Analysis — MA131

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Chapter 1

Inequalities

1.1 What are Inequalities?

An inequality is a statement involving one of the order relationships $<$, $>$, $\leq$, $\geq$. Inequalities can be split into two types:

(i) those whose truth depends on the value of the variables involved, e.g. $x^2 > 4$ is true if and only if $x < -2$ or $x > 2$;

(ii) those which are always true, e.g. $(x-3)^2 + y^2 \geq 0$ is true for all real values of x and y.

We begin here by looking at how to deal with inequalities of the first type. Later on in sections 1.6 and 1.7 we will see some examples of the second type.

With the first type of inequality our task is usually to find the set of values for which the inequality is true; this is called *solving* the inequality. The set we find is called the *solution set* and the numbers in the set are called *solutions*.

The basic statement $x < y$ can be interpreted in two simple ways.

Caution

0 is not a positive number

$$x < y \iff y - x \text{ is positive} \iff \begin{cases} \text{The point representing } x \text{ on the standard number line is to the left of the} \\ \text{point representing } y. \end{cases}$$

It can be shown that with this interpretation we have the following familiar basic rules for manipulating inequalities based on "$<$". Similar definitions and rules apply to $>$, $\leq$, and $\geq$.

Remember these rules

These rules are important. You should know them by heart.

Rule	Example (based on "$<$")
Adding the same number to each side preserves the inequality.	$x < y \iff x + a < y + a$
Multiplying both sides by a positive number preserves the inequality.	If a is positive then $x < y \implies ax < ay$
Multiplying both sides by a negative number reverses the inequality.	If b is negative then $x < y \implies bx > by$
Inequalities of the same type are *transitive*.	$x < y$ and $y < z \implies x < z$.

We *solve* an inequality involving variables by finding all the values of those variables that make the inequality true. Some solutions are difficult to find and not all inequalities have solutions.

Exercise 1

1. Show that $x = 0$ is a solution of $\frac{(x-2)(x-4)}{(x+3)(x-7)} < 0$.
2. Solve the inequalities:
   (a) $x^2 > 4$; (b) $x - 2 \leq 1 + x$; (c) $-2 < 3 - 2x < 2$.
3. Write down an inequality that has no solution.

1.2 Using Graphs

Graphs can often indicate the solutions to an inequality. The use of graphs should be your first method for investigating an inequality.

Exercise 2 Draw graphs to illustrate the solutions of the following inequalities.

1. $x^3 < x$; 2. $1/x < x < 1$.

In the second case you will need to plot the graphs of $y = 1/x$, $y = x$ and $y = 1$.

1.3 Case Analysis

You solve inequalities by using the basic rules given in section 1.1. When solving inequalities which involve products, quotients and modulus signs (more on these later) you often have to consider separate cases. Have a good look at the following examples.

Examples

1. Solve $x^2 < 1$.

   First we notice that $x^2 < 1 \iff x^2 - 1 < 0 \iff (x+1)(x-1) < 0$. We can see at once that there are two possible cases:

   (a) $x + 1 > 0$ and $x - 1 < 0 \iff x > -1$ and $x < 1 \iff -1 < x < 1$;
   (b) $x + 1 < 0$ and $x - 1 > 0 \iff x < -1$ and $x > 1$ Impossible!

   It follows that $-1 < x < 1$.

2. Solve $\frac{1}{x} + \frac{1}{x+1} > 0$.

   To get an idea of the solutions of this inequality it is a good idea to draw graphs of $\frac{1}{x}$ and $\frac{-1}{x+1}$ on the same axis because $\frac{1}{x} + \frac{1}{x+1} > 0 \iff \frac{1}{x} < \frac{-1}{x+1}$. It is useful to note that $\frac{1}{x} + \frac{1}{x+1} = \frac{2x+1}{x(x+1)}$. We look for the points where the denominator changes sign (at $x = -1$ and $x = 0$) and choose our cases accordingly. For the values $x = 0$ or $x = -1$ the inequality is meaningless so we rule these values out straight away.

Products

The product xy of two real numbers is positive if and only if x and y are either *both* positive or *both* negative. Their product is negative if and only if they have opposite signs.

Sign Language

We use the double implication sign ($\iff$) to ensure that we find only the solution set and not some larger set to which it belongs. For instance, suppose we wished to solve $2x < -1$. We could quite correctly write

$$2x < -1 \implies 2x < 0 \implies x < 0,$$

but clearly it is not true that $x < 0 \implies 2x < -1$.

(a) Consider only $x < -1$. In this case x and $x+1$ are negative and $x(x+1)$ is positive. So $\frac{2x+1}{x(x+1)} > 0 \iff 2x+1 > 0 \iff x > -1/2$ which is impossible for this case.

(b) Consider only $-1 < x < 0$. Then $x(x+1)$ is negative so $\frac{2x+1}{x(x+1)} > 0 \iff 2x+1 < 0 \iff x < -1/2$. So we have solutions for the x under consideration exactly when $-1 < x < -1/2$.

(c) Consider only $x > 0$. Then $x(x+1)$ is positive so as in case 1 we require $x > -1/2$. So the solutions for those x under consideration are exactly $x > 0$.

It follows that the solution set is exactly those x such that either$-1 < x < -1/2$ or $x > 0$.

1.4 Taking Powers

Exercise 3 Is the following statement true for all x and y: "If $x < y$ then $x^2 < y^2$"? What about this statement: "If $x^2 < y^2$ then $x < y$"?

You probably suspect that the following is true:

Power Rule
If x and y are *positive* real numbers then, for each natural number n, $x < y$ if and only if $x^n < y^n$.

Example This is another way of saying that if x is positive then the function x^n is strictly increasing. We would like to prove this useful result. Of course we are looking for an arithmetic proof that does not involve plotting graphs of functions but uses only the usual rules of arithmetic. The proof must show both that $x < y \implies x^n < y^n$ *and* that $x^n < y^n \implies x < y$. Notice that these are two *different* statements.

Caution
The Power Rule doesn't work if x or y are negative.

Exercise 4

1. Use mathematical induction to prove that if both x and y are positive then $x < y \implies x^n < y^n$.

2. Now try to prove the *converse*, that if both x and y are positive then $x^n < y^n \implies x < y$.

   If inspiration doesn't strike today, stay tuned during *Foundations*, especially when the word *contrapositive* is mentioned.

Contrapositive
The contrapositive of the statement $p \implies q$ is the statement *not q* $\implies$ *not p*. These are equivalent, but sometimes one is easier to prove than the other.

1.5 Absolute Value (Modulus)

At school you were no doubt on good terms with the modulus, or absolute value, sign and were able to write useful things like $|2| = 2$ and $|-2| = 2$. What you may not have seen written explicitly is the definition of the absolute value, also known as the absolute value function.

Definition

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

Exercise 5

1. Check that this definition works by substituting in a few positive and negative numbers, not to mention zero.
2. Plot a graph of the absolute value function.

Proposition
The following are key properties of the absolute value function $|\cdot|$.

1. $||x|| = |x|$.
2. $|xy| = |x||y|$.
3. $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$.

Proof. It is easy to prove 1. directly from the definition since $|x|$ in the left hand side is always positive. To prove 2. we consider the three cases, where one of x or y is negative and the other is positive, where both are positive, and when both are negative. For example, if both x and y are negative then xy is positive and

$$|xy| = xy = -|x|(-|y|) = |x||y|.$$

The proof of 3. uses the fact that

$$1 = x \cdot \frac{1}{x} \underset{\text{by 2.}}{\Longrightarrow} 1 = |x| \left|\frac{1}{x}\right| \Longrightarrow \frac{1}{|x|} = \left|\frac{1}{x}\right|.$$

Now we can write

$$\left|\frac{x}{y}\right| = |x| \left|\frac{1}{y}\right| = |x| \frac{1}{|y|} = \frac{|x|}{|y|}.$$

■

In Analysis, intervals of the real line are often specified using absolute values. The following result makes this possible:

Theorem *Interval Property*
If x and b are real numbers and $b > 0$, then $|x| < b$ if and only if $-b < x < b$.

Proof. Suppose $|x| < b$. For $x \geq 0$ this means that $x < b$ and for $x < 0$ this means that $-x < b$ which is the same as $x > -b$. Together these prove half the result. Now suppose $-b < x < b$. Then $-b < |x| < b$ if $x \geq 0$ by definition. If $x < 0$ then $-b < -|x| < b$ and it follows again that $-b < |x| < b$. ■

Corollary
If y, a and b are real numbers and $b > 0$, then $|y - a| < b$ if and only if $a - b < y < a + b$.

Proof. Substitute $x = y - a$ in the interval property. ■

This corollary justifies the graphical way of thinking of the modulus sign $|a - b|$ as the distance along the real line between a and b. This can make solving simple inequalities involving the absolute value sign very easy. To solve the inequality $|x - 3| < 1$ you need to find all the values of x that are within distance 1 from the number 3, i.e. the solution set is $2 < x < 4$.

Exercise 6 Solve the inequalities:

1. $|x - 2| > 1$; 2. $|x + 5| < 3$; 3. $|6x - 12| > 3$.

[Hint: don't forget that $|x + 5| = |x - (-5)|$ is just the distance between x and -5 and that $|6x - 12| = 6|x - 2|$ is just six times the distance between x and 2.]

If the above graphical methods fail then expressions involving absolute values can be hard to deal with. Two arithmetic methods are to try to get rid of the modulus signs by Case Analysis or by squaring. We illustrate these methods in the following very simple example.

Squaring
The method of squaring depends upon the equivalence: if $b > a \geq 0$, then

$$a < |x| < b \iff a^2 < x^2 < b^2$$

Example Solve the inequality $|x + 4| < 2$. Squaring:

$$\begin{aligned} 0 \leq |x + 4| < 2 &\iff (x + 4)^2 < 4 \\ &\iff x^2 + 8x + 16 < 4 \\ &\iff x^2 + 8x + 12 < 0 \\ &\iff (x + 2)(x + 6) < 0 \\ &\iff -6 < x < -2. \end{aligned}$$

Case Analysis:

1. Consider $x > -4$. Then $|x + 4| < 2 \iff x + 4 < 2 \iff x < -2$. So solutions for this case are $-4 < x < 2$.

2. Consider $x \leq -4$. Then $|x + 4| < 2 \iff -x - 4 < 2 \iff x > -6$. So solutions for this case are $-6 < x \leq -4$.

So the solution set is $-6 < x < -2$.

Exercise 7 Solve the following inequalities:

1. $|x-1| + |x-2| \geq 5$; 2. $|x-1| \cdot |x+1| > 0$.

1.6 The Triangle Inequality

Here is an essential inequality to put in your mathematical toolkit. It comes in handy in all sorts of places.

Dummy Variables
The Triangle Inequality holds for all values, so you can stick into it any numbers or variables you like. The x and y are just dummies.

The Triangle Inequality
For all real numbers x and y, $|x+y| \leq |x| + |y|$.

Exercise 8

1. Put a variety of numbers into the Triangle Inequality and convince yourself that it really works.
2. Write out the triangle inequality when you take $x = a - b$ and $y = b - c$.
3. Prove the Triangle Inequality.
   [Hint: Square both sides.]

Have A Go
You can also prove the Triangle Inequality by Case Analysis. The proof is longer, but it is a good test of whether you can really handle inequalities.

1.7 Arithmetic and Geometric Means

Given two numbers a and b the arithmetic mean is just the average value that you are used to, namely $(a+b)/2$. Another useful average of two positive values is given by the geometric mean. This is defined to take the value $\sqrt{ab}$.

Exercise 9 Calculate the arithmetic and the geometric mean for the numbers 0, 10 and 1, 9 and 4, 6 and 5, 5.

Exercise 10

1. Show, for positive a and b, that $\frac{a+b}{2} - \sqrt{ab} = \frac{(\sqrt{a}-\sqrt{b})^2}{2}$.
2. Show that the arithmetic mean is always greater than or equal to the geometric mean. When can they be equal?

Definition
Suppose we have a list of n positive numbers $a_1, a_2, ..., a_n$. We can define the arithmetic and geometric means by:

Arithmetic Mean $= \frac{a_1+a_2+...+a_n}{n}$;

Geometric Mean $= \sqrt[n]{a_1 a_2 ... a_n}$.

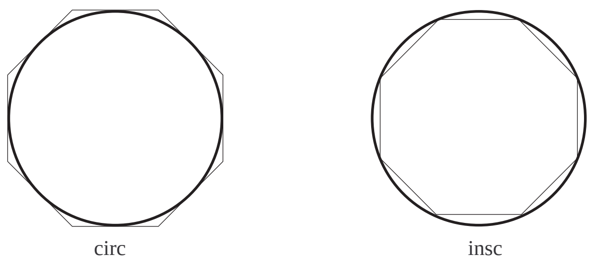
Exercise 11

1. Calculate both means for the numbers 1, 2, 3 and for 2, 4, 8.
2. It is a true inequality that the arithmetic mean is always greater than or equal to the geometric mean. There are many proofs, none of them are completely straightforward. Puzzle for a while to see if you can prove this result. Let your class teacher know if you succeed.

   [Hint: The case $n = 4$ is a good place to start.]

1.8 * Archimedes and π *

Archimedes used the following method for calculating π.



The area of a circle of radius 1 is π. Archimedes calculated the areas

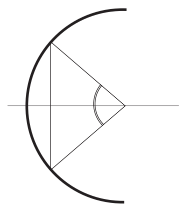
$A_n =$ area of the circumscribed regular polygon with n sides,
$a_n =$ area of the inscribed regular polygon with n sides.

The area of the circle is between that of the inscribed and circumscribed polygons, so $a_n \leq \pi \leq A_n$ for any n. Archimedes claimed that the following two formulae hold:

$$a_{2n} = \sqrt{a_n A_n}; \qquad A_{2n} = \frac{2A_n a_{2n}}{A_n + a_{2n}}.$$

Exercise 12 What are the values of A_4 and a_4? Use Archimedes formulae and a calculator to find $a_8, A_8, a_{16}, A_{16}, a_{32}, A_{32}, a_{64}, A_{64}$. How many digits of π can you be sure of?

To prove the formulae, Archimedes used geometry, but we can find a short proof using trigonometry.



Exercise 13 Use trigonometry to show that $A_n = n\tan\left(\frac{\pi}{n}\right)$ and $a_n = \frac{n}{2}\sin\left(\frac{2\pi}{n}\right)$. Now use the double angle formulae to prove Archimedes' formulae.

Check Your Progress
By the end of this chapter you should be able to:

- Solve inequalities using case analysis and graphs.
- Define the absolute value function and manipulate expressions containing absolute values.
- Interpret the set $\{x : |x - a| < b\}$ as an interval on the real line.
- Prove that if x and y are positive real numbers and n is a natural number then $x \leq y$ iff $x^n \leq y^n$.
- State and prove the Triangle Inequality.

Chapter 2

Sequences I

2.1 Introduction

A sequence is a list of numbers in a definite order so that we know which number is in the first place, which number is in the second place and, for any natural number n, we know which number is in the n^{th} place.

