Problem 1. Let $X$ be a separable infinite-dimensional Hilbert space. Show that:
(a) Every orthonormal sequence converges weakly to 0.
(b) The unit sphere $S = \{ x : \|x\| = 1 \}$ is weakly dense in the unit ball $B = \{ x : \|x\| \leq 1 \}$.
(Note: These properties also hold for nonseparable Hilbert spaces.)

Problem 2. Let $X$ and $Y$ be two Hilbert spaces. Define the direct sum of $X$ and $Y$ by
$$X \oplus Y = \{(x, y) : x \in X, y \in Y \},$$
with the inner product
$$(x, y), (x', y')_{X \oplus Y} = (x, x')_X + (y, y')_Y.$$
Show that $X \oplus Y$ is a Hilbert space. Find the orthogonal complement of the subspace $\{(x, 0) : x \in X \}$.

Problem 3. Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ denote the circle, where operations $x + y$ and $x - y$ are taken modulo $2\pi$. Recall that the convolution of two functions is defined by
$$(f \ast g)(x) = \int_{\mathbb{T}} f(x - y)g(y)dy.$$ 
Let $\varphi_n \geq 0$ be a function on $\mathbb{T}$ that satisfies
$$\int_{\mathbb{T}} \varphi_n(x)dx = 1, \quad \text{and} \quad \lim_{n \to \infty} \int_{|x| > \delta} \varphi_n(x)dx = 0$$
for any $\delta > 0$. Show that, for any $f \in C(\mathbb{T})$, $\varphi_n \ast f$ converges uniformly to $f$, i.e. that
$$\lim_{n \to \infty} \sup_{x \in \mathbb{T}} |(\varphi_n \ast f)(x) - f(x)| = 0.$$

Problem 4.
(a) Show that the function $\varphi_n(x) = c_n(1 + \cos x)^n$, where $c_n$ is chosen so that $\int_{\mathbb{T}} \varphi_n = 1$, satisfies the properties of Problem 3.
(b) Check that $\{ e^{ikx} \}_{k \in \mathbb{Z}}$ is an orthonormal set in $L^2(\mathbb{T})$. 


(c) Show that \{e^{ikx}\}_{k \in \mathbb{Z}} is an orthonormal basis for \(L^2(\mathbb{T})\). For this, show that
\[
\varphi_n(x) = \sum_{k=-n}^{n} a_{nk} e^{ikx}, \quad \text{with} \quad a_{nk} = 2^{-n} c_n \left(\frac{2n}{n+k}\right).
\]
Then show that for any \(f \in C(\mathbb{T})\),
\[
(\varphi_n * f)(x) = \sum_{k=-n}^{n} b_k e^{ikx}, \quad \text{with} \quad b_k = a_{nk} \int_{\mathbb{T}} e^{-iky} f(y) dy.
\]
Then use Problem 3 and Proposition 3.7 of the course to conclude that \{e^{ikx}\}_{k \in \mathbb{Z}} is an orthonormal basis.

**Problem 5.** Show that \(\zeta(2) = \sum_{n \geq 1} n^{-2} = \pi^2/6\). Hint: Use the fact that \{e^{ikx}\}_{k \in \mathbb{Z}} is an orthonormal basis for \(L^2(\mathbb{T})\), and that the function \(f(x) = x\) has Fourier coefficients proportional to \(1/n\).