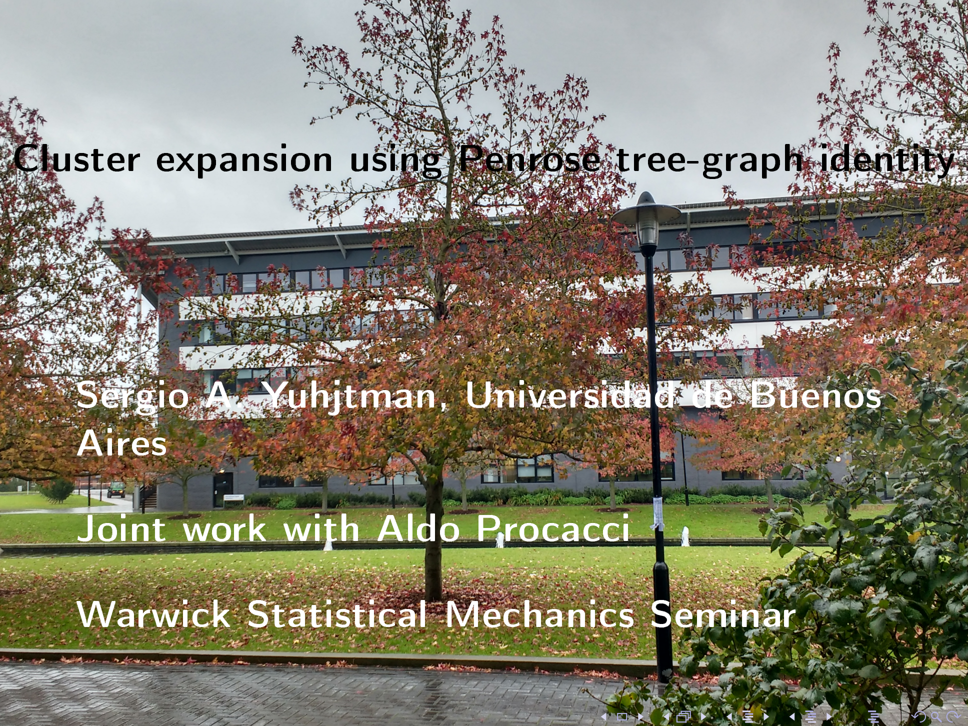

Cluster expansion using Penrose tree-graph identity

Sergio A. Yuhjtman, Universidad de Buenos Aires

Joint work with Aldo Procacci

Warwick Statistical Mechanics Seminar

Statistical mechanics:

systems formed by large collection of microscopic objects, with some rule of interaction.

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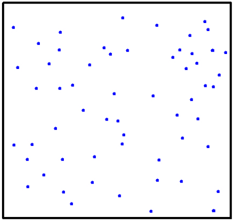

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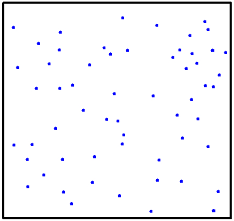

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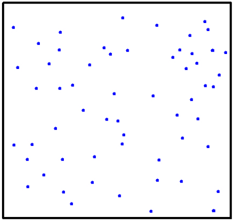

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divided by $$Z(\Lambda) = \sum_{m \geq 0} \int_{\Lambda^m} d\vec{y} \; \frac{z^m}{m!} e^{-\beta \sum_{i,j} V(y_i - y_j)}$$

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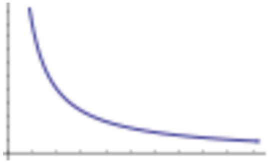

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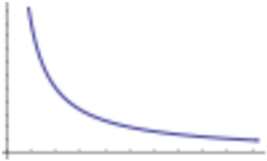

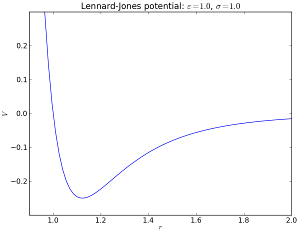

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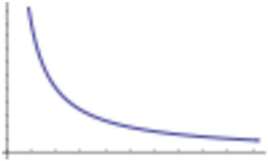

