MA3F4 – Linear Analysis

Assignment 9

Problem 1. Prove Proposition 4.14 (a), (b), (c), and (e).

Problem 2. Prove Proposition 4.14 (d), i.e. if \((T_n)\) is a sequence of compact operators that converges to \(T\) w.r.t. the operator norm, then \(T\) is compact.

Hints:

- A criterion for compactness is that for any bounded sequence \((x_m)\), \((Tx_m)\) contains a convergent subsequence.
- Given a bounded sequence \((x_m)\), use Cantor diagonal process to get a subsequence \((y_m)\) such that \((T_ny_m)\) converges for any fixed \(n\).
- Show that \((Ty_m)\) converges.

Problem 3. Let \(X\) be a separable Hilbert space. An operator \(T : X \to X\) is Hilbert-Schmidt if there exists an orthonormal basis \((e_n)\) such that \(\sum_n \|Te_n\|^2 < \infty\).

(a) Show that Hilbert-Schmidt operators are compact (hence bounded).

We define the norm of a Hilbert-Schmidt operator \(T\) by
\[
\|T\|_{\text{HS}} = \left( \sum_{n \geq 1} \|Te_n\|^2 \right)^{1/2}.
\]

(b) Show that \(\| \cdot \|_{\text{HS}}\) is a norm.
(c) Show that the Hilbert-Schmidt norm does not depend on the choice of the orthonormal basis.

Problem 4. Let \(T\) be the multiplication operator by a function \(g\). That is, we define \(T : L^2(\mathbb{R}) \to L^2(\mathbb{R})\) by
\[
Tf(x) = g(x)f(x),
\]
where \(g\) is a fixed function, that we suppose to be continuous and bounded. Prove that
(a) \(T\) is bounded;
(b) the spectrum is \(\sigma(T) = \{g(x) : x \in \mathbb{R}\}\);
(c) give an example of a continuous, bounded function \(g\) such that \(T\) has eigenvalues;
(d) can you find a function \(g \not\equiv 0\) such that \(T\) is compact?

Problem 5. Let \(K : L^2([0,1]) \to L^2([0,1])\) be the integral operator defined by
\[
Kf(x) = \int_0^x f(y)dy.
\]
(a) Find the adjoint operator \(K^*\).
(b) Show that \(\|K\| = \frac{2}{\pi}\).
(c) Show that $0 \in \sigma_c(K)$.

(d) The lottery question. Give a correct solution to Michael Doré by Tuesday, and enter the lottery for a bottle wine! Show that $\sigma(K) = \sigma_c(K) = \{0\}$.

For (d), you may use the notion of spectral radius. The spectral radius of a bounded operator $T$ is defined by $r(T) = \sup_{\alpha \in \sigma(T)} |\alpha|$. Prove that it satisfies

$$r(T) = \lim_{n \to \infty} \|T^n\|^{1/n}.$$ 

Then show that $r(K) = 0$. 