All the sequences in this course are infinite and contain only real numbers. For example:

$$1, 2, 3, 4, 5, \ldots$$
$$-1, 1, -1, 1, -1, \ldots$$
$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$$
$$\sin(1), \sin(2), \sin(3), \sin(4), \ldots$$

In general we denote a sequence by:

$$(a_n) = a_1, a_2, a_3, a_4, \ldots$$

Notice that for each natural number, n, there is a term a_n in the sequence; thus a sequence can be thought of as a function $a : \mathbb{N} \to \mathbb{R}$ given by $a(n) = a_n$. Sequences, like many functions, can be plotted on a graph. Let's denote the first three sequences above by (a_n), (b_n) and (c_n), so the n^{th}terms are given by:

$$\begin{aligned} a_n &= n; \\ b_n &= (-1)^n; \\ c_n &= \frac{1}{n}. \end{aligned}$$

Initially

Sometimes you will see a_0 as the initial term of a sequence. We will see later that, as far as convergence is concerned, it doesn't matter where you start the sequence.

Figure 2.1 shows roughly what the graphs look like.

Another representation is obtained by simply labelling the points of the sequence on the real line, see figure 2.2. These pictures show types of behaviour that a sequence might have. The sequence (a_n) "goes to infinity", the sequence (b_n) "jumps back and forth between -1 and 1", and the sequence (c_n) "converges to 0". In this chapter we will decide how to give each of these phrases a precise meaning.

Sine Time

What do you think the fourth sequence, $\sin(n)$, looks like when you plot it on the real line?

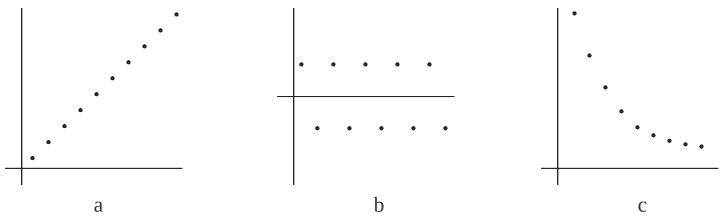


Figure 2.1: Graphing sequences as functions $\mathbb{N} \to \mathbb{R}$.

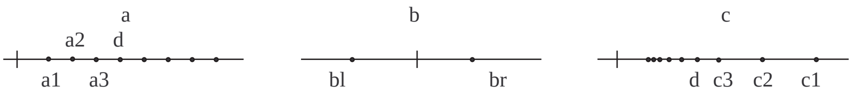


Figure 2.2: Number line representations of the sequences in figure 2.1.

Exercise 1 Write down a formula for the n^{th}term of each of the sequences below. Then plot the sequence in each of the two ways described above.

1. $1, 3, 5, 7, 9, \dots$
2. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \dots$
3. $0, -2, 0, -2, 0, -2, \dots$
4. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots$

2.2 Increasing and Decreasing Sequences

Labour Savers

Note that:
strictly increasing $\Longrightarrow$ increasing (and not decreasing)
strictly decreasing $\Longrightarrow$ decreasing (and not increasing)
increasing $\Longrightarrow$ monotonic
decreasing $\Longrightarrow$ monotonic.

Definition
A sequence (a_n) is:

strictly increasing	if, for all n, $a_n < a_{n+1}$;
increasing	if, for all n, $a_n \le a_{n+1}$;
strictly decreasing	if, for all n, $a_n > a_{n+1}$;
decreasing	if, for all n, $a_n \ge a_{n+1}$;
monotonic	if it is increasing *or* decreasing *or* both;
non-monotonic	if it is neither increasing *nor* decreasing.

Example Recall the sequences (a_n), (b_n) and (c_n), given by $a_n = n$, $b_n = (-1)^n$ and $c_n = \frac{1}{n}$. We see that:

1. for all n, $a_n = n < n + 1 = a_{n+1}$, therefore (a_n) is strictly increasing;
2. $b_1 = -1 < 1 = b_2$, $b_2 = 1 > -1 = b_3$, therefore (b_n) is neither increasing nor decreasing, i.e. non-monotonic;
3. for all n, $c_n = \frac{1}{n} > \frac{1}{n+1} = c_{n+1}$, therefore (c_n) is strictly decreasing.

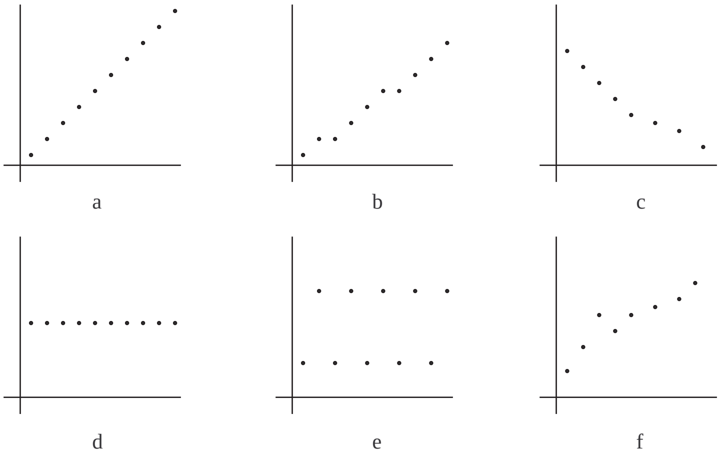


Figure 2.3: Which sequences are monotonic?

Exercise 2 Test whether each of the sequences defined below has any of the following properties: increasing; strictly increasing; decreasing; strictly decreasing; non-monotonic. [A graph of the sequence may help you to decide, but use the formal definitions in your proof.]

1. $a_n = -\frac{1}{n}$ 2. $a_{2n-1} = n, a_{2n} = n$ 3. $a_n = 1$
4. $a_n = 2^{-n}$ 5. $a_n = \sqrt{n+1} - \sqrt{n}$ 6. $a_n = \sin n$

Hint: In part 5, try using the identity $a - b = \frac{a^2-b^2}{a+b}$.

Be Dotty

When you are graphing your sequences, remember not to "join the dots". Sequences are functions defined on the *natural numbers* only.

2.3 Bounded Sequences

Definition

A sequence (a_n) is:

bounded above if there exists U such that, for all n, $a_n \leq U$;
U is an *upper bound* for (a_n);
bounded below if there exists L such that, for all n, $a_n \geq L$;
L is a *lower bound* for (a_n);
bounded if it is both bounded above *and* bounded below.

Boundless Bounds

If U is an upper bound then so is any number greater than U. If L is a lower bound then so is any number less than L. *Bounds are not unique.*

Example

1. The sequence $\left(\frac{1}{n}\right)$ is bounded since $0 < \frac{1}{n} \leq 1$.
2. The sequence (n) is bounded below but is not bounded above because for each value C there exists a number n such that $n > C$.

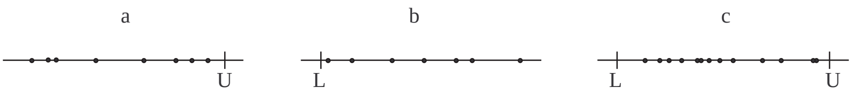


Figure 2.4: Sequences bounded above, below and both.

Bounds for Monotonic Sequences

Each increasing sequence (a_n) is bounded *below* by a_1.
Each decreasing sequence (a_n) is bounded *above* by a_1.

Exercise 3 Decide whether each of the sequences defined below is bounded above, bounded below, bounded. If it is none of these things then explain why. Identify upper and lower bounds in the cases where they exist. Note that, for a positive real number x, $\sqrt{x}$, denotes the positive square root of x.

1.	$\frac{(-1)^n}{n}$	2.	$\sqrt{n}$	3.	$a_n = 1$
4.	$\sin n$	5.	$\sqrt{n+1} - \sqrt{n}$	6.	$(-1)^n n$

Exercise 4

1. A sequence (a_n) is known to be increasing.
   (a) Might it have an upper bound?
   (b) Might it have a lower bound?
   (c) Must it have an upper bound?
   (d) Must is have a lower bound?

   Give a numerical example to illustrate each possibility or impossibility.

2. If a sequence is not bounded above, must it contain
   (a) a positive term,
   (b) an infinite number of positive terms?

2.4 Sequences Tending to Infinity

We say a sequence tends to infinity if its terms eventually exceed any number we choose.

Definition
A sequence (a_n) *tends to infinity* if, for every $C > 0$, there exists a natural number N such that $a_n > C$ for all $n > N$.

We will use three different ways to write that a sequence (a_n) tends to infinity, $(a_n) \to \infty$, $a_n \to \infty$, as $n \to \infty$ and $\lim_{n\to\infty} a_n = \infty$.
Example

1. $\left(\frac{n}{3}\right) \to \infty$. Let $C > 0$. We want to find N such that if $n > N$ then $\frac{n}{3} > C$. Note that $\frac{n}{3} > C \Leftrightarrow n > 3C$. So choose $N \geq 3C$. If $n > N$ then $\frac{n}{3} > \frac{N}{3} \geq C$. In the margin draw a graph of the sequence and illustrate the positions of C and N.

2. $(2^n) \to \infty$. Let $C > 0$. We want to find N such that if $n > N$ then $2^n > C$.

   Note that $2^n > C \Leftrightarrow n > \log_2 C$. So choose $N \geq \log_2 C$. If $n > N$ then $2^n > 2^N \geq 2^{\log_2 C} = C$.

Exercise 5 When does the sequence $(\sqrt{n})$ eventually exceed 2, 12 and 1000? Then prove that $(\sqrt{n})$ tends to infinity.

Is Infinity a Number?
We have not defined "infinity" to be any sort of number - in fact, we have not defined infinity at all. We have side-stepped any need for this by defining the phrase "tends to infinity" as a self-contained unit.

Exercise 6 Select values of C to demonstrate that the following sequences do not tend to infinity.

1. $1,\ 1,\ 2,\ 1,\ 3,\ \ldots,\ n,\ 1,\ \ldots$
2. $-1,\ 2,\ -3,\ 4,\ \ldots,\ (-1)^n n,\ \ldots$
3. $11,\ 12,\ 11,\ 12,\ \ldots,\ 11,\ 12,\ \ldots$

Exercise 7 Think of examples to show that:

1. an increasing sequence need not tend to infinity;
2. a sequence that tends to infinity need not be increasing;
3. a sequence with no upper bound need not tend to infinity.

Theorem
Let (a_n) and (b_n) be two sequences such that $b_n \geq a_n$ for all n. If $(a_n) \to \infty$ then $(b_n) \to \infty$.

Proof. Suppose $C > 0$. We know that there exists N such that $a_n > C$ whenever $n > N$. Hence $b_n \geq a_n > C$ whenever $n > N$. ■

Example We know that $n^2 \geq n$ and $(n) \to \infty$, hence $(n^2) \to \infty$.

Definition
A sequence (a_n) *tends to minus infinity* if, for every $C < 0$, there exists a number N such that $a_n < C$ for all $n > N$.

The corresponding three ways to write that (a_n) tends to minus infinity are $(a_n) \to -\infty$,$a_n \to -\infty$, *as* $n \to \infty$ and $\lim_{n\to\infty} a_n = -\infty$

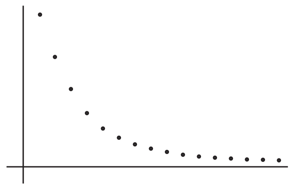


Figure 2.5: Does this look like a null sequence?

Example You can show that $(a_n) \to -\infty$ if and only if $(-a_n) \to \infty$. Hence, $(-n)$, $\left(\frac{-n}{2}\right)$ and $(-\sqrt{n})$ all tend to minus infinity.

Theorem
Suppose $(a_n) \to \infty$ and $(b_n) \to \infty$. Then $(a_n + b_n) \to \infty$, $(a_n b_n) \to \infty$, $(ca_n) \to \infty$ when $c > 0$ and $(ca_n) \to -\infty$ when $c < 0$.

Proof. We'll just do the first part here. Suppose $(a_n) \to \infty$ and $(b_n) \to \infty$. Let $C > 0$. Since $(a_n) \to \infty$ and $C/2 > 0$ there exists a natural number N_1 such that $a_n > C/2$ whenever $n > N_1$. Also, since $(b_n) \to \infty$ and $C/2 > 0$ there exists a natural number N_2 such that $b_n > C/2$ whenever $n > N_2$. Now let $N = \max\{N_1, N_2\}$. Suppose $n > N$. Then

$$n > N_1 \text{ and } n > N_2 \text{ so that } a_n > C/2 \text{ and } b_n > C/2.$$

This gives that

$$a_n + b_n > C/2 + C/2 = C.$$

This is exactly what it means to say that $(a_n + b_n) \to \infty$.

Try doing the other parts in your portfolio. [Hint: for the second part use $\sqrt{C}$ instead of $C/2$ in a proof similar to the above.] ■

2.5 Null Sequences

If someone asked you whether the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots, \frac{1}{n}, \ldots$$

"tends to zero", you might draw a graph like figure 2.5 and then probably answer "yes". After a little thought you might go on to say that the sequences

$$1, 0, \frac{1}{2}, 0, \frac{1}{3}, , 0, \frac{1}{4}, 0, \ldots, \frac{1}{n}, 0, \ldots$$

and

$$-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \ldots, (-1)^n \frac{1}{n}, \ldots$$

also "tend to zero".

Is Zero Allowed?

We are going to allow zeros to appear in sequences that "tend to zero" and not let their presence bother us. We are even going to say that the sequence

$$0, 0, 0, 0, 0, \ldots$$

"tends to zero".

We want to develop a precise definition of what it means for a sequence to "tend to zero". As a first step, notice that for each of the sequences above, every positive number is eventually an upper bound for the sequence and every negative number is eventually a lower bound. (So the sequence is getting "squashed" closer to zero the further along you go.)

Exercise 8

1. Use the sequences below (which are not null) to demonstrate the inadequacy of the following attempts to define a null sequence.

   (a) A sequence in which each term is strictly less than its predecessor.

   (b) A sequence in which each term is strictly less than its predecessor while remaining positive.

   (c) A sequence in which, for sufficiently large n, each term is less than some small positive number.

   (d) A sequence in which, for sufficiently large n, the absolute value of each term is less than some small positive number.

   (e) A sequence with arbitrarily small terms.

   I. $2,\ 1,\ 0,\ -1,\ -2,\ -3,\ -4,\ \ldots,\ -n,\ \ldots$

   II. $2,\ \frac{3}{2},\ \frac{4}{3},\ \frac{5}{4},\ \frac{6}{5},\ \ldots,\ \frac{n+1}{n},\ \ldots$

   III. $2,\ 1,\ 0,\ -1,\ -0.1,\ -0.1,\ -0.1,\ \ldots,\ -0.1,\ \ldots$

   IV. $2,\ 1,\ 0,\ -0.1,\ 0.01,\ -0.001,\ 0.01,\ -0.001,\ \ldots,\ 0.01,\ -0.001,\ \ldots$

   V. $1,\ \frac{1}{2},\ 1,\ \frac{1}{4},\ 1,\ \frac{1}{8},\ \ldots$

2. Examine the sequence

$$-1,\ \frac{1}{2},\ -\frac{1}{3},\ \frac{1}{4},\ -\frac{1}{5},\ \ldots,\ \frac{(-1)^n}{n},\ \ldots$$

   (a) Beyond what stage in the sequence are the terms between -0.1 and 0.1?