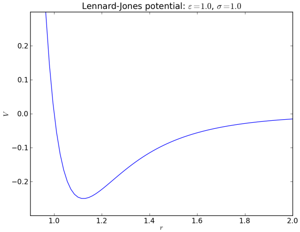

Low interactions: β or z small

Partition function:

$$\Lambda \subset \mathbb{R}^d, \qquad Z(\Lambda) = \sum_{m \geq 0} \frac{z^m}{m!} \int_{\Lambda^m} d\vec{y}\; e^{-\beta \sum_{i,j} V(y_i - y_j)}$$

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Uniform convergence radius for $\Lambda \to \mathbb{R}^d$?

1963: Penrose, Ruelle First positive answer.

Later: Better bounds but more hypothesis, using trees & graphs.

Here: Better bounds, no extra hypothesis.

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Because $\alpha(g_1)\alpha(g_2) = \alpha(g_1 \cup g_2)$ and

the series of *Exp* combines connected components in every possible way.

We need to bound

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$$\sum_{g\in\mathcal{C}_X}\prod_{s\in E_g}(e^{-V_s}-1)=\sum_{\tau\in\mathcal{T}_X}\Big(\prod_{s\in\tau}(e^{-V_s}-1)\prod_{s\in M(\tau)\setminus\tau}e^{-V_s}\Big)$$

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proof:

$$\prod_{s\in M(\tau)\backslash\tau}\Big((e^{-V_s}-1)+1\Big)=\sum_{g'\subset M(\tau)\backslash\tau}\Big(\prod_{s\in g'}(e^{-V_s}-1)\Big)$$

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Proposition: $T^{-1}(\tau) = [\tau, M(\tau)]$

where M is easy to describe.

$\mathcal{C}_X \xrightarrow{T} \mathcal{T}_X$ minimum spanning tree map.

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If $\tau \subset g \subset M(\tau)$ then $T(g) = \tau$.

Penrose tree-graph identity for minimum spanning trees wrt V.

$$\sum_{g \in \mathcal{C}_X} \prod_{s \in g} (e^{-V_s} - 1) = \sum_{\tau \in \mathcal{T}_X} \prod_{s \in \tau} (e^{-V_s} - 1) \prod_{s \in M(\tau) \setminus \tau} e^{-V_s}$$

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Now we bound:

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Further trick

$$= \sum_{\tau \in \mathcal{T}_X} \prod_{s \in \tau} (1 - e^{-|V_s|}) \prod_{s \in M(\tau) \setminus \tau^+} e^{-V_s}$$

Stability hypothesis

the potential V has to satisfy

$$\sum_{\{x,y\}\in X^{(2)}} V(x-y) \geq -B|X|$$

for any finite configuration $X \subset \mathbb{R}^d$, for some constant $B \geq 0$.

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If $V \geq 0$ ✓

If not, negative part has to be controlled.

$$\sum_{\tau \in \mathcal{T}_X} \left(\prod_{s \in \tau} (1 - e^{-|V_s|}) \prod_{s \in M(\tau) \setminus \tau^+} e^{-V_s} \right)$$

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Claim: $\displaystyle\sum_{s \in M(\tau) \setminus \tau^+} V_s \geq -B|X|$

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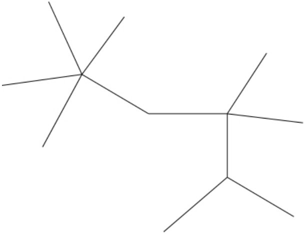
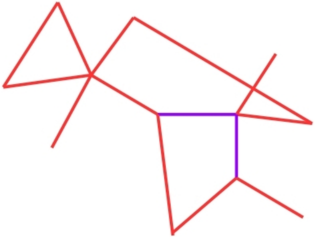
<u>Claim:</u> $\sum_{s \in M(\tau) \setminus \tau^+} V_s \geq -B|X|$

