Assignment 7 — Solutions

Problem 2.
(a) We have
\[ \|T f\|^2 = \int |g(x)|^2 |f(x)|^2 \, dx \leq \sup |g(x)|^2 \|f\|^2, \]
so that \( \|T\| \leq \sup |g(x)| \). It will actually follow from (b) that \( \|T\| = \sup |g(x)| \).
(b) Let us first see that if \( \lambda \not\in \{g(x) : x \in X\} \), we have \( \lambda \in \rho(T) \). Consider the operator \( S \) defined by
\[ Sf(x) = \frac{1}{g(x) - \lambda} f(x). \]
It is bounded since \( \|S\| \leq \sup \frac{1}{|g(x) - \lambda|} < \infty \). One also sees that
\[ S(T - \lambda) = (T - \lambda) S = 1. \]
Then \( T - \lambda \) has a bounded inverse, so that \( \lambda \in \rho(T) \).
We prove now that if \( \lambda \in \{g(x) : x \in X\} \), we have \( \lambda \in \sigma(T) \). Since the spectrum of a bounded operator is closed (Corollary 5.8), and together with the result that we just proved, we obtain that \( \{g(x) : x \in X\} = \sigma(T) \).
We use the following property. For any bounded operator,
\[ \inf_{\|x\|=1} \|(T - \lambda)x\| = 0 \implies \lambda \in \sigma(T). \]
We actually proved in the course (Proposition 5.12) that the two properties above are equivalent for self-adjoint operators. But the implication holds for general bounded operators.
Let \( x_0 \) such that \( g(x_0) = \lambda \), and consider the functions \( f_n \) defined by
\[ f_n(x) = \begin{cases} \sqrt{n} & \text{if } |x - x_0| < \frac{1}{2n}, \\ 0 & \text{otherwise.} \end{cases} \]
Then \( \|f_n\| = 1 \), and
\[ \|(T - \lambda)f_n\|^2 = n \int_{x_0 - \frac{1}{2n}}^{x_0 + \frac{1}{2n}} |g(x) - \lambda|^2 \, dx \leq \sup_{|x - x_0| < \frac{1}{2n}} |g(x) - \lambda|^2. \]
The latter goes to 0 as \( n \to \infty \) since \( g \) is continuous. Then \( \lambda \in \sigma(T) \).
(c) A trivial example: \( g(x) \equiv 1 \), so that \( T \) is the identity and 1 is the (unique) eigenvalue. More generally, \( \lambda \) is an eigenvalue iff \( g^{-1}(\{\lambda\}) \) has nonzero Lebesgue measure, in which case any function whose support is in \( g^{-1}(\{\lambda\}) \) is an eigenvector.
(d) \( T \) is not compact, unless \( g \equiv 0 \). Suppose that \( |g(x)| > \varepsilon \) for all \( x \) in a neighbourhood of \( x_0 \), and consider the functions \( f_n \) defined above. We show that \( (Tf_n) \)
has no converging subsequence. For $m < n$ large enough, we have

$$
\|Tf_m - Tf_n\|^2 = \int_{|x| < \frac{1}{2m}} |g(x)|^2(n - m)dx + \int_{\frac{1}{2m} < |x| < \frac{1}{2n}} |g(x)|^2mdx
\geq \varepsilon^2 \left[ \frac{1}{n}(n - m) + \left( \frac{1}{m} - \frac{1}{n} \right)m \right]
= (1 - \frac{m}{n})2\varepsilon^2.
$$

Suppose that $(Tf_{n_k})$ is a converging subsequence. Then it is Cauchy and $\|Tf_{n_k} - Tf_{n_k'}\|$ is as small as we want for $k, k'$ large enough. But the above bound implies that

$$
\|Tf_{n_k} - Tf_{n_k'}\| > (1 - \frac{n_k}{n_k'})2\varepsilon^2,
$$

which is not small if $n_k' > 2n_k$. Then $(Tf_{n_k})$ does not converge.