   (b) Beyond what stage in the sequence are the terms between -0.01 and 0.01?

   (c) Beyond what stage in the sequence are the terms between -0.001 and 0.001?

   (d) Beyond what stage in the sequence are the are the terms between $-\varepsilon$ and ε, where ε is a given positive number?

You noticed in Exercise 8 (2.) that for every value of ε, no matter how tiny, the sequence was eventually sandwiched between ε and $-\varepsilon$ (i.e. within ε of zero). We use this observation to create our definition. See figure 2.6

Definition
A sequence (a_n) *tends to zero* if, for each $\varepsilon > 0$, there exists a natural number N such that $|a_n| < \varepsilon$ for all $n > N$.

εrror.
The choice of ε, the Greek e, is to stand for 'error', where the terms of a sequence are thought of as sucessive attempts to hit the target of 0.

Make Like an Elephant
This definition is the trickiest we've had so far. Even if you don't understand it yet
Memorise It!
In fact, memorise all the other definitions while you're at it.

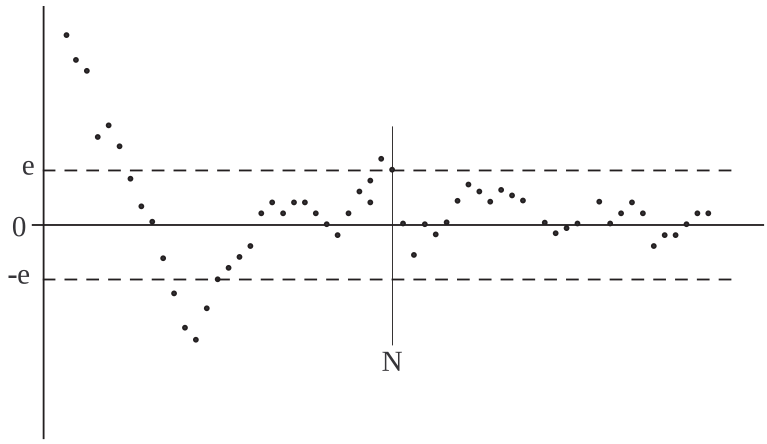


Figure 2.6: Null sequences; first choose ε, then find N.

The three ways to write a sequence tends to zero are, $(a_n) \to 0$, $a_n \to 0$, as $n \to \infty$, and $\lim_{n\to\infty} a_n = 0$. We also say (a_n) *converges to zero*, or (a_n) is a *null sequence*.

Archimedean Property
One property of the real numbers that we don't often give much thought to is this:
Given any real number x there is an integer N such that $N > x$.
Where have we used this fact?

Example The sequence $(a_n) = \left(\frac{1}{n}\right)$ tends to zero. Let $\varepsilon > 0$. We want to find N such that if $n > N$, then $|a_n| = \frac{1}{n} < \varepsilon$.

Note that $\frac{1}{n} < \varepsilon \Leftrightarrow n > \frac{1}{\varepsilon}$. So choose a natural number $N \geq \frac{1}{\varepsilon}$. If $n > N$, then $|a_n| = \frac{1}{n} < \frac{1}{N} \leq \varepsilon$.

Exercise 9 Prove that the sequence $\left(\frac{1}{\sqrt{n}}\right)$ tends to zero.

Exercise 10 Prove that the sequence $(1, 1, 1, 1, 1, 1, 1, \dots)$ does *not* tend to zero. (Find a value of ε for which there is no corresponding N.)

Lemma
If $(a_n) \to \infty$ then $\left(\frac{1}{a_n}\right) \to 0$.

Exercise 11 Prove this lemma.

Exercise 12 Think of an example to show that the opposite statement,

$$\text{if } (a_n) \to 0 \text{ then } \left(\frac{1}{a_n}\right) \to \infty,$$

is *false*, even if $a_n \neq 0$ for all n.

Lemma *Absolute Value Rule*
$(a_n) \to 0$ if and only if $(|a_n|) \to 0$.

Proof. The absolute value of $|a_n|$ is just $|a_n|$, i.e.$||a_n|| = |a_n|$. So $|a_n| \to 0$ iff for each $\varepsilon > 0$ there exists a natural number N such that $|a_n| < \varepsilon$ whenever $n > N$. But, by definition, this is exactly what $(a_n) \to 0$ means. ■

Example We showed before that $\left(\frac{1}{n}\right) \to 0$. Now $\frac{1}{n} = \left|\frac{(-1)^n}{n}\right|$. Hence $\left(\frac{(-1)^n}{n}\right) \to 0$.

Theorem *Sandwich Theorem for Null Sequences*
Suppose $(a_n) \to 0$. If $0 \le |b_n| \le a_n$ then $(b_n) \to 0$.

Example

1. Clearly $0 \le \frac{1}{n+1} \le \frac{1}{n}$. Therefore $\left(\frac{1}{n+1}\right) \to 0$.
2. $0 \le \frac{1}{n^{3/2}} \le \frac{1}{n}$. Therefore $\left(\frac{1}{n^{3/2}}\right) \to 0$.

Exercise 13 Prove that the following sequences are null using the result above. Indicate what null sequence you are using to make your Sandwich.

1. $\left(\dfrac{\sin n}{n}\right)$ 2. $\left(\sqrt{n+1} - \sqrt{n}\right)$

2.6 Arithmetic of Null Sequences

Theorem
Suppose $(a_n) \to 0$ and $(b_n) \to 0$. Then for all numbers c and d:

$$(ca_n + db_n) \to 0 \qquad \textit{Sum Rule} \text{ for Null Sequences;}$$
$$(a_n b_n) \to 0 \qquad \textit{Product Rule} \text{ for Null Sequences.}$$

Examples

- $\left(\frac{1}{n^2}\right) = \left(\frac{1}{n} \cdot \frac{1}{n}\right) \to 0$ (Product Rule)
- $\left(\frac{2n-5}{n^2}\right) = \left(\frac{2}{n} - \frac{5}{n^2}\right) \to 0$ (Sum Rule)

The Sum Rule and Product Rule are hardly surprising. If they failed we would surely have the wrong definition of a null sequence. So proving them carefully acts as a test to see if our definition is working.

Exercise 14

1. If (a_n) is a null sequence and c is a constant number, prove that $(c \cdot a_n)$ is a null sequence. [Hint: Consider the cases $c \neq 0$ and $c = 0$ in turn].

2. Deduce that $\frac{10}{\sqrt{n}}$ is a null sequence.

3. Suppose that (a_n) and (b_n) are both null sequences, and $\varepsilon > 0$ is given.

   (a) Must there be an N_1 such that $|a_n| < \frac{1}{2}\varepsilon$ when $n > N_1$?
   (b) Must there be an N_2 such that $|b_n| < \frac{1}{2}\varepsilon$ when $n > N_2$?
   (c) Is there an N_0 such that when $n > N_0$ both $n > N_1$ and $n > N_2$?
   (d) If $n > N_0$ must $|a_n + b_n| < \varepsilon$?
   You have proved that the termwise *sum* of two null sequences is null.
   (e) If the sequence (c_n) is also null, what about $(a_n + b_n + c_n)$? What about the sum of k null sequences?

Exercise 15 Suppose (a_n) and (b_n) are both null sequences, and $\varepsilon > 0$ is given.

1. Must there be an N_1 such that $|a_n| < \varepsilon$ when $n > N_1$?

2. Must there be an N_2 such that $|b_n| < 1$ when $n > N_2$?

3. Is there an N_0 such that when $n > N_0$ both $n > N_1$ and $n > N_2$?

4. If $n > N_0$ must $|a_n b_n| < \varepsilon$?

   You have proved that the termwise *product* of two null sequences is null.

5. If the sequence (c_n) is also null, what about $(a_n b_n c_n)$? What about the product of k null sequences?

Example To show that $\left(\frac{n^2+2n+3}{n^3}\right)$ is a null sequence, note that $\frac{n^2+2n+3}{n^3} = \frac{1}{n} + \frac{2}{n^2} + \frac{3}{n^3}$. We know that $\left(\frac{1}{n}\right) \to 0$ so $\left(\frac{1}{n^2}\right)$ and $\left(\frac{1}{n^3}\right)$ are null by the Product Rule. It follows that $\left(\frac{n^2+2n+3}{n^3}\right)$ is null by the Sum Rule.

2.7 * Application - Estimating π *

Recall Archimedes' method for approximating π: A_n and a_n are the areas of the circumscribed and inscribed regular n sided polygon to a circle of radius 1. Archimedes used the formulae

$$a_{2n} = \sqrt{a_n A_n} \qquad A_{2n} = \frac{2A_n a_{2n}}{A_n + a_{2n}}$$

to estimate π.

Exercise 16 Why is the sequence $a_4, a_8, a_{16}, a_{32}, \ldots$ increasing? Why are all the values between 2 and π? What similar statements can you make about the sequence $A_4, A_8, A_{16}, A_{32}, \ldots$?

Using Archimedes' formulae we see that

$$
\begin{aligned}
A_{2n} - a_{2n} &= \frac{2A_n a_{2n}}{A_n + a_{2n}} - a_{2n} \\
&= \frac{A_n a_{2n} - a_{2n}^2}{A_n + a_{2n}} \\
&= \frac{a_{2n}}{A_n + a_{2n}} (A_n - a_{2n}) \\
&= \frac{a_{2n}}{A_n + a_{2n}} \left(A_n - \sqrt{a_n A_n}\right) \\
&= \left(\frac{a_{2n}\sqrt{A_n}}{(A_n + a_{2n})(\sqrt{A_n} + \sqrt{a_n})}\right)(A_n - a_n)
\end{aligned}
$$

Exercise 17 Explain why $\left(\frac{a_{2n}\sqrt{A_n}}{(A_n+a_{2n})(\sqrt{A_n}+\sqrt{a_n})}\right)$ is never larger than 0.4. [Hint: use the bounds from the previous question.] Hence show that the error $(A_n - a_n)$ in calculating π reduces by at least 0.4 when replacing n by $2n$. Show that by calculating $A_{2^{10}}$ and $a_{2^{10}}$ we can estimate π to within 0.0014. [Hint: recall that $a_n \leq \pi \leq A_n$.]

Check Your Progress
By the end of this chapter you should be able to:

- Explain the term "sequence" and give a range of examples.
- Plot sequences in two different ways.
- Test whether a sequence is (strictly) increasing, (strictly) decreasing, monotonic, bounded above or bounded below - and formally state the meaning of each of these terms.
- Test whether a sequence "tends to infinity" and formally state what that means.
- Test whether a sequence "tends to zero" and formally state what that means.
- Apply the Sandwich Theorem for Null Sequences.
- Prove that if (a_n) and (b_n) are null sequences then so are $(|a_n|)$, (ca_n+db_n) and (a_nb_n).

Chapter 3

Sequences II

3.1 Convergent Sequences

Plot a graph of the sequence $(a_n) = \frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots, \frac{n+1}{n}, \ldots$ To what limit do you think this sequence tends? What can you say about the sequence $(a_n - 1)$? For $\epsilon = 0.1$, $\epsilon = 0.01$ and $\epsilon = 0.001$ find an N such that $|a_n - 1| < \epsilon$ whenever $n > N$.

Good N-ough

Any N that works is good enough - it doesn't have to be the smallest possible N.

Definition
Let $a \in \mathbb{R}$. A sequence (a_n) *tends to* a if, for each $\epsilon > 0$, there exists a natural number N such that $|a_n - a| < \epsilon$ for all $n > N$.

Recycle

Have a closer look at figure 3.1, what has been changed from figure 6 of workbook 2? It turns out that this definition is very similar to the definition of a null sequence.

Elephants Revisited

A null sequence is a special case of a convergent sequence. So **memorise** this definition and get the other one for free.

See figure 3.1 for an illustration of this definition.

We use the notation $(a_n) \to a$, $a_n \to a$, as $n \to \infty$ and $\lim_{n\to\infty} a_n = a$ and say that (a_n) *converges to* a, or the *limit* of the sequence (a_n) as n tends to infinity is a.

Example Prove $(a_n) = \left(\frac{n}{n+1}\right) \to 1$.

Let $\epsilon > 0$. We have to find a natural number N so that

$$|a_n - 1| = \left|\frac{n}{n+1} - 1\right| < \epsilon$$

when $n > N$. We have

$$\left|\frac{n}{n+1} - 1\right| = \left|-\frac{1}{n+1}\right| = \frac{1}{n+1} < \frac{1}{n}.$$

Hence it suffices to find N so that $\frac{1}{n} < \epsilon$ whenever $n > N$. But $\frac{1}{n} < \epsilon$ if and only if $\frac{1}{\epsilon} < n$ so it is enough to choose N to be a natural number with $N > \frac{1}{\epsilon}$. Then, if $n > N$ we have

$$|a_n - 1| = \left|\frac{n}{n+1} - 1\right| = \left|-\frac{1}{n+1}\right| = \frac{1}{n+1} < \frac{1}{n} < \frac{1}{N} < \epsilon.$$

Lemma
$(a_n) \to a$ if and only if $(a_n - a) \to 0$.

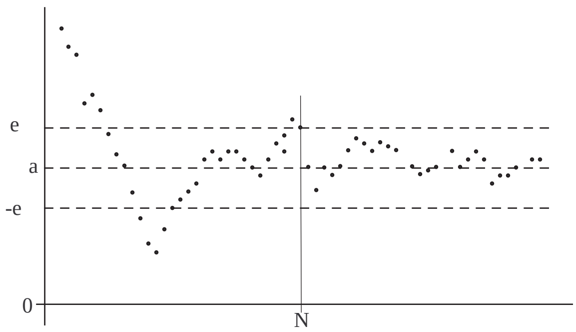


Figure 3.1: Convergent sequences; first choose ε, then find N.

Proof. We know that $(a_n - a) \to 0$ means that for each $\epsilon > 0$, there exists a natural number N such that $|a_n - a| < \epsilon$ when $n > N$. But this is exactly the definition of $(a_n) \to a$. ■

We have spoken of *the* limit of a sequence but can a sequence have more than one limit? The answer had better be "No" or our definition is suspect.

Theorem *Uniqueness of Limits*
A sequence cannot converge to more than one limit.

Exercise 1 Prove the theorem by assuming $(a_n) \to a$, $(a_n) \to b$ with $a < b$ and obtaining a contradiction. [Hint: try drawing a graph of the sequences with a and b marked on]

Theorem
Every convergent sequence is bounded.

Exercise 2 Prove the theorem above.

