Problem 1. Let $T \in B(X)$, and $\alpha, \beta \in \rho(T)$. Let $R_\alpha = (T - \alpha \mathbb{I})^{-1}$ denote the resolvent.

(a) Show that it satisfies the Hilbert relation (or resolvent equation)

$$R_\alpha - R_\beta = (\alpha - \beta)R_\alpha R_\beta.$$  

(b) Show that $R_\alpha R_\beta = R_\beta R_\alpha$.

Problem 2. Let $T : D(T) \to X$ be densely-defined, and let $\sigma'(T)$ denote the set of approximate eigenvalues; precisely,

$$\sigma'(T) = \{ \lambda \in \mathbb{C} : \inf_{x \in D(T) \atop \|x\|=1} \| (T - \lambda \mathbb{I})x \| = 0 \}.$$  

(a) Show that, if $T$ is self-adjoint, $\sigma'(T) = \sigma(T)$. (Compare with Prop. 5.12 for bounded operators.)

(b) More generally, show that

$$\sigma_p(T) \cup \sigma_c(T) \subset \sigma'(T) \subset \sigma(T).$$

Problem 3. An example where the domain affects the spectrum. Let $X = \ell^2(\mathbb{N})$, $x = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$, and $M = \text{span}\{x\}$. Let $P$ be the orthogonal projection onto $M$.

(a) Suppose that $D(P) = \ell^2$. Show that $1 \in \sigma_p(P)$.

(b) Suppose that $D(P)$ is the set of elements of $\ell^2$ with finitely many non zero entries. Show that $1 \in \sigma_r(P)$.

Problem 4. Generalisation of the above example. Let $S \subset T$ be densely-defined operators, with $T$ an extension of $S$. Prove that

$$\sigma_p(S) \subset \sigma_p(T);$$

$$\sigma_r(S) \supset \sigma_r(T);$$

$$\sigma_c(S) \subset \sigma_p(T) \cup \sigma_c(T).$$

Problem 5. This question is optional, but it is puzzling.

Let $\ell_0$ be the space of all sequences of complex numbers $(x_1, x_2, \ldots)$ with finitely many nonzero entries. Can you find a norm such that $\ell_0$ is complete? If yes, give it. If not, prove there exists none.

(I heard that Baire category theorem might help.)