Assignment 4

Problem 1. Recall that a map between topological spaces is open iff the image of each open set is open. Show that a linear map between normed spaces is open iff the image of the unit ball (around 0) contains a ball around 0.

Problem 2. Let $X, Y, Z$ be normed spaces, and $S : X \to Y$ and $T : Y \to Z$ be operators.

(a) Show that $\|T \circ S\| \leq \|S\| \|T\|$. 
(b) Give an example where $\|T \circ S\| < \|S\| \|T\|$.

Problem 3. Let $X$ be the space of sequences $x = (x_1, x_2, \ldots)$ of complex numbers with finitely many nonzero terms. We consider the norm $\|x\| = \sup_i |x_i|$. Define $T : X \to X$ by

$$Tx = (x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \ldots)$$

Show that $T$ is linear and bounded. Show that $T$ is bijective but that $T^{-1}$ is unbounded. Does this contradict the Inverse Mapping Theorem?

Problem 4. Let $T : X \to Y$ be a bounded operator between Banach spaces $X$ and $Y$. Show that, if $T$ is bijective, there exist constants $c_1$ and $c_2$ such that

$$c_1 \|x\| \leq \|Tx\| \leq c_2 \|x\|$$

for all $x \in X$.

Problem 5. Here is an exercise that belongs more to analysis than to functional analysis, but it is a beautiful application of Baire Category Theorem. Show that there exist continuous functions on $[0, 1]$ that are nowhere differentiable.

To that purpose, introduce the set $A_n$ of functions $f \in C([0, 1], \mathbb{R})$ such that there exists $x_0$ (that depends on $f$) such that $|f(x) - f(x_0)| \leq n|x - x_0|$ for all $x \in [0, 1]$.

(a) Show that $A_n$ is nowhere dense in $C([0, 1])$ (with the sup norm).
(b) Show that if $f$ is differentiable at some $x \in [0, 1]$, then $f \in \bigcup_n A_n$.
(c) Use Baire Theorem to show that $\bigcup_n A_n \not\subseteq C([0, 1])$. 