3.2 "Algebra" of Limits

Theorem
$a, b \in \mathbb{R}$. Suppose $(a_n) \to a$ and $(b_n) \to b$. Then

$$(ca_n + db_n) \to ca + db \qquad \textit{Sum Rule} \text{ for Sequences}$$
$$(a_n b_n) \to ab \qquad \textit{Product Rule} \text{ for Sequences}$$
$$\left(\frac{a_n}{b_n}\right) \to \frac{a}{b}, \text{ if } b \neq 0 \qquad \textit{Quotient Rule} \text{ for Sequences}$$

Connection
It won't have escaped your notice that the Sum Rule for *null sequences* is just a special case of the Sum Rule for *sequences*. The same goes for the Product Rule.
Why don't we have a Quotient Rule for null sequences?

Polly Want a Cracker?
If you have a parrot, teach it to say:
"The limit of the sum is the sum of the limits."
"The limit of the product is the product of the limits."
"The limit of the quotient is the quotient of the limits."

There is another useful way we can express all these rules: *If* (a_n) *and* (b_n) *are convergent then*

$$\lim_{n\to\infty}(ca_n + db_n) = c\lim_{n\to\infty} a_n + d\lim_{n\to\infty} b_n \qquad \textit{Sum Rule}$$
$$\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n \qquad \textit{Product Rule}$$
$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty}(a_n)}{\lim_{n\to\infty}(b_n)}, \text{ if } \lim_{n\to\infty}(b_n) \neq 0 \qquad \textit{Quotient Rule}$$

Example In full detail

$$\begin{aligned}
\lim_{n\to\infty}\frac{(n^2+1)(6n-1)}{2n^3+5} &= \lim_{n\to\infty}\frac{\left(1+\frac{1}{n^2}\right)\left(6-\frac{1}{n}\right)}{2+\frac{5}{n^3}} \\
&\quad \text{using the Quotient Rule} \\
&= \frac{\lim_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(6-\frac{1}{n}\right)\right]}{\lim_{n\to\infty}\left(2+\frac{5}{n^3}\right)} \\
&\quad \text{using the Product and Sum Rules} \\
&= \frac{\left(1+\lim_{n\to\infty}\left(\frac{1}{n^2}\right)\right)\left(6-\lim_{n\to\infty}\left(\frac{1}{n}\right)\right)}{2+5\lim_{n\to\infty}\left(\frac{1}{n^3}\right)} \\
&= \frac{(1+0)(6-0)}{2+0} \\
&= 3
\end{aligned}$$

Unless you are asked to show where you use each of the rules you can keep your solutions simpler. Either of the following is fine:

$$\lim_{n\to\infty}\frac{(n^2+1)(6n-1)}{2n^3+5} = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n^2}\right)\left(6-\frac{1}{n}\right)}{2+\frac{5}{n^3}} = \frac{(1+0)(6-0)}{2+0} = 3$$

or

$$\frac{(n^2+1)(6n-1)}{2n^3+5} = \frac{\left(1+\frac{1}{n^2}\right)\left(6-\frac{1}{n}\right)}{2+\frac{5}{n^3}} \to \frac{(1+0)(6-0)}{2+0} = 3$$

Bigger and Better

By induction, the Sum and Product Rules can be extended to cope with any *finite* number of convergent sequences. For example, for three sequences:

$$\lim_{n\to\infty}(a_n b_n c_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n \cdot \lim_{n\to\infty} c_n$$

Exercise 3 Show that

$$(a_n - a)(b_n - b) + a(b_n - b) + b(a_n - a) = a_n b_n - ab$$

Exercise 4 Use the identity in Exercise 3 and the rules for *null* sequences to prove the Product Rule for sequences.

Exercise 5 Write a proof of the Quotient Rule. You might like to structure your proof as follows.

1. Note that $(bb_n) \to b^2$ and show that $bb_n > \frac{b^2}{2}$ for sufficiently large n.

Don't Worry

You can still use the Quotient Rule if some of the b_ns are zero. The fact that $b \neq 0$ ensures that there can only be a finite number of these.
Can you see why?
And "eventually", the sequence leaves them behind.

2. Then show that eventually $0 \le \left|\frac{1}{b_n} - \frac{1}{b}\right| \le \frac{2}{b^2}|b - b_n|$ and therefore $\left(\frac{1}{b_n}\right) \to \frac{1}{b}$.
3. Now tackle $\frac{a_n}{b_n} = a_n \frac{1}{b_n}$.

Cunning Required
Do you know a cunning way to rewrite

$$1 + 2 + 3 + \cdots + n\,?$$

Exercise 6 Find the limit of each of the sequences defined below.

1. $\frac{7n^2+8}{4n^2-3n}$
2. $\frac{2^n+1}{2^n-1}$
3. $\frac{(\sqrt{n}+3)(\sqrt{n}-2)}{4\sqrt{n}-5n}$
4. $\frac{1+2+\cdots+n}{n^2}$

3.3 Further Useful Results

Connection
The Sandwich Rule for *null sequences* represents the case when $l = 0$.

Theorem *Sandwich Theorem for Sequences*
Suppose $(a_n) \to l$ and $(b_n) \to l$. If $a_n \le c_n \le b_n$ then $(c_n) \to l$.

This improved Sandwich theorem can be tackled by rewriting the hypothesis as $0 \le c_n - a_n \le b_n - a_n$ and applying the earlier Sandwich theorem.

Exercise 7 Prove the Sandwich Theorem for sequences.

There are going to be many occasions when we are interested in the behaviour of a sequence "after a certain point", regardless of what went on before that. This can be done by "chopping off" the first N terms of a sequence (a_n) to get a shifted sequence (b_n) given by $b_n = a_{N+n}$. We often write this as (a_{N+n}), so that

$$(a_{N+n}) = a_{N+1},\ a_{N+2},\ a_{N+3},\ a_{N+4},\ \ldots$$

which starts at the term a_{N+1}. We use it in the definition below.

Definition
A sequence (a_n) satisfies a certain property *eventually* if there is a natural number N such that the sequence (a_{N+n}) satisfies that property.

For instance, a sequence (a_n) is *eventually bounded* if there exists N such that the sequence (a_{N+n}) is bounded.

Max and Min
In your proof you may well use the fact that each finite set has a maximum and a minimum. *Is this true of infinite sets?*

Lemma
If a sequence is eventually bounded then it is bounded.

Exercise 8 Prove this lemma.

The next result, called the Shift Rule, tells you that a sequence converges if and only if it converges eventually. So you can chop off or add on any *finite* number of terms at the beginning of a sequence without affecting the convergent behaviour of its infinite "tail".

Theorem *Shift Rule*
Let N be a natural number. Let (a_n) be a sequence. Then $a_n \to a$ if and only if the "shifted" sequence $a_{N+n} \to a$.

Proof. Fix $\epsilon > 0$. If $(a_n) \to a$ we know there exists N_1 such that $|a_n - a| < \epsilon$ whenever $n > N_1$. When $n > N_1$, we see that $N+n > N_1$, therefore $|a_{N+n}-a| < \epsilon$. Hence $(a_{N+n}) \to a$. Conversely, suppose that $(a_{N+n}) \to a$. Then there exists N_2 such that $|a_{N+n} - a| < \epsilon$ whenever $n > N_2$. When $n > N + N_2$ then $n - N > N_2$ so $|a_n - a| = |a_{N+(n-N)} - a| < \epsilon$. Hence $(a_n) \to a$. ■

Corollary *Sandwich Theorem with Shift Rule*
Suppose $(a_n) \to l$ and $(b_n) \to l$. If eventually $a_n \leq c_n \leq b_n$ then $(c_n) \to l$.

Example We know $1/n \to 0$ therefore $1/(n+5) \to 0$.

Exercise 9 Show that the Shift Rule also works for sequences which tend to infinity: $(a_n) \to \infty$ if and only if $(a_{N+n}) \to \infty$.

If all the terms of a convergent sequence sit within a certain interval, does its limit lie in that interval, or can it "escape"? For instance, if the terms of a convergent sequence are all positive, is its limit positive too?

Lemma
Suppose $(a_n) \to a$. If $a_n \geq 0$ for all n then $a \geq 0$.

Exercise 10 Prove this result. [Hint: Assume that $a < 0$ and let $\epsilon = -a > 0$. Then use the definition of convergence to arrive at a contradiction.]

Exercise 11 Prove or disprove the following statement:

"Suppose $(a_n) \to a$. If $a_n > 0$ for all n then $a > 0$."

Theorem *Inequality Rule*
Suppose $(a_n) \to a$ and $(b_n) \to b$. If (eventually) $a_n \leq b_n$ then $a \leq b$.

Exercise 12 Prove this result using the previous Lemma. [Hint: Consider $(b_n - a_n)$.]

Limits on Limits

Limits cannot escape from closed intervals. They can escape from open intervals - but only as far as the end points.

Corollary *Closed Interval Rule*
Suppose $(a_n) \to a$. If (eventually) $A \le a_n \le B$ then $A \le a \le B$.

If $A < a_n < B$ it is *not* the case that $A < a < B$. For example $0 < \frac{n}{n+1} < 1$ but $\frac{n}{n+1} \to 1$.

3.4 Subsequences

Caution

Note that the subsequence (a_{n_i}) is indexed by i not n. In all cases $n_i \ge i$. (Why is this?) *Remember these facts when subsequences appear!*

A subsequence of (a_n) is a sequence consisting of some (or all) of its terms in their original order. For instance, we can pick out the terms with even index to get the subsequence

$$a_2, a_4, a_6, a_8, a_{10}, \dots$$

or we can choose all those whose index is a perfect square

$$a_1, a_4, a_9, a_{16}, a_{25}, \dots$$

In the first case we chose the terms in positions 2,4,6,8,… and in the second those in positions 1,4,9,16,25,…

In general, if we take any strictly increasing sequence of natural numbers $(n_i) = n_1, n_2, n_3, n_4, \dots$ we can define a subsequence of (a_n) by

$$(a_{n_i}) = a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, \dots$$

Definition
A *subsequence* of (a_n) is a sequence of the form (a_{n_i}), where (n_i) is a strictly increasing sequence of natural numbers.

Prove the obvious

It may seem obvious that every subsequence of a convergent sequence converges, but you should still check that you know how to prove it!

Effectively, the sequence (n_i) "picks out" which terms of (a_n) get to belong to the subsequence. Think back to the definition of convergence of a sequence. Why is it immediate from the definition that if a sequence (a_n) converges to a then all subsequence (a_{n_i}) converge to a? This is a fact which we will be using constantly in the rest of the course.

Notice that the shifted sequence (a_{N+n}) is a subsequence of (a_n).

Exercise 13 Let $(a_n) = (n^2)$. Write down the first four terms of the three subsequences (a_{n+4}), (a_{3n-1}) and (a_{2^n}).

Here is another result which we will need in later chapters.

Exercise 14 Suppose we have a sequence (a_n) and are trying to prove that it converges. Assume that we have have shown that the subsequences (a_{2n}) and (a_{2n+1}) both converge to the same limit a. Prove that $(a_n) \to a$ converges.

Exercise 15 Answer "Yes" or "No" to the following questions, but be sure that you know why and that you aren't just guessing.

1. A sequence (a_n) is known to be increasing, but not strictly increasing.
   (a) Might there be a strictly increasing subsequence of (a_n)?
   (b) Must there be a strictly increasing subsequence of (a_n)?
2. If a sequence is bounded, must every subsequence be bounded?
3. If the subsequence $a_2,\ a_3,\ \ldots,\ a_{n+1},\ \ldots$ is bounded, does it follow that the sequence (a_n) is bounded?
4. If the subsequence $a_3,\ a_4,\ \ldots,\ a_{n+2},\ \ldots$ is bounded does it follow that the sequence (a_n) is bounded?
5. If the subsequence $a_{N+1},\ a_{N+2},\ \ldots,\ a_{N+n},\ \ldots$ is bounded does it follow that the sequence (a_n) is bounded?

Lemma
Every subsequence of a bounded sequence is bounded.

Proof. Let (a_n) be a bounded sequence. Then there exist L and U such that $L \le a_n \le U$ for all n. It follows that if (a_{n_i}) is a subsequence of (a_n) then $L \le a_{n_i} \le U$ *for all* i. Hence (a_{n_i}) is bounded. ■

You might be surprised to learn that every sequence, no matter how bouncy and ill-behaved, contains an increasing or decreasing subsequence.

Theorem
Every sequence has a monotonic subsequence.

Sine Time Again
The fact that a sequence has a guaranteed monotonic subsequence doesn't mean that the subsequence is easy to find.
Try identifying an increasing or decreasing subsequence of $\sin n$ and you'll see what I mean.

3.5 * Application - Speed of Convergence *

Often sequences are defined *recursively*, that is, later terms are defined in terms of earlier ones. Consider a sequence (a_n) where $a_0 = 1$ and $a_{n+1} = \sqrt{a_n + 2}$, so the sequence begins $a_0 = 1, a_1 = \sqrt{3}, a_2 = \sqrt{\sqrt{3} + 2}$.

Exercise 16 Use induction to show that $1 \le a_n \le 2$ for all n.

Now assume that (a_n) converges to a limit, say, a. Then:

$$a = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \left((a_{n+1})^2 - 2\right) = \left(\lim_{n\to\infty} a_{n+1}\right)^2 - 2 = a^2 - 2$$

So to find a we have to solve the quadratic equation $a^2 - a - 2 = 0$. We can rewrite this as $(a + 1)(a - 2) = 0$, so either $a = -1$ or $a = 2$. But which one is it? The Inequality Theorem comes to our rescue here. Since $a_n \ge 1$ for all n it follows that $a \ge 1$, therefore $a = 2$. We will now investigate the speed that a_n approaches 2.

Exercise 17 Show that $2 - a_{n+1} = \frac{2-a_n}{2+\sqrt{2+a_n}}$. Use this identity and induction to show that $(2 - a_n) \leq \frac{1}{(2+\sqrt{3})^n}$ for all n. How many iterations are needed so that a_n is within 10^{-100} is its limit 2?

An excellent method for calculating square roots is the Newton-Raphson method which you may have met at A-level. When applied to the problem of calculating $\sqrt{2}$ this leads to the sequence given by: $a_0 = 2$ and $a_{n+1} = \frac{1}{a_n} + \frac{a_n}{2}$.

Exercise 18 Use a calculator to calculate a_1, a_2, a_3, a_4. Compare them with $\sqrt{2}$.

Exercise 19 Use induction to show that $1 \leq a_n \leq 2$ for all n. Assuming that (a_n) converges, show that the limit must be $\sqrt{2}$.

We will now show that the sequence converges to $\sqrt{2}$ like a bat out of hell.

Exercise 20 Show that $(a_{n+1} - \sqrt{2}) = \frac{(a_n-\sqrt{2})^2}{2a_n}$. Using this identity show by induction that $|a_n - \sqrt{2}| \leq \frac{1}{2^{2^n}}$. How many iterations do you need before you can guarentee to calculate $\sqrt{2}$ to within an error of 10^{-100} (approximately 100 decimal places)?