(as if the graph $M(\tau) \setminus \tau^+$ was the complete graph)

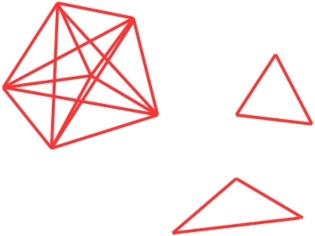
Tree: τ



$\sum_{s \in M(\tau) \setminus \tau^+} V_s \geq$



$\geq \sum_{cc} \sum_s V_s \geq -B|X|$



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For $|X| = n$,

$$\int (\ldots) \leq z^n \; \frac{n^{n-2}}{n!} \Big(\int_{\mathbb{R}^d} (1 - e^{-\beta|V(x)|}) dx \Big)^{n-1} e^{\beta B n} |D|$$

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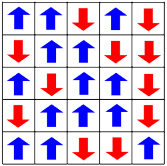
$$R \geq \left(e^{\beta B + 1} \int_{\mathbb{R}^d} (1 - e^{-\beta|V|}) \right)^{-1}$$

Ising Model



Energy of a configuration: $$H(\sigma) = \sum_{\substack{i,j\in\Lambda \\ i\sim j}} |\sigma_i - \sigma_j|$$

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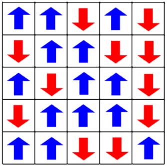

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$$p_\Lambda(\sigma) = \frac{e^{-\beta H(\sigma)}}{\sum_{\sigma'} e^{-\beta H(\sigma')}} \qquad \text{(Boltzmann-Gibbs)}$$

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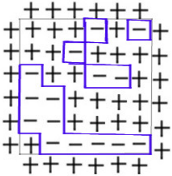
(so if $d \geq 2$, we have a *phase transition*)

With cluster expansion we can understand both statements, and obtain explicit formulas.

For β large:



Ising configuration = polymer configuration.

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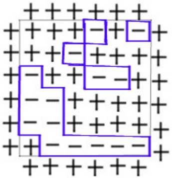

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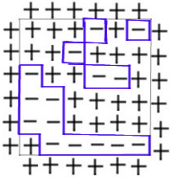

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Polymer activity: $e^{-\beta \text{ length}}$

Interaction: contact forbidden.

β small: polymers are defined in a different way.

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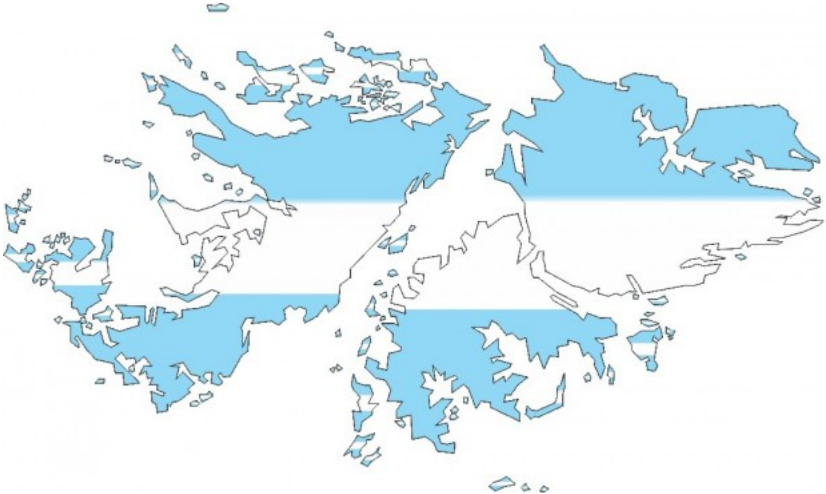
$$\log Z(\Lambda) = \int_{\Gamma_\Lambda} dX \sum_{g \in \mathcal{C}_X} \prod_{s \in E_g} (e^{-V_s} - 1)$$

The same method works



Thank you!