Assignment 6 — Analysis I — Daniel Ueltschi

Due: Monday, 16 November, 3:00pm.

Problem 1. Find \( \sup A \) and \( \inf A \) where \( A \) is the set defined by

a. \( A = \{ x \in \mathbb{R} : x^4 < 16 \} \)

b. \( A = \{ x \in \mathbb{R} : x^4 \leq 16 \} \)

c. \( A = \{ x \in \mathbb{Q} : x = \frac{1}{n} + 2^{-n}, n \in \mathbb{N} \} \)

d. \( A = \{ x \in \mathbb{R} : |x| < 3 \text{ and } x^2 > 2 \} \)

e. \( A = \{ x = \frac{m}{n+1}, m, n \in \mathbb{N} \} \)

f. \( A = \{ x \in \mathbb{Q} : 0 < \sqrt{x} < 3 \} \)

Problem 2. Prove the following theorem (Cauchy). Suppose that \( (a_n) \rightarrow a \).

Then the sequence \( (b_n) \) defined by

\[
b_n = \frac{a_1 + a_2 + \ldots + a_n}{n}
\]

is convergent and \( (b_n) \rightarrow a \).

Problem 3. Find the limit of the sequence \( (a_n) \) defined by

\[
a_n = \frac{1 + \sqrt[3]{2} + \sqrt[3]{3} + \ldots + \sqrt[n]{n}}{n}
\]

(The result of the previous exercise may help!)

Problem 4. A sequence \( (a_n) \) is known to have a finite limit superior and a finite limit inferior

(a) Might it have an upper bound?

(b) Must it have an upper bound?

(c) Might it be convergent?

(d) Must it be convergent?

Problem 5. Find a sequence \( (a_n) \) such that

a. \( \lim \inf (a_n) = \lim \sup (a_n) \)

b. \( \lim \inf (a_n) = 4 \lim \sup (a_n) \)

c. \( 1 + \lim \inf (a_n) = \lim \sup (a_n) \)

d. \( \lim \inf (a_n) = 2 \lim \sup (a_n) + 1 \)

Problem 6. Let \( (a_n) \) and \( (b_n) \) be two bounded sequences of natural numbers. Prove the following:

(a) If \( c \geq 0 \) then \( \lim \sup (ca_n) = c \lim \sup (a_n) \).

(b) \( \lim \sup (a_n + b_n) \leq \lim \sup (a_n) + \lim \sup (b_n) \).

Notice that the following properties also hold, and that they can be proved in a similar way: (i) \( \lim \inf (ca_n) = c \lim \inf (a_n) \); (ii) if \( c \leq 0 \), \( \lim \inf (ca_n) = c \lim \sup (a_n) \) and \( \lim \sup (ca_n) = c \lim \inf (a_n) \); (iii) \( \lim \inf (a_n + b_n) \geq \lim \inf (a_n) + \lim \inf (b_n) \).

Problem 7. Give an example of a sequence \( (a_n) \) which is not convergent, but such that \( a_{n+1} - a_n \rightarrow 0 \).