Assignment 3

Problem 1. Consider the functional $\delta : C([0, 1]) \to \mathbb{R}$ such that $\delta(f) = f(0)$. Find the norm of $\delta$, when $C([0, 1])$ is equipped with the $L^p$ norm, $1 \leq p \leq \infty$.

Problem 2. Let $X$ be a normed linear space. Show the following:
   (a) If $\|x_n - x\| \to 0$, then $x_n \to x$ weakly.
   (b) If $X$ is finite-dimensional, then weak convergence is equivalent to norm convergence.
   (c) If $X$ is infinite-dimensional, then weak convergence does not imply norm convergence. Give an example.
   (d) If $x_n$ converges weakly to both $x$ and $y$, then $x = y$.

Problem 3. Show that $(\ell^p)^* = \ell^q$ with $\frac{1}{p} + \frac{1}{q} = 1$, when $1 \leq p < \infty$. More precisely, show that
   (i) any $a \in \ell^q$ defines a continuous linear functional $f_a$ on $\ell^p$ by $f_a(x) = \sum_n a_n x_n$, $x \in \ell^p$;
   (ii) to any functional $f \in (\ell^p)^*$ there corresponds a sequence $a \in \ell^q$ such that $f = f_a$;
   (iii) the operator norm of $f_a$ is equal to the $\ell^q$ norm of $a$.

Are the $\ell^p$ spaces reflexive?

Problem 4. Let $c \in \ell^\infty$ be the space of convergent sequences, and $c_0$ be the space of sequences that converge to 0. Show that
   (i) $c$ and $c_0$ are Banach spaces;
   (ii) $(c_0)^* = c^* = \ell^1$.

Are $c_0$ and $c$ reflexive?

Problem 5. Prove that $\ell^\infty$ is not separable.