 Assignment 8

**Problem 1.** Let $X$ be a Hilbert space. Show that if $T$ is a bounded, positive definite operator on $X$, then $(X, \langle \cdot, T \cdot \rangle)$ is a Hilbert space iff there exists $c > 0$ such that

$$
(x, Tx) \geq c \|x\|^2
$$

for any $x$. Hint: One direction is easy. For the other direction, consider the inclusion map

$$
\iota : (X, \langle \cdot, \cdot \rangle) \to (X, \langle \cdot, T \cdot \rangle)
$$

$x \mapsto x$,

and use the inverse mapping theorem. (Thanks to Michael Doré for the hint!)

**Problem 2.** Let $T \in \mathcal{B}(X)$, and $\alpha, \beta \in \rho(T)$. Let $R_\alpha = (T - \alpha I)^{-1}$ denote the resolvent.

(a) Show that it satisfies the Hilbert relation (or resolvent equation)

$$
R_\alpha - R_\beta = (\alpha - \beta)R_\alpha R_\beta.
$$

(b) Show that $R_\alpha R_\beta = R_\beta R_\alpha$.

**Problem 3.** (Shift operators) We consider the right and left shift operators on $\ell^2(\mathbb{N})$:

$$
S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots),
$$

$$
T(x_1, x_2, \ldots) = (x_2, x_3, \ldots).
$$

(a) Find $\|S\|$, $\|T\|$, $S^*$, $T^*$, $S^{-1}$, $T^{-1}$.

(b) Find ran $S$, ran $T$, ker $S$, ker $T$, and check that

$$
\text{ran } S = (\ker T)^\perp, \quad \text{ran } T = (\ker S)^\perp.
$$

(c) Find the spectrum of $S$ and $T$.

**Problem 4.** Let $T \in \mathcal{B}(X)$. Show that

(a) If $u_1, \ldots, u_n \in X$ are eigenvectors of $T$ corresponding to distinct eigenvalues, then $\{u_1, \ldots, u_n\}$ forms a linearly independent set.

(b) If $T = T^*$, and $M$ is an invariant subspace (that is, $T(M) \subset M$), then $M^\perp$ is also invariant.

**Problem 5.** The lottery question. Give a correct solution to Michael Doré by Tuesday, and enter the lottery for a bottle wine!

Let $\ell_0$ be the space of all sequences of complex numbers $(x_1, x_2, \ldots)$ with finitely many nonzero entries. Can you find a norm such that $\ell_0$ is complete? If yes, give it. If not, prove there exists none.

(I heard that Baire category theorem might help.)