Sequences as in Exercise 17 are said to converge *exponentially* and those as in Exercise 20 are said to converge *quadratically* since the error is squared at each iteration. The standard methods for calculating π were exponential (just as is the Archimedes method) until the mid 1970s when a quadratically convergent approximation was discovered.

Check Your Progress
By the end of this chapter you should be able to:

- Define what it means for a sequence to "converge to a limit".
- Prove that every convergent sequence is bounded.
- State, prove and use the following results about convergent sequences: If $(a_n) \to a$ and $(b_n) \to b$ then:

  Sum Rule: $(ca_n + db_n) \to ca + db$

  Product Rule: $(a_n b_n) \to ab$

  Quotient Rule: $(a_n/b_n) \to a/b$ if $b \neq 0$

  Sandwich Theorem: if $a = b$ and $a_n \leq c_n \leq b_n$ then $(c_n) \to a$

  Closed Interval Rule: if $A \leq a_n \leq B$ then $A \leq a \leq B$
- Explain the term "subsequence" and give a range of examples.

Chapter 4

Sequences III

4.1 Roots

We can use the results we've established in the last workbook to find some interesting limits for sequences involving roots. We will need more technical expertise and low cunning than have been required hitherto. First a simple inequality.

Bernoulli's Inequality
When $x > -1$ and n is a natural number,

$$(1+x)^n \geq 1 + nx.$$

Strictly Speaking
Bernoulli's Inequality is actually *strict* unless $x = 0$, $n = 0$ or $n = 1$.

Exercise 1 Sketch a graph of both sides of Bernoulli's inequality in the cases $n = 2$ and $n = 3$.

For *non-negative* values of x Bernoulli's inequality can be easily proved using the Binomial Theorem, which expands the left-hand side:

$$\begin{aligned}(1+x)^n &= 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{6}x^3 \\ &\quad + \cdots + nx^{n-1} + x^n \\ &\geq 1 + nx.\end{aligned}$$

Binomial Theorem
For all real values x and y

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

What difficulties do we have with this line of argument if $x < 0$?

Exercise 2 Finish off the following proof of Bernoulli's Inequality for $x > -1$ using *mathematical induction*. Note down where you use the fact that $x > -1$.

Proof. We want to show that $(1+x)^n \geq 1 + nx$ where $x > -1$ and n is a natural number. This is true for $n = 1$ since $(1+x)^1 = 1 + x$.

Now assume $(1+x)^k \geq 1 + kx$. Then

$$(1+x)^{k+1} = (1+x)^k(1+x).$$

The Bernoulli Boys
Bernoulli's Inequality is named after Jacques Bernoulli, a Swiss mathematician who used it in a paper on infinite series in 1689 (though it can be found earlier in a 1670 paper by an Englishman called Isaac Barrow).

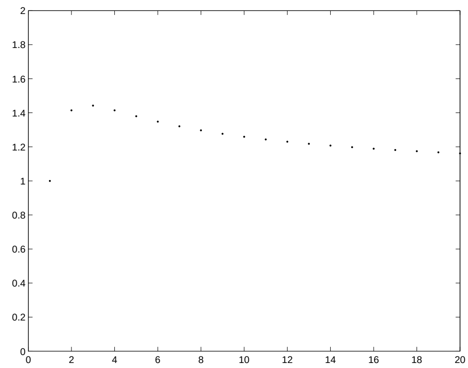


Figure 4.1: The sequence $(n^{1/n})$.

Exercise 3 Use a calculator to explore the sequences $(2^{1/n})$, $(10^{1/n})$ and $(1000^{1/n})$. Repeated use of the square root button will give a subsequence in each case.

Roots
$x^{1/n} = \sqrt[n]{x}$ is the positive n^{th}root of x.

Proposition
If $x > 0$ then $(x^{1/n}) \to 1$.

Example $100000000000^{1/n} \to 1$ and also $0.000000000001^{1/n} \to 1$.

To prove this result you might follow the following fairly cunning steps (although other proofs are very welcome):

Exercise 4

1. First assume that $x \geq 1$ and deduce that $x^{1/n} \geq 1$.
2. Let $a_n = x^{1/n} - 1$ and use Bernoulli's inequality to show that $x \geq 1 + na_n$.
3. Use the Sandwich Rule to prove that (a_n) is a null sequence.
4. Deduce that $(x^{1/n}) \to 1$.
5. Show that $(x^{1/n}) \to 1$ when $0 < x < 1$ by considering $(1/x^{1/n})$.

Auto Roots
$n^{1/n} = \sqrt[n]{n}$ is the n^{th}root of n. Natural numbers approach unity by rooting themselves.

Exercise 5 Use a calulator to explore the limit of $(2^n + 3^n)^{1/n}$. Now find the limit of the sequence $(x^n + y^n)^{1/n}$ when $0 \leq x \leq y$. (Try to find a sandwich for your proof.)

What happens if we first write $n = (n^{1/n})^n = (1+(n^{1/n}-1))^n$ and apply Bernoulli's Inequality? Do we get anywhere?

Proposition
$(n^{1/n}) \to 1$.

See figure 4.1 for a graph of this sequence.

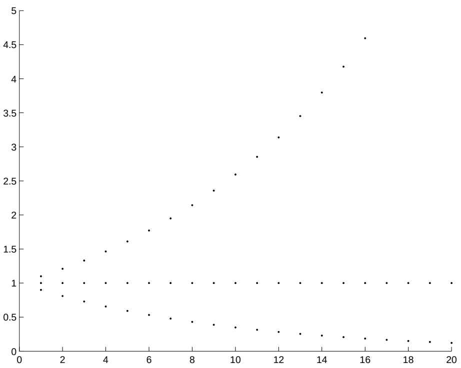


Figure 4.2: The sequence (x^n) with three slightly different values of x.

Proof. The proof is similar to that of the previous lemma but we have to be cunning and first show that $(n^{1/2n}) \to 1$. Since $n \geq 1$ we have $n^{1/2n} \geq 1$. Therefore,

$$\begin{aligned}\sqrt{n} &= (n^{1/2n})^n = (1 + (n^{1/2n} - 1))^n \\ &\geq 1 + n(n^{1/2n} - 1) > n(n^{1/2n} - 1)\end{aligned}$$

using Bernoulli's inequality. Rearranging, we see that $1 \leq n^{1/2n} < \frac{1}{\sqrt{n}} + 1$. So $(n^{1/2n}) \to 1$ by the Sandwich Theorem. Hence $(n^{1/n}) = (n^{1/2n})^2 \to 1$ by the Product Rule. ■

4.2 Powers

Exercise 6 Explore, with a calculator if necessary, and then write down a conjectured limit for the power sequence (x^n). (Warning: you should get four different possible answers depending on the value of x.)

To prove your conjectures you can use Bernoulli's inequality again. Note that if $x > 1$ then $x^n = (1 + (x-1))^n \geq 1 + n(x-1)$. To prove your conjecture for $0 < x < 1$ look at the sequence $1/x^n$ and then use Exercise 11 of Chapter 2. Then treat all other values of x such as $x = 0$, $x = 1$, $-1 < x < 0$, and $x \leq -1$.

Many sequences are not exactly powers but grow or shrink at least as fast as a sequence of powers so that we can compare them with (or sandwich them by) a geometric sequence. A useful idea to formalise this is to consider the ratio of two successive terms: a_{n+1}/a_n. If this is close to a value x then the sequence (a_n) might behave like the sequence (x^n). We explore this idea.

Ratio Lemma, "light" version

Let (a_n) be a sequence of positive numbers. Suppose $0 < l < 1$ and $\frac{a_{n+1}}{a_n} \leq l$ for all n. Then $a_n \to 0$.

Proof. We have that

$$a_n \leq l a_{n-1} \leq l^2 a_{n-2} \leq ... \leq l^{n-1} a_1.$$

Then $0 \leq a_n \leq l^{n-1} a_1$. Since $l^{n-1} a_1 \to 0$ as $n \to \infty$, and using the Sandwich Theorem (Chapter 3), we obtain that $a_n \to 0$. ■

Exercise 7 Consider the sequence $a_n = 1 + \frac{1}{n}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{1 + \frac{1}{n+1}}{1 + \frac{1}{n}} = \frac{\frac{n+2}{n+1}}{\frac{n+1}{n}} = \frac{n(n+2)}{(n+1)^2} = \frac{n^2+n}{n^2+2n+1} < 1,$$

for all n. Then $a_n \to 0$ by the Ratio Lemma. Is that really so? If not, what is wrong?

A much more powerful of the above lemma is as follows:

Ratio Lemma, full version
Let (a_n) be a sequence of nonzero numbers. Suppose $0 < l < 1$ and $|\frac{a_{n+1}}{a_n}| \leq l$ eventually. Then $a_n \to 0$.

The precise definition of "$|\frac{a_{n+1}}{a_n}| \leq l$ eventually" is that there exists N such that $|\frac{a_{n+1}}{a_n}| \leq l$ for all $n > N$.
Proof. This proof follows the same lines as those for the "light" version of the lemma, but it looks more difficult. We know that $|a_{n+1}| \leq l|a_n|$ for all $n > N$. Then if $n > N$,

$$|a_n| \leq l|a_{n-1}| \leq l^2|a_{n-2}| \leq ... \leq l^{n-N-1}|a_{N+1}| = l^n \frac{|a_{N+1}|}{l^{N+1}}.$$

By choosing C large enough, we have $|a_n| \leq l^n C$ for all $n \leq N$. Suppose also that $C \geq \frac{|a_{N+1}|}{l^{N+1}}$. Then we have for all n,

$$0 \leq |a_n| \leq l^n C.$$

Since $l^n C \to 0$ as $n \to \infty$, we have $a_n \to 0$ by the Sandwich Theorem (Chapter 3). ■

Examples

1. Show that $\frac{n^2}{2^n} \to 0$
2. Show that $\frac{n!}{1000^n} \to \infty$

Proof

1. The ratio of successive terms is $\frac{a_{n+1}}{a_n} = \frac{(n+1)^2/2^{n+1}}{n^2/2^n} = \frac{1}{2}\left(1 + \frac{1}{n}\right)^2 \to \frac{1}{2}$. So, taking $\epsilon = \frac{1}{4}$ in the definition of convergence, we have $\frac{1}{4} \leq \frac{a_{n+1}}{a_n} \leq \frac{3}{4}$ for large n. The Ratio Lemma then implies that $\frac{n^2}{2^n} \to 0$.

Powerful Powers
All increasing power sequences grow faster than any polynomial sequence.

Powerless Powers
All power sequences are powerless against the factorial sequence $(n!)$.

2. Let $a_n = \frac{1000^n}{n!}$. Then $\frac{a_{n+1}}{a_n} = \frac{1000^{n+1}/(n+1)!}{1000^n/n!} = \frac{1000}{n+1} \leq \frac{1}{2}$ for all $n \geq 1999$. The Ratio Lemma says that $\frac{1000^n}{n!} \to 0$ so that $\frac{n!}{1000^n} \to \infty$.

In both of the examples above we showed that $\left(\frac{a_{n+1}}{a_n}\right) \to a$ for $0 \leq a < 1$, and then used this to show that $\frac{a_{n+1}}{a_n} \leq l$ eventually. The following corollary to version 2 of the Ratio Lemma allows us to cut out some of this work.

Corollary
Let $a_0, a_1, a_2, \ldots$ be a sequence of positive numbers. If $\left(\frac{a_{n+1}}{a_n}\right) \to a$ with $0 \leq a < 1$ then $(a_n) \to 0$.

Proof. If $\frac{a_{n+1}}{a_n} \to a$, then for all $\varepsilon > 0$ there exists N such that $|\frac{a_{n+1}}{a_n} - a| < \varepsilon$ for all $n > N$. We choose $\varepsilon = \frac{1-a}{2}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| < a + \varepsilon = \frac{a}{2} + \frac{1}{2} < 1$$

for all $n > N$. Then $a_n \to 0$ by the Ratio Lemma (full version). ■

Exercise 8 State whether the following sequences tend to zero or infinity. Prove your answers:

1. $\frac{n^{1000}}{2^n}$ 2. $\frac{1.0001^n}{n}$ 3. $\frac{n!}{n^{1000}}$ 4. $\frac{(n!)^2}{(2n)!}$

Exercise 9 Try using the Ratio Lemma to prove that the sequence $\frac{1}{n} \to 0$. Why does the lemma tell you nothing?

The sequences (n^k) for $k = 1, 2, 3, \ldots$ and (x^n) for $x > 1$ and $(n!)$ all tend to infinity. Which is quickest? The above examples suggest some general rules which we prove below.

Exercise 10 Prove that $\left(\frac{x^n}{n!}\right) \to 0$ for all values of x.

Notice that this result implies that $(n!)^{1/n} \to \infty$, since for any value of x, eventually we have that $\left(\frac{x^n}{n!}\right) < 1$ giving that $x < (n!)^{1/n}$.

Exercise 11 Prove that $\left(\frac{n!}{n^n}\right) \to 0$ as $n \to \infty$.

Proof. We use the Ratio Lemma. Let $a_n = \frac{n!}{n^n}$. We have

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \frac{(n+1)!}{n!} \frac{n^n}{(n+1)^{n+1}} = \frac{n^n}{(n+1)^n} = \frac{1}{(1+\frac{1}{n})^n} \leq \frac{1}{2}.$$

The last inequality follows from $(1 + \frac{1}{n})^n \geq 2$ (Bernoulli's inequality). Then $a_n \to 0$ by the Ratio Lemma. Alternatively, we could have remarked that

$$a_n = \frac{n!}{n^n} = \frac{n}{n} \cdot \frac{(n-1)}{n} \cdot \frac{(n-2)}{n} \cdots \frac{2}{n} \cdot \frac{1}{n} \leq \frac{1}{n}.$$

Then $a_n \to 0$ by the Sandwich Theorem. ■

Exercise 12 Find the limit of the sequence $\left(\frac{x^n}{n^k}\right)$ as $n \to \infty$ for all values of $x > 0$ and $k = 1, 2, \ldots$

Exercise 13 Find the limits of the following sequences. Give reasons.

1. $\left(\frac{n^4 11^n + n^9 9^n}{7^{2n}+1}\right)$
2. $\left(\left(4^{10} + 2^n\right)^{1/n}\right)$
3. $\left(\frac{3n^3 + n\cos^2 n}{n^2 + \sin^2 n}\right)$
4. $\left(\left(3n^2 + n\right)^{1/n}\right)$

4.3 Application - Factorials

Factorials $n!$ occur throughout mathematics and especially where counting arguments are used. The last section showed that the factorial sequence $(n!)$ is more powerful than any power sequence (x^n), but earlier you showed that $\frac{n!}{n^n} \to 0$. One gains much information about the speed of divergence of $n!$ with Stirling's formula: $n! \approx n^n e^{-n}$. Here is an amazingly precise version of Stirling's formula:

Stirling & de Moivre

The Stirling formula is due to Abraham de Moivre (1667–1754), a French Protestant who lived in England because of persecutions from the Catholic king Louis XIV. De Moivre could improve the formula thanks to comments by the Scottish James Stirling (1692–1770), who was from a Jacobite family and therefore had difficulties with the English (Jacobites were supporters of the king James II of Britain, who was deposed because of his Catholic faith and went into exile in France).

