-)}$$

$$\mathcal{J}_\epsilon(z - z') \sim |z - z'|^{\beta^2/(2\pi)}$$

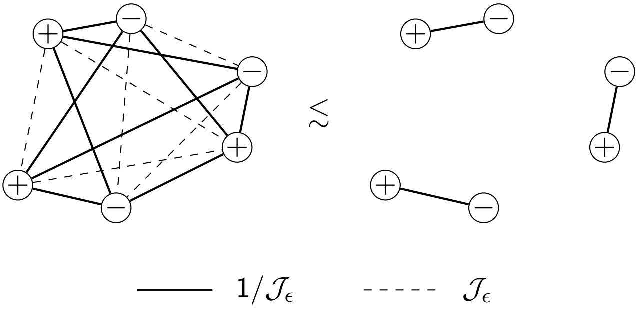


Da Prato - Debussche method: bound on higher moments

We can show that

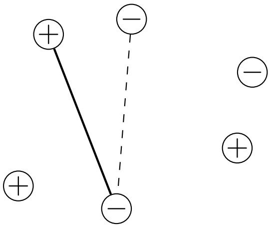
$$\frac{\prod_{i\neq j}\mathcal{J}_\epsilon(z_i^+ - z_j^+)\prod_{i\neq j}\mathcal{J}_\epsilon(z_i^- - z_j^-)}{\prod_{i,j}\mathcal{J}_\epsilon(z_i^+ - z_j^-)} \lesssim \frac{1}{\prod_{(i,j)\in\mathcal{S}}\mathcal{J}_\epsilon(z_i^+ - z_j^-)}$$

where $\mathcal{S}$ is a pairing of positive-negative charges.



Da Prato - Debussche method: bound on higher moments

- A cancellation occurs when two opposite charges are close, while a third charge is far away.



- Motivated by this - Multiscale analysis

Conclusion: For all $\beta^2 < 8\pi$, $\Psi_\epsilon \to \Psi \in C^{-\beta^2/(4\pi)}$.
Therefore if $\beta^2 < 4\pi$, $\Psi \in C^\gamma$ with $\gamma > -1$, and

$$\partial_t v = \frac{1}{2}\Delta v + \zeta\Big(Im(\Psi)\cos(\beta v) + Re(\Psi)\sin(\beta v)\Big)$$

is well-posed.

Theory of regularity structure and $\beta^2 \geq 4\pi$

If $\Psi \in C^\gamma$ with $\gamma \leq -1$,

$$\partial_t v = \frac{1}{2}\Delta v + \Psi f(v)$$

"Young's theorem - Schauder's estimate" argument breaks down.

A Stochastic ODE example:

$$dX_t = f(X_t)\, dB_t$$

- If $dB \in C^\gamma(\mathbb{R}_+)$ with $\gamma > -\frac{1}{2}$, Young's theorem applies for $X \in C^{\frac{1}{2}}$; Fix Point Argument in $C^{\frac{1}{2}}$
- For B Brownian motion, $dB \in C^\gamma(\mathbb{R}_+)$ with $\gamma < -\frac{1}{2}$; the argument breaks down - one needs extra information to define the product $f(X_t)dB_t$.
- Extra information given by rough path theory.

Theory of regularity structure and $\beta^2 \geq 4\pi$

For smooth function f

$$dX_t = f(X_t)\, dB_t$$

- X locally "looks like" Brownian motion (So does $f(X)$.)

$$X_t - X_{t_0} = g_{t_0} \cdot (B_t - B_{t_0}) + \text{sth. smoother}$$

- Only need to define one product $B\, dB$.

Theory of regularity structure and $\beta^2 \geq 4\pi$

$$dX_t = f(X_t)\, dB_t \qquad \partial_t v = \frac{1}{2}\Delta v + f(v)\,\Psi$$

$$dB \in C^{-1/2-\varepsilon} \qquad \Psi \in C^{-1-\varepsilon}$$

The solutions, at small scale, behave like

$$X \sim B = \int_0^t dB_s \qquad \text{analogous} \qquad v \sim K * \Psi$$

where $K = (\partial_t - \frac{1}{2}\Delta)^{-1}$.

