Assignment 6 — Analysis I — Daniel Ueltschi

Due: Monday, 16 November, 3:00pm.

Problem 1. Find $\sup A$ and $\inf A$ where A is the set defined by

a. $A = \{x \in R : x^4 < 16\}$
b. $A = \{x \in R : x^4 \leq 16\}$
c. $A = \{x \in Q : x = \frac{1}{n} + 2^{-n}, n \in N\}$
d. $A = \{x \in R : |x| < 3 \text{ and } x^2 > 2\}$
e. $A = \{x = \frac{m}{2^n+1},\ \ m, n \in N\}$
f. $A = \{x \in Q : 0 < \sqrt{x} < 3\}$

Problem 2. Prove the following theorem (Cauchy). Suppose that $(a_n) \to a$. Then the sequence (b_n) defined by

$$b_n = \frac{a_1 + a_2 + ... + a_n}{n}$$

is convergent and $(b_n) \to a$.

Problem 3. Find the limit of the sequence (a_n) defined by

$$a_n = \frac{1 + \sqrt[2]{2} + \sqrt[3]{3} + ... + \sqrt[n]{n}}{n}.$$

(The result of the previous exercise may help!)

Problem 4. A sequence (a_n) is known to have a finite limit superior and a finite limit inferior

(a) Might it have un upper bound?
(b) Must it have an upper bound?
(c) Might it be convergent?
(d) Must it be convergent?

Problem 5. Find a sequence (a_n) such that

a. $\liminf(a_n) = \limsup(a_n)$
b. $\liminf(a_n) = 4 \limsup(a_n)$
c. $1 + \liminf(a_n) = \limsup(a_n)$
d. $\liminf(a_n) = 2 \limsup(a_n) + 1$

Problem 6. Let (a_n) and (b_n) be two bounded sequences of natural numbers. Prove the following:

(a) If $c \geq 0$ then $\limsup(ca_n) = c \limsup(a_n)$.

(b) $\limsup(a_n + b_n) \leq \limsup(a_n) + \limsup(b_n)$.

Notice that the following properties also hold, and that they can be proved in a similar way: (i) $\liminf(ca_n) = c \liminf(a_n)$; (ii) if $c \leq 0$, $\liminf(ca_n) = c \limsup(a_n)$ and $\limsup(ca_n) = c \liminf(a_n)$; (iii) $\liminf(a_n + b_n) \geq \liminf(a_n) + \liminf(b_n)$.

Problem 7. Give an example of a sequence (a_n) which is not convergent, but such that $a_{n+1} - a_n \to 0$.