Theorem
For all $n \geq 1$,

$$\sqrt{2\pi n}\; n^n e^{-n}\left(1 + \tfrac{1}{12n}\right) \leq \; n! \; \leq \sqrt{2\pi n}\; n^n e^{-n}\left(1 + \tfrac{1}{12n} + \tfrac{1}{288n^2}\right).$$

In this section we will get a partial proof of the above theorem by using two clever but very useful tricks. The first trick is to change the product $n! = 2 \cdot 3 \cdot 4 \cdots (n-1) \cdot n$ into a sum by taking logarithms: $\log n! = \log 2 + \log 3 + \cdots + \log(n-1) + \log n$.

The second trick, which we shall use repeatedly in future sections, is to approximate the sum by an integral, see figure 4.3. Here is a graph of the function $\log(x)$ with a series of blocks, each of width one, lying underneath the graph. The sum of the areas of all the blocks is $\log 2 + \log 3 + \cdots + \log(n-1)$. But the area of the blocks is less than the area under the curve between $x = 1$ and $x = n$. So we have:

$$\begin{aligned} \log n! &= \log 2 + \log 3 + \cdots + \log(n-1) + \log n \\ &\leq \int_1^n \log x dx + \log n \\ &= [x \log x - x]_1^n + \log n \\ &= (n+1)\log n - n + 1. \end{aligned}$$

Taking exponentials of both sides we get the wonderful upper bound

$$n! \leq n^{n+1} e^{-n+1}.$$

This should be compared to claim of the theorem, namely that $n! \approx n^{n+\frac{1}{2}} e^{-n}$.

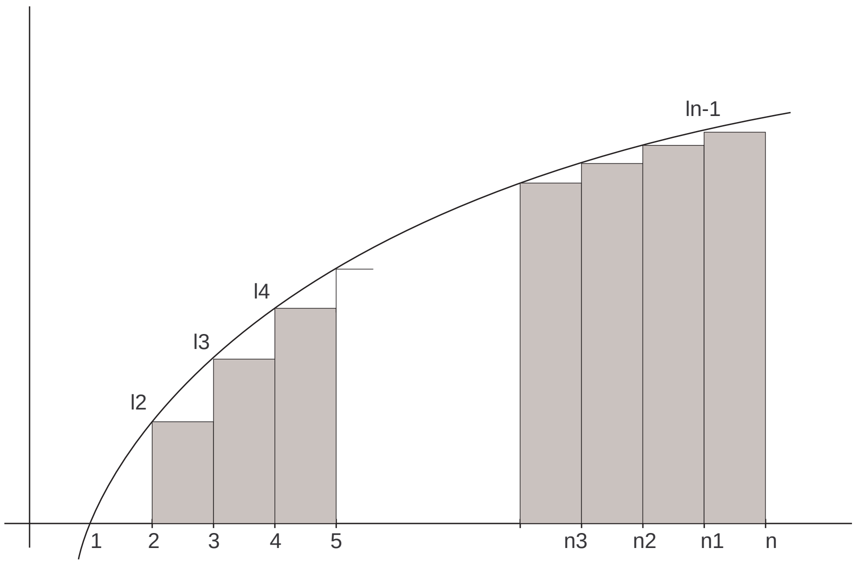


Figure 4.3: Approximating $\int_1^n \log x dx$ from below.

Exercise 14 Use figure 4.4 to obtain a lower bound on $n!$. In this case the area of the blocks is greater than the area under the graph.

Exercise 15 Use your upper and lower bounds on $n!$ to find the following limits: (i) $\left(\frac{n!2^n}{n^n}\right)$ (ii) $\left(\frac{n!4^n}{n^n}\right)$

4.4 * Application - Sequences and Beyond *

In Chapter 2 we defined a sequence as an infinite list of numbers. However, the concept of an infinite list of other objects is also useful in mathematics. With that in mind, we define a *sequence of objects* to be an infinite list of those objects.

For example, let P_n be the regular n-sided polygon of area 1. Then (P_n) is a sequence of shapes.

The main question we asked about sequences was whether they converged or not. To examine convergence in general, we need to be able to say when two objects are close to each other. This is not always an easy thing to do. However, given a sequence of objects, we may be able to derive sequences of numbers and examine those sequences to learn something about the original sequence. For example, given a sequence of tables, we could look at the number of legs of each table. This gives us a new sequence of numbers which is related to the original sequence of tables. However, different related sequences can behave in wildly different ways.

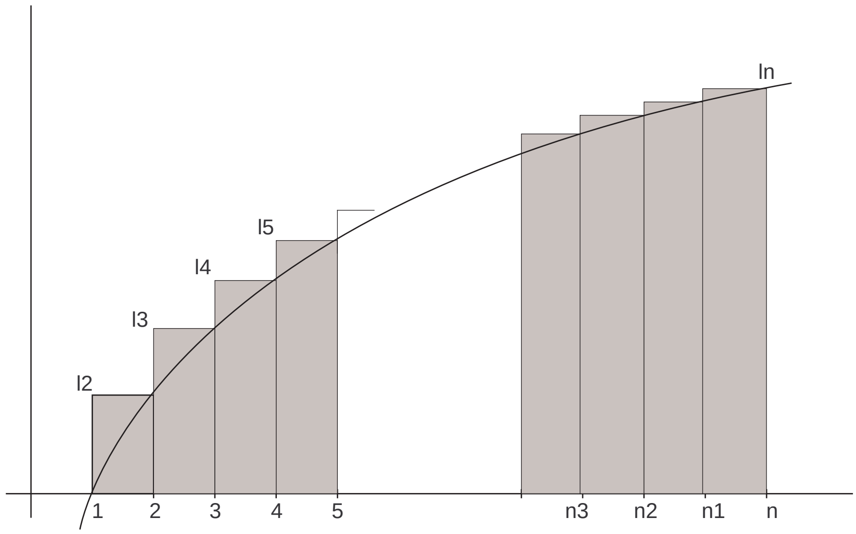


Figure 4.4: Approximating $\int_1^n \log x \mathrm{d}x$ from above.

We shall demonstrate this by considering the set of solid shapes in $\mathbb{R}^2$. Given a sequence of shapes, (S_n), two obvious related sequences are the sequence of perimeters, $(p(S_n))$, and the sequence of areas, $(a(S_n))$.

Example Let P_n be the regular n-sided polygon centred at the origin which fits exactly inside the unit circle. This sequence starts with P_3, which is the equilateral triangle.

Let $a_n = a(P_n)$ be the sequence of areas of the polygons. In Chapter 1 we saw that $a_n = \frac{n}{2}\sin\left(\frac{2\pi}{n}\right)$ and in Chapter 2 we saw that this converges to π as $n \to \infty$ which is the area of the unit circle.

Exercise 16 Let $p_n = p(P_n)$ be the perimeter of P_n. Show that $p_n = 2n\sin\left(\frac{\pi}{n}\right) = 2n\sin\left(\frac{2\pi}{2n}\right) = 2a_{2n}$.

Exercise 17 Show that $\lim p_n = 2\pi$.

We see that in both cases we get what we would expect, namely that as the shape looks more and more like the circle, so also the area and perimeter tend to those of the circle.

Exercise 18 Consider the sequence of shapes in figure 4.5. Each is produced from the former by replacing each large step by two half-sized ones. Draw the "limiting" shape. Given that the original shape is a square of area 1, what is the perimeter and area of the n^{th}shape? Compare the limits of these sequences with the perimeter and area of the limiting shape.

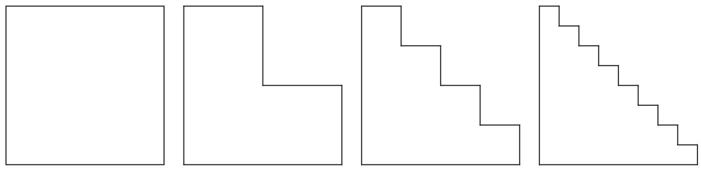


Figure 4.5:

A famous example of this type of behaviour is the Koch curve, see figure 4.6. The initial figure is an equilateral triangle of area A_1 and perimeter p_1. To each side of the triangle is attached another equilateral triangle at the trisection points of the triangle. This process is then applied to each side of the resulting figure and so on.

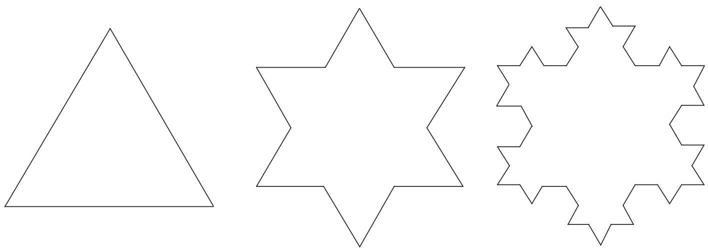


Figure 4.6:

Let p_n be the perimeter of the shape at the n^{th}stage and A_n the area.

Exercise 19

1. What is the number of sides of the shape at the n^{th}stage (for $n = 1$ the answer is 3).
2. Show that $p_{n+1} = \frac{4}{3}p_n$.
3. Prove that $(p_n) \to \infty$.
4. Show that, in making the $(n+1)^{\text{th}}$ shape, each little triangle being added has area $\frac{1}{9^n}A_1$.
5. Show that $A_{n+1} = A_n + \frac{3}{9}\left(\frac{4}{9}\right)^{n-1} A_1$.
6. Prove that $(A_n) \to \frac{8}{5}A_1$. (Recall the sum of a geometric series.)

Check Your Progress
By the end of this chapter you should be able to:

- Understand, memorise, prove, and use a selection of standard limits involving roots, powers and factorials.

Chapter 5

Completeness I

Completeness is the key property of the real numbers that the rational numbers lack. Before examining this property we explore the rational and irrational numbers, discovering that both sets populate the real line more densely than you might imagine, and that they are inextricably entwined.

5.1 Rational Numbers

Definition
A real number is *rational* if it can be written in the form $\frac{p}{q}$, where p and q are integers with $q \neq 0$. The set of rational numbers is denoted by $\mathbb{Q}$. A real number that is not rational is termed *irrational*.

Example $\frac{1}{2}, -\frac{5}{6}, 100, \frac{567877}{-1239}, \frac{8}{2}$ are all rational numbers.

Exercise 1

1. What do you think the letter $\mathbb{Q}$ stands for?
2. Show that each of the following numbers is rational: 0, -10, 2.87, 0.0001, -8^{-9}, 0.6666....
3. Prove that between any two distinct rational numbers there is another rational number.
4. Is there a smallest positive rational number?
5. If a is rational and b is irrational, are $a + b$ and ab rational or irrational? What if a and b are both rational? Or both irrational?

A sensible question to ask at this point is this: are all real numbers rational? In other words, can any number (even a difficult one like π or e) be expressed as a simple fraction if we just try hard enough? For good or ill this is not the case, because, as the Greeks discovered:

Historical Roots
The proof that $\sqrt{2}$ is irrational is attributed to Pythagoras *ca.* $580 - 500$ BC who is well known to have had a triangle fetish.

What does $\sqrt{2}$ have to do with triangles?

Theorem
$\sqrt{2}$ is irrational.

Euler's constant

Euler's γ constant is defined by

$$\gamma = \lim_{n\to\infty}\Big[\sum_{k=1}^{n}\frac{1}{k} - \log n\Big] = 0.5772...$$

It is *not known* whether γ is rational or irrational. It is only known that, if $\gamma = \frac{p}{q}$, q is larger than $10^{10'000}$.

This theorem assures us that *at least one* real number is not rational. You will meet the famous proof of this result in the Foundations course. Later in the course you will prove that e is irrational. The proof that π is irrational is also not hard but somewhat long and you will probably not meet it unless you hunt for it.

We now discover that, despite the fact that some numbers are irrational, the rationals are spread so thickly over the real line that you will find one wherever you look.

Exercise 2

1. Illustrate on a number line those portions of the sets

$$\{m|\, m \in \mathbb{Z}\}, \quad \{m/2|\, m \in \mathbb{Z}\}, \quad \{m/4|\, m \in \mathbb{Z}\}, \quad \{m/8|m \in \mathbb{Z}\}$$

   that lie between ± 3. Is each set contained in the set which follows in the list? What would an illustration of the set $\{m/2^n|\, m \in \mathbb{Z}\}$ look like for some larger positive integer n?

2. Find a rational number which lies between 57/65 and 64/73 and may be written in the form $m/2^n$, where m is an integer and n is a non-negative integer.

Integer Part

If x is a real number then $[x]$, the *integer part* of x, is the unique integer such that

$$[x] \le x < [x] + 1.$$

For example

$$[3.14] = 3 \text{ and } [-3.14] = -4.$$

Open Interval

For $a < b \in \mathbb{R}$, the open interval (a, b) is the set of all numbers *strictly* between a and b: $(a, b) = \{x \in \mathbb{R} : a < x < b\}$

Chalk and Cheese

Though the rationals and irrationals share certain properties, do not be fooled into thinking that they are two-of-a-kind. You will learn later that the rationals are "countable", you can pair them up with the natural numbers. The irrationals, however, are manifestly "uncountable"

Theorem
Between any two distinct real numbers there is a rational number.

I.e. if $a < b$, there is a rational $\frac{p}{q}$ with $a < \frac{p}{q} < b$.

Proof. Consider the set of numbers of the form $\frac{p}{q}$ with q fixed, and p any integer. Assume that there are no such numbers between a and b. Let $\frac{p}{q}$ be the number immediately before a. Then $\frac{p+1}{q}$ is the number immediately after b. We necessarily have

$$\frac{p+1}{q} - \frac{p}{q} \ge b - a \iff \frac{1}{q} \ge b - a.$$

If we choose q sufficiently large, then the above inequality is wrong. Then there is at least one rational number between a and b. ■

Corollary
Let $a < b$. There is an infinite number of rational numbers in the open interval (a, b).

Proof. One can think of many proofs. One could proceed as above, but proving that there are more than N numbers between a and b, for arbitrarily large N. But we can also use the theorem directly. We know that there must be a rational number, say c_1, between a and b. Then there is another rational number, say c_2, between c_1 and b. Then there is c_3, etc... All those numbers are distinct and they are between a and b. ■

We have shown that the rational numbers are spread densely over the real line. What about the irrational numbers?

Exercise 3 Prove that between any two distinct *rational* numbers there is an irrational number. [Hint: Use $\sqrt{2}$ and consider the distance between your two rationals.]

Theorem
Between any two distinct real numbers there is an irrational number.