- Only need to define one product $\Psi\,(K * \Psi)$.
- A whole theory (Theory of regularity structures recently developed by Martin Hairer) behind this "analogy".

Regularity structure and $\beta^2 \geq 4\pi$: moments of $\Psi\,(K * \Psi)$

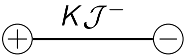
- First moment:

$$\mathbb{E}\left[\Psi(z)\int_{\mathbb{R}^{2+1}} K(z-w)\bar{\Psi}(w)dw\right] = \int_{\mathbb{R}^{2+1}} K(z-w)\mathcal{J}(z-w)^{-1}dw$$

For $\beta^2 \geq 4\pi$ non-integrable singularity at $z \approx w$, since

$$K(z-w) \sim |z-w|^{-2}$$

$$\mathcal{J}(z-w)^{-1} \sim |z-w|^{-\beta^2/(2\pi)}$$



- Suggest renormalization: define the product to be

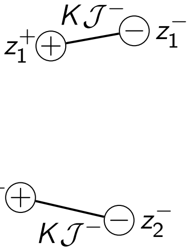
$$\Psi(K * \Psi) \stackrel{\text{def}}{=} \lim_{\epsilon \to 0}\left[\Psi_\epsilon(K * \bar{\Psi}_\epsilon) - \int K\mathcal{J}_\epsilon^{-1}\right]$$

Regularity structure and $\beta^2 \geq 4\pi$: moments of $\Psi\,(K * \Psi)$

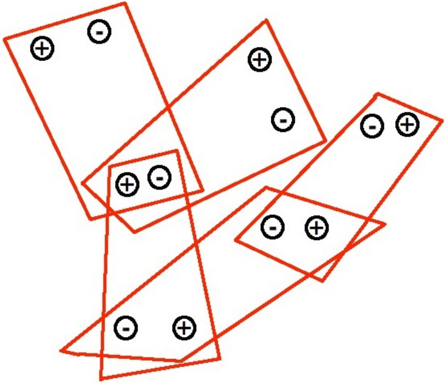
Second moment:

$$\int_{\mathbb{R}^{2+1}} \int_{\mathbb{R}^{2+1}} K(z_1^+ - z_1^-) K(z_2^+ - z_2^-) \frac{1}{\mathcal{J}(z_1^+ - z_1^-)\mathcal{J}(z_2^+ - z_2^-)} \times \left(\frac{\mathcal{J}(z_1^+ - z_2^+)\mathcal{J}(z_1^- - z_2^-)}{\mathcal{J}(z_1^+ - z_2^-)\mathcal{J}(z_1^- - z_2^+)} - 1 \right) dz_1^- \, dz_2^-$$

- Singularities: $z_1^+ \approx z_1^-$ or $z_2^+ \approx z_2^-$
- But in either of the two cases, the second line vanishes.



Theory of regularity structure and $\beta^2 \geq 4\pi$: moments of $\Psi\,(K * \Psi)$



- Renormalization of $\Psi(K * \bar{\Psi}) \Rightarrow$ change of the equation

$$\partial_t v_\epsilon = \frac{1}{2}\Delta v_\epsilon + \zeta\Big(Im(\Psi_\epsilon)\cos(\beta v_\epsilon) - \zeta C_\epsilon \cos(\beta v_\epsilon)\cos'(\beta v_\epsilon)\Big)$$
$$+ \zeta\Big(Re(\Psi_\epsilon)\sin(\beta v_\epsilon) - \zeta C_\epsilon \sin(\beta v_\epsilon)\sin'(\beta v_\epsilon)\Big)$$

Larger values of β^2

- At $\beta^2 = 4\pi$, $\Psi \in C^{-1}$ - need $\Psi \cdot K\Psi$
- At $\beta^2 = 16\pi/3$, $\Psi \in C^{-4/3}$ - need $\Psi \cdot K(\Psi \cdot K\Psi)$
- At $\beta^2 = 6\pi$, $\Psi \in C^{-3/2}$ - need $\Psi \cdot K(\Psi \cdot K(\Psi \cdot K\Psi))$
- ......

Infinite thresholds:

$$0 < 4\pi < \frac{16\pi}{3} < 6\pi < \ldots < \frac{8(n-1)}{n}\pi < \ldots \to 8\pi$$