Assignment 9

Problem 1. Let $X, Y$ be Hilbert spaces, and $U : X \rightarrow Y$ a unitary map. Let $T : D(T) \rightarrow X$ be a densely-defined operator in $X$. We define the operator $\tilde{T}$ in $Y$ by

$$D(\tilde{T}) = UD(T) = \{Ux : x \in D(T)\};$$

$$\tilde{T} = U^{-1}TU.$$

The goal of this exercise is to observe that $T$ and $\tilde{T}$ are closely related. Precisely,

(a) $D(\tilde{T})$ is dense in $Y$, and $D(\tilde{T}) = Y$ iff $D(T) = X$.

(b) $\|\tilde{T}\| = \|T\|$ (both may be infinite).

(c) $\tilde{T}$ is closed iff $T$ is closed. Also, $U^{-1}TU = \tilde{T}$.

(d) $D(\tilde{T}^*) = UD(T^*)$ and $U^{-1}T^*U = \tilde{T}^*$.

(e) $\tilde{T}$ is symmetric iff $T$ is symmetric, and $T$ is self-adjoint iff $T$ is self-adjoint.

Problem 2. We have seen in Assignment 5 that the Fourier functions $e_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}$ form an orthonormal basis for $L^2(\mathbb{T})$, where $\mathbb{T}$ is the one-dimensional torus $[0, 2\pi]$. Then any $f \in L^2(\mathbb{T})$ can be written as

$$f = \sum_{k \in \mathbb{Z}} \hat{f}_k e_k.$$ 

The Fourier coefficients $\hat{f}_k$ are uniquely determined (actually, $\hat{f}_k = (e_k, f)$) and they satisfy $\sum_k |\hat{f}_k|^2 = \|f\|^2 < \infty$. Thus the Fourier transform can be viewed as a map

$$U : L^2(\mathbb{T}) \rightarrow \ell^2(\mathbb{Z}),$$

$$f \mapsto Uf = (\hat{f}_k).$$

(a) Check that $U$ is a unitary map.

(b) If $f \in C^1(\mathbb{T})$, check that

$$\langle \hat{f}' \rangle_k = ik \hat{f}_k.$$ 

Let $D = -\frac{d^2}{dx^2}$ the differential operator with domain $D(D) = C^1(\mathbb{T})$, and let $M$ be the multiplication operator in $\ell^2(\mathbb{Z})$, $M(a_k) = (ka_k)$, with domain $UD(D)$. Show that

(c) $M = U^{-1}DU$.

(d) $D$ and $M$ are symmetric.

(e) Describe the closure $\overline{M}$, and check that $\overline{M}$ is self-adjoint.

(f) Conclude that $D$ and $M$ are both essentially self-adjoint.