Proof. We can proceed as in the proof of the previous theorem. Consider the set of numbers of the form $\frac{p}{q} + \sqrt{2}$ with q fixed, and p any integer; all those numbers are irrational. Assume that there are no such numbers between a and b. Let $\frac{p}{q} + \sqrt{2}$ be the number immediately before a. Then $\frac{p+1}{q} + \sqrt{2}$ is the number immediately after b. We necessarily have

$$\frac{p+1}{q} + \sqrt{2} - \big(\frac{p}{q} + \sqrt{2}\big) \geq b - a \quad \Longleftrightarrow \quad \frac{1}{q} \geq b - a.$$

If we choose q sufficiently large, then the above inequality is wrong. Then there is at least one irrational number between a and b. ■

Corollary
Let $a < b$. There is an infinite number of irrational numbers in the open interval (a, b).

Whatever method you used to prove the last corollary will work for this one too. Can you see why?

Is It Love?
We have shown that between any two rationals there is an infinite number of irrationals, and that between any two irrationals there is an infinite number of rationals. So the two sets are intimately and inextricably entwined.

Try to picture the two sets on the real line.

5.2 Least Upper Bounds and Greatest Lower Bounds

Definition
A non-empty set A of real numbers is

bounded above	if there exists U such that $a \leq U$ for all $a \in A$; U is an *upper bound* for A.
bounded below	if there exists L such that $a \geq L$ for all $a \in A$; L is a *lower bound* for A.
bounded	if it is both bounded above *and* below.

Boundless Bounds
If U is an upper bound then so is any number greater than U. If L is a lower bound then so is any number less than L.

Bounds are not unique

Exercise 4 For each of the following sets of real numbers decide whether the set is bounded above, bounded below, bounded or none of these:

1. $\{x : x^2 < 10\}$ 2. $\{x : x^2 > 10\}$ 3. $\{x : x^3 > 10\}$ 4. $\{x : x^3 < 10\}$

Definition
A number u is a *least upper bound* of A if
1. u is an upper bound of A and
2. if U is any upper bound of A then $u \leq U$.

A number l is a *greatest lower bound* of A if
1. l is a lower bound of A and
2. if L is any lower bound of A then $l \geq L$.

The least upper bound of a set A is also called the *supremum* of A and is denoted by $\sup A$, pronounced "soup A".
The greatest lower bound of a set A is also called the *infimum* of A and is denoted by $\inf A$.
Example Let $A = \{\frac{1}{n} : n = 2, 3, 4, \dots\}$. Then $\sup A = 1/2$ and $\inf A = 0$.

Exercise 5 Check that 0 is a lower bound and 2 is an upper bound of each of these sets

1. $\{x|\, 0 \leq x \leq 1\}$ 2. $\{x|\, 0 < x < 1\}$ 3. $\{1 + 1/n|\, n \in \mathbb{N}\}$
4. $\{2 - 1/n|\, n \in \mathbb{N}\}$ 5. $\{1 + (-1)^n/n|\, n \in \mathbb{N}\}$ 6. $\{q|\, q^2 < 2,\ q \in \mathbb{Q}\}$.

For which of these sets can you find a lower bound greater than 0 and/or an upper bound less that 2? Identify the greatest lower bound and the least upper bound for each set.

Can a least upper bound or a greatest lower bound for a set A belong to the set? Must a least upper bound or a greatest lower bound for a set A belong to the set?

We have been writing *the* least upper bound so there had better be only one.

Exercise 6 Prove that a set A can have at most *one* least upper bound.

5.3 Axioms of the Real Numbers

Despite their exotic names, the following fundamental properties of the real numbers will no doubt be familiar to you. They are listed below. Just glimpse through them to check they are well known to you.

- For $x, y \in \mathbb{R}$, $x + y$ is a real number

  closure under addition

- For $x, y, z \in \mathbb{R}$, $(x + y) + z = x + (y + z)$

  associativity of addition

- For $x, y \in \mathbb{R}$, $x + y = y + x$

  commutativity of addition

- There exists a number 0 such that for $x \in \mathbb{R}$, $x + 0 = x = 0 + x$

  existence of an additive identity

- For $x \in \mathbb{R}$ there exists a number $-x$ such that $x + (-x) = 0 = (-x) + x$

  existence of additive inverses

- For $x, y \in \mathbb{R}$, xy is a real number

  closure under multiplication

- For $x, y, z \in \mathbb{R}$, $(xy)z = x(yz)$

  associativity of multiplication

- For $x, y \in \mathbb{R}$, $xy = yx$

  commutativity of multiplication

- There exists a number 1 such that $x \cdot 1 = x = 1 \cdot x$ for all $x \in \mathbb{R}$.

  existence of multiplicative identity

- For $x \in \mathbb{R}, x \neq 0$ there exists a number x^{-1} such that $x \cdot x^{-1} = 1 = x^{-1} \cdot x$

  existence of multiplicative inverses

- For $x, y, z \in \mathbb{R}$, $x(y + z) = xy + xz$

  distributive law

- For $x, y \in \mathbb{R}$, exactly one of the following statements is true: $x < y$, $x = y$ or $x > y$

  trichotomy

- For $x, y, z \in \mathbb{R}$, if $x < y$ and $y < z$ then $x < z$

  transitivity

- For $x, y, z \in \mathbb{R}$, if $x < y$ then $x + z < y + z$

  adding to an inequality

- For $x, y, z \in \mathbb{R}$, if $x < y$ and $z > 0$ then $zx < zy$

  multiplying an inequality

There is one last axiom, without which the reals would not behave as expected:

Completeness Axiom
Every non-empty subset of the reals that is bounded above has a least upper bound.

If you lived on a planet where they only used the rational numbers then all the axioms would hold *except* the completeness axiom. The set $\{x \in \mathbb{Q} : x^2 \leq 2\}$ has rational upper bounds $1.5, 1.42, 1.415, \ldots$ but no rational least upper bound.

Of course, living in the reals we can see that the least upper bound is $\sqrt{2}$. This sort of problem arises because the rationals are riddled with holes and the completeness axiom captures our intuition that the real line has no holes in it - it is complete.

Exercise 7 If A and B denote bounded sets of real numbers, how do the numbers $\sup A$, $\inf A$, $\sup B$, $\inf B$ relate if $B \subset A$?

Give examples of unequal sets for which $\sup A = \sup B$ and $\inf A = \inf B$.

The following property of the supremum is used frequently throughout Analysis.

Possible Lack of Attainment

Notice that $\sup A$ and $\inf A$ need not be elements of A.

Lemma
Suppose a set A is non-empty and bounded above. For every $\epsilon > 0$, there exists $a \in A$ such that

$$\sup A - \epsilon < a \leq \sup A.$$

Proof. *Ab absurdo.* If the lemma is wrong, then there exists $\varepsilon > 0$ such that the interval $(\sup A - \varepsilon, \sup A]$ contains no number of A. Since A has no number greater than $\sup A$, that means that all numbers of A are less (or equal) than $\sup A - \varepsilon$. Then $\sup A - \varepsilon$ is an upper bound for A. It is smaller than $\sup A$, which contradicts the fact that $\sup A$ is the least upper bound. ■

Exercise 8 Suppose A is a non-empty set of real numbers which is bounded below. Define the set $-A = \{-a : a \in A\}$.

1. Sketch two such sets A and $-A$ on the real line. Notice that they are reflected about the origin. Mark in the position of $\inf A$.

2. Prove that $-A$ is a non-empty set of real numbers which is bounded below, and that $\sup(-A) = -\inf A$. Mark $\sup(-A)$ on the diagram.

Different Versions of Completeness

This Theorem has been named 'Greatest lower bounds *version*' because it is an equivalent version of the Axiom of Completeness. Between now and the end of the next chapter we will uncover 5 more versions!

Theorem *Greatest lower bounds version*
Every non-empty set of real numbers which is bounded below has a greatest lower bound.

Proof. Suppose A is a non-empty set of real numbers which is bounded below. Then $-A$ is a non-empty set of real numbers which is bounded above. The completeness axiom tells us that $-A$ has a least upper bound $\sup(-A)$. From Exercise 8 we know that $A = -(-A)$ has a greatest lower bound, and that $\inf A = -\sup(-A)$. ■

5.4 Consequences of Completeness - Bounded Monotonic Sequences

The mathematician Weierstrass was the first to pin down the ideas of completeness in the 1860's and to point out that all the deeper results of analysis are based upon completeness. The most immediately useful consequence is the following theorem:

Theorem *Increasing sequence version*
Every bounded increasing sequence is convergent.

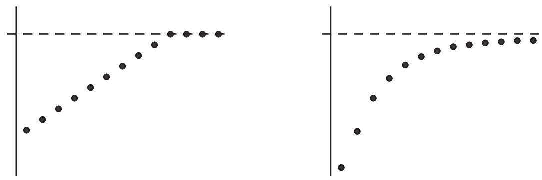


Figure 5.1: Bounded increasing sequences must converge.

Figure 5.1 should make this reasonable. Plotting the sequence on the real line as the set $A = \{a_1, a_2, a_3, \dots\}$ we can guess that the limit should be $\sup A$.
Proof. Let (a_n) be a bounded increasing sequence. We show that $a_n \to \sup A$. Let ε be any positive number. By the above lemma, there exists $a_N \in A$ such that $\sup A - \varepsilon < a_N \le \sup A$. Since (a_n) is increasing, we have

$$\sup A - \varepsilon < a_n \le \sup A$$

for all $n > N$. Then $|a_n - \sup A| < \varepsilon$. This holds for every $\varepsilon > 0$, so that $a_n \to \sup A$. ■

Check that your proof has used the completeness axiom, the fact that the sequence is increasing, and the fact that the sequence is bounded above. If you have not used each of these then your proof must be wrong!

Corollary *Decreasing sequence version*
Every bounded decreasing sequence is convergent.

Proof. The sequence $(-a_n)$ is bounded and increasing, then it converges to a number $-a$. Then $a_n \to a$ by the theorem of Section 2.6. ■

Example In Chapter 3, we considered a recursively defined sequence (a_n) where

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = \sqrt{a_n + 2}.$$

We showed by induction that $a_n \ge 1$ for all n (because $a_1 = 1$ and $a_k \ge 1 \implies a_{k+1} = \sqrt{a_k + 2} \ge \sqrt{3} \ge 1$) and that $a_n \le 2$ for all n (because $a_1 \le 2$ and $a_k \le 2 \implies a_{k+1} = \sqrt{a_k + 2} \le \sqrt{4} = 2$). So (a_n) is bounded.

We now show that the sequence is increasing.

$$a_n^2 - a_n - 2 = (a_n - 2)(a_n + 1) \le 0 \text{ since } 1 \le a_n \le 2$$
$$\therefore a_n^2 \le a_n + 2$$
$$\therefore a_n \le \sqrt{a_n + 2} = a_{n+1}.$$

Decreasing?
To see whether a sequence (a_n) is decreasing, try testing

$$a_{n+1} - a_n \le 0$$

or, when terms are positive,

$$\frac{a_{n+1}}{a_n} \le 1.$$

Hence (a_n) is increasing and bounded. It follows from Theorem 5.4 that (a_n) is convergent. Call the limit a. Then $a^2 = \lim_{n\to\infty} a_{n+1}^2 = \lim_{n\to\infty} a_n + 2 = a + 2$ so that $a^2 - a - 2 = 0 \implies a = 2$ or $a = -1$. Since $(a_n) \in [1, 2]$ for all n we know from results in Chapter 3 that $a \in [1, 2]$, so the limit must be 2.

Exercise 9 Consider the sequence (a_n) defined by

$$a_1 = \frac{5}{2} \quad \text{and} \quad a_{n+1} = \frac{1}{5}\left(a_n^2 + 6\right).$$

Show by induction that $2 < a_k < 3$. Show that (a_n) is decreasing. Finally, show that (a_n) is convergent and find its limit.

Exercise 10 Explain why every monotonic sequence is either bounded above or bounded below. Deduce that an increasing sequence which is bounded above is bounded, and that a decreasing sequence which is bounded below is bounded.

Exercise 11 If (a_n) is an increasing sequence that is *not* bounded above, show that $(a_n) \to \infty$. Make a rough sketch of the situation.

The two theorems on convergence of bounded increasing or decreasing sequences give us a method for showing that monotonic sequences converge even though we may not know what the limit is.

5.5 * Application - k^{th} Roots *

So far, we have taken it for granted that every positive number a has a unique positive k^{th} root, that is there exists $b > 0$ such that $b^k = a$, and we have been writing $b = a^{1/k}$. But how do we know such a root exists? We now give a careful proof. Note that even square roots do not exist if we live just with the rationals - so any proof must use the Axiom of Completeness.

Stop Press
$\sqrt{2}$ exists!!!
Mathematicians have at last confirmed that $\sqrt{2}$ is really there.

Phew! What a relief.

Theorem
Every positive real number has a unique positive k^{th} root.

Suppose a is a positive real number and k is a natural number. We wish to show that there exists a unique positive number b such that $b^k = a$. The idea of the proof is to define the set $A = \{x > 0 : x^k > a\}$ of numbers that are too big to be the k^{th} root. The infimum of this set, which we will show to exist by the greatest lower bound characterisation of completeness in this chapter, *should* be the k^{th} root. We must check this.

Note that the greatest lower bound characterisation is an immediate consequence of the completeness axiom. It is indeed equivalent to the completeness axiom, and some authors give it as the completeness axiom.

Fill in the gaps in the following proof:

Exercise 12 Show that the set A is non-empty [Hint: Show that $1 + a \in A$].

By definition the set A is bounded below by 0. So the greatest lower bound characterisation of completeness implies that $b = \inf A$ must exist. Argue that for each natural number n there exists $a_n \in A$ such that $b \leq a_n < b + \frac{1}{n}$.

Exercise 13 Show that $a_n^k \to b^k$ and conclude that $b^k \geq a$.

We will now show that $b^k \leq a$, by contradiction. Assume $b^k > a$. Then $0 < \frac{a}{b^k} < 1$ so we may choose $\delta > 0$ so that $\delta < \frac{b}{k}\left(1 - \frac{a}{b^k}\right)$.

Exercise 14 Now achieve a contradiction by showing that $b - \delta \in A$. (Hint: use Bernoulli's Inequality.)

We have shown that $b^k = a$. Prove that there is no other positive k^{th} root.

Arbitrary Exponents

The existence of n^{th} roots suggests one way to define the number a^x when $a > 0$ and x is *any* real number.
If $x = m/n$ is rational and $n \geq 1$ then

$$a^x = \left(a^{1/n}\right)^m$$

If x is irrational then we know there is a sequence of rationals (x_i) which converges to x. It is possible to show that the sequence (a^{x_i}) also converges and we can try to define:

$$a^x = \lim_{i \to \infty} a^{x_i}$$

Check Your Progress
By the end of this chapter you should be able to:

- Prove that there are an infinite number of rationals and irrationals in every open interval.
- State and understand the definitions of least upper bound and greatest lower bound.
- Calculate $\sup A$ and $\inf A$ for sets on the real line.
- State and use the Completeness Axiom in the form "every non-empty set A which is bounded above has a least upper bound ($\sup A$)".

Chapter 6

Completeness II

6.1 An Interesting Sequence

Exercise 1 Consider the sequence $a_n = \left(1 + \frac{1}{n}\right)^n$. Show that $\frac{a_{n+1}}{a_n} = \left(1 + \frac{1}{n+1}\right)\left(1 - \frac{1}{(n+1)^2}\right)^n$ and then use Bernoulli's inequality to show that $a_{n+1} \geq a_n$.

Show that $\left(1 + \frac{1}{2n}\right)^n = \frac{1}{\left(1 - \frac{1}{2n+1}\right)^n}$ and then use Bernoulli's inequality to show that $\left(1 + \frac{1}{2n}\right)^n \leq 2$. Hence show that (a_{2n}) is bounded. Using the fact that (a_n) is increasing, show that it is bounded and hence convergent.

Exercise 2 Show that $\left(1 - \frac{1}{n}\right) = \frac{1}{\left(1 + \frac{1}{n-1}\right)}$ and hence that $\lim_{n\to\infty}\left(1 - \frac{1}{n}\right)^n$ exists.

Exercise 3 Criticise the following argument: $\left(1 + \frac{1}{n}\right)^n \to (1)^n = 1$.

> $e = 2.718...$
>
> Newton showed already in 1665 that
>
> $$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$
>
> The constant was named by Euler, who proved that
>
> $$e = \lim_{n\to\infty} (1 + \tfrac{1}{n})^n.$$
>
> It is also the limit of $n/\sqrt[n]{n!}$ (compare with Stirling's formula!). It is known that e is irrational (Euler, 1737) and transcendental (Hermite, 1873).

6.2 Consequences of Completeness - General Bounded Sequences

We showed in Chapter 3 that every subsequence of a bounded sequence is bounded. We also saw that every sequence has a monotonic subsequence (see Section 3.4). We can now tie these facts together.

Exercise 4

1. Find an upper bound and a lower bound for the sequences with n^{th} term

$$(a)\ (-1)^n, \qquad (b)\ (-1)^n\left(1 + \frac{1}{n}\right).$$

   Is either sequence convergent? In each case find a convergent subsequence. Is the convergent subsequence monotonic?

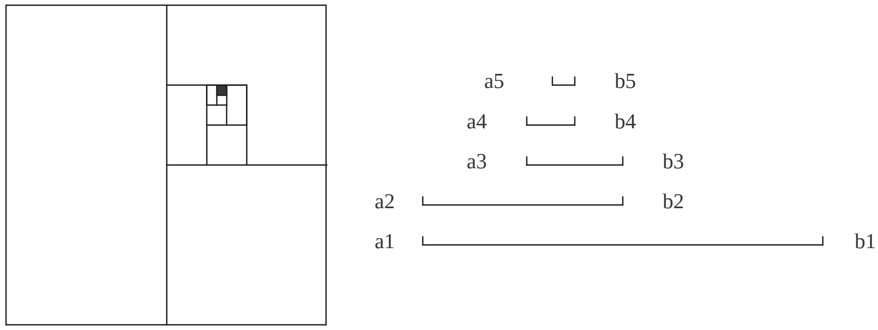


Figure 6.1: Lion Hunting.

2. Look back at your proof that every convergent sequence is bounded (Workbook 3). Is it true that every bounded sequence is convergent?

Theorem *Bolzano-Weierstrass*
Every bounded sequence has a convergent subsequence.

Example The weird, oscillating sequence $(\sin n)$ is far from being convergent. But, since $-1 \leq \sin n \leq 1$, we are guaranteed that it has a convergent subsequence.

Proof. Recall the theorem of Section 3.4: every sequence has a monotonic subsequence. If the sequence is bounded, the subsequence is also bounded, and it converges by the theorem of Section 5.4. ■

There is another method of proving the Bolzano-Weierstrass theorem called Lion Hunting - a technique useful elsewhere in analysis. The name refers to a method trapping a lion hiding in a square jungle. Build a lion proof fence dividing the jungle in half. Shortly, by listening for screams, it will be apparent in which half the lion is hiding. Build a second fence dividing this region in half. Repeating this procedure quickly traps the lion in a manageable area, see figure 6.1.

We use this idea to find a limit point for a sequence on the real line. We will illustrate this on a sequence (x_n) all of whose values lie in $[a_1, b_1] = [0, 1]$. At least one of the two intervals $[0, 1/2]$ and $[1/2, 1]$ must contain infinitely many of the points of (x_n). Choosing this half (or choosing at random if both contain infinitely many points) we label it as the interval $[a_2, b_2]$. Then we split this interval into two and we can find one of these halves which contains infinitely many of the points (x_n), and we label it $[a_3, b_3]$. We continue in this way: at the k^{th} step we start with an interval $[a_k, b_k]$ containing infinitely many points. One of the intervals $\left[a_k, \frac{a_k+b_k}{2}\right]$ or $\left[\frac{a_k+b_k}{2}, b_k\right]$ still has infinitely many points and we label this as $[a_{k+1}, b_{k+1}]$.

Exercise 5 Explain why (a_n) and (b_n) converge to a limit L. Explain why it is possible to find a subsequence (x_{n_i}) so that $x_{n_k} \in [a_k, b_k]$ and show that this subsequence is convergent.

6.3 Cauchy Sequences

Recall the notion of convergence: $a_n \to a$ if and only if for every $\varepsilon > 0$, there exists N such that $|a_n - a| < \varepsilon$ for all $n > N$. This definition has one drawback, namely that we need to know a in order to prove that the sequence converges. In Chapter 5 we found a criterion for convergence that does not involve the actual limit:

Convergence Test
A monotonic sequence converges if and only if it is bounded.

Exercise 6 Have you proved both the "if" and the "only if" parts of this test?

Is there a similar test that works for general non-monotonic sequences?

Exercise 7 Cleverclog's Test says that a sequence converges if and only if $a_{n+1} - a_n \to 0$. Give an example to show that Cleverclog's test is completely false (alas).

There is a test for convergence of a general sequence, which does not involve the limit, which we shall discover in this section.

Definition
A sequence (a_n) has the *Cauchy property* if, for each $\epsilon > 0$ there exists a natural number N such that $|a_n - a_m| < \epsilon$ for all $n, m > N$.

We use the shorthand "a Cauchy sequence" for a sequence with the Cauchy property. In words, the Cauchy property means that for any positive ϵ, no matter how small, we can find a point in the sequence beyond which any two of the terms are less than ϵ apart. So the terms are getting more and more "clustered" or "crowded".

Example $\left(\frac{1}{n}\right)$ is a Cauchy sequence. Fix $\epsilon > 0$. We have to find a natural number N such that if $n, m > N$ then

$$\left|\frac{1}{n} - \frac{1}{m}\right| < \epsilon.$$

But

$$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \left|\frac{1}{n}\right| + \left|\frac{1}{m}\right| = \frac{1}{n} + \frac{1}{m}.$$

Thus, if $\frac{1}{n} < \frac{\epsilon}{2}$ and $\frac{1}{m} < \frac{\epsilon}{2}$ we will have what we need. These two conditions hold when both n and m are greater than $\frac{2}{\epsilon}$. Hence we choose N to be a natural number with $N > \frac{2}{\epsilon}$. Then we have, for $n, m > N$

$$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \left|\frac{1}{n}\right| + \left|\frac{1}{m}\right| = \frac{1}{n} + \frac{1}{m} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This shows that $\left(\frac{1}{n}\right)$ is a Cauchy sequence.

Exercise 8 Suppose $(a_n) \to a$. Show that $|a_n - a_m| \leq |a_n - a| + |a - a_m|$. Use this fact to prove that (a_n) is Cauchy.

This shows that every convergent sequence is Cauchy.

The beauty of the Cauchy property is that it is sufficient to ensure the convergence of a sequence - without having to know or show just what the limit is.

Big Bangers

Bolzano (1781-1848), Cauchy (1789-1857) and Weierstrass (1815-1897) all helped fuel the analytical Big Bang of the 19^{th} century. Both the Bolzano-Weierstrass Theorem and the theorem stating that every Cauchy sequence converges were discovered by Bolzano, a humble Czech priest. But it took Weierstrass and Cauchy to broadcast them to the world.

Theorem
Every Cauchy sequence is convergent.

Exercise 9 Let (a_n) be a Cauchy sequence. By putting $\varepsilon = 1$ in the Cauchy criterion prove that every Cauchy sequence is bounded. Now use the Bolzano-Weierstrass Theorem together with the identity

$$|a_n - a| \leq |a_n - a_{n_i}| + |a_{n_i} - a|$$

to prove that every Cauchy sequence is convergent.

Combining the last two results we have the following general test:

Convergence Test
A sequence is convergent if and only if it has the Cauchy property.

The previous theorem will be one of the most used results in your future analysis courses. Here we give only one application. A sequence (a_n) is called *strictly contracting* if for some number $0 < l < 1$, called the contraction factor,

$$|a_{n+1} - a_n| \leq l\,|a_n - a_{n-1}| \text{ for all } n = 1, 2, 3, \ldots$$

In words, the distances between successive terms are decreasing by a factor l.

Exercise 10 Define a sequence by $a_0 = 1$ and $a_{n+1} = \cos(a_n/2)$. Use the inequality $|\cos(x) - \cos(y)| \leq |x - y|$ (which you may assume) to show that (a_n) is strictly contracting with contracting factor $l = 1/2$.

Exercise 11 The aim of this question is to show that a strictly contracting sequence (a_n) is Cauchy. Show by induction on n that $|a_{n+1} - a_n| \leq |a_1 - a_0| l^n$. Then suppose that $n > m$ and use the triangle inequality in the form:

$$|a_n - a_m| \leq |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \ldots |a_{m+1} - a_m|$$

to show that (a_n) is Cauchy.

If we apply this to the contracting sequence found in Assignment 10 we see that the sequence given by $a_0 = 1$ and $a_{n+1} = \cos(a_n/2)$ defines a Cauchy sequence. So by the previous theorem it must converge, say $(a_n) \to a$.

Exercise 12 Using the sequence and inequality given in Exercise 10, show that $\cos(a_n/2) \to \cos(a/2)$. Hence show that the sequence (a_n) converges to the unique solution of $x = \cos(x/2)$.

6.4 The Many Faces of Completeness

We have proved that the results below (except for the *infinite decimal sequences version* which is proved in the next section) are all consequences of the Axiom of Completeness. In fact, all of them are logically equivalent to this Axiom and to each other. This means you can prove any one of them from any other of them. So any one of them can be used as an alternative formulation of the Completeness Axiom and indeed you will find many books that use one of the results 1,2,3,4,5 or 6 as their axiom of completeness.

Completeness Axiom
Every non-empty set A of real numbers which is bounded above has a least upper bound, called $\sup A$.

Equivalent Conditions 1. 2. and 3. were proved in Chapter 5 (Completeness I).

1. Every non-empty set A of real numbers which is bounded below has a *greatest* lower bound, called $\inf A$.
2. Every bounded increasing sequence is convergent.
3. Every bounded decreasing sequence is convergent.
4. Every bounded sequence has a convergent subsequence.
5. Every Cauchy sequence is convergent.
6. Every infinite decimal sequence is convergent.

6.5 * Application - Classification of Decimals *

In this section we are going to take a close look at decimal representations for real numbers. We use expansions in base 10 (why?) but most of the results below hold for other bases: binary expansions (base 2) or ternary expansions (base 3) ...

Exercise 13

Dotty Notation

Don't forget the notation for repeating decimals:
A single dot means that that digit is repeated forever, so that $0.82\dot{3}$ stands for the infinite decimal $0.8233333\ldots$.
Two dots means that the sequence of digits between the dots is repeated forever, so $1.8\dot{2}4\dot{3}$ stands for $1.8243243243243\ldots$.

1. Write each of the fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{8}$, $\frac{1}{9}$, $\frac{1}{10}$, as a decimal. Which of them have finite decimal representations?

2. Find the decimal representation of $\frac{1}{17}$. Is your answer exacly $\frac{1}{17}$?

The easiest decimal representations are the finite ones - the ones that have only a finite number of decimal places, like 342.5017. A positive finite decimal has the form $d_0.d_1d_2\dots d_n$ where d_0 is a non-negative integer and each of the $d_1, d_2, \dots, d_n$ is one of the integers $0, 1, 2, \dots, 9$. Then $d_0.d_1d_2\dots d_n$ is defined to be the number:

$$d_0 + \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n}$$

or, written more compactly, $\sum_{j=0}^{n} d_j 10^{-j}$.

Exercise 14 What changes are needed when defining a negative finite decimal?

The definition of an infinite decimal requires a bit more care.

Two too many

Don't get confused between the sequence (d_n) and the sequence of sums

$$\left(\sum_{j=0}^{n} d_j 10^{-j}\right)$$

The sequence (d_n) consists of the *digits* of the decimal number. The sequence of sums is the sequence of which we take the limit.

Definition
A positive real number x has a representation as an *infinite decimal* if there is a non-negative integer d_0 and a sequence (d_n) with $d_n \in \{0, 1, \dots, 9\}$ for each n, such that the sequence with n^{th}term defined by:

$$d_0 + \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n} = \sum_{j=0}^{n} d_j 10^{-j}$$

converges to x. In this case, we write:

$$x = d_0.d_1d_2d_3\dots$$

A negative real number x has a representation as the *infinite decimal* $(-d_0).d_1d_2d_3d_4\dots$ if $-x$ has a representation as the infinite decimal $d_0.d_1d_2d_3d_4\dots$.

Example Writing $\pi = 3.1415926\dots$ means that π is the limit of the sequence (3.1, 3.14, 3.141, 3.1415, ...).

We could equally have said that the decimal expansion $d_0.d_1d_2d_3\dots$ with $d_0 \geq 0$, represents a real number x if the sequence of sums $\left(\sum_{k=0}^{n} d_k 10^{-k}\right)$ converges to x.

It is almost obvious that every real number has a decimal representation. For example, if x is positive we can find the decimal digits as follows. Define d_0

to be the largest integer less than or equal to x. Then define iteratively:

$$\begin{aligned}
d_1 &= \max\{j : d_0 + \frac{j}{10} \le x\} \\
d_2 &= \max\{j : d_0 + \frac{d_1}{10} + \frac{j}{10^2} \le x\} \\
\dots \\
d_n &= \max\{j : d_0 + \frac{d_1}{10} + \frac{d_2}{10^2} + \dots + \frac{d_{n-1}}{10^{n-1}} + \frac{j}{10^n} \le x\}
\end{aligned}$$

It is easy to check that each digit is in $\{0, 1, 2, \dots, 9\}$. Moreover, after N digits we must have $x - \frac{1}{10^N} < \sum_{n=0}^{N} d_n 10^{-n} \le x$ so that x is the limit of the sequence of sums by the Sandwich Rule.

Up and Down

Notice that every *non-negative* infinite decimal is the limit of *increasing* finite decimals, because you are always adding an additional non-negative term as you go.
However, every *negative* infinite decimal is the limit of *decreasing* finite decimals.

*** Consequences of Completeness for Decimals ***

At the moment, whenever we talk about a decimal expansion, $d_0.d_1d_2d_3\dots$, we need to show that the sequence of sums converges. What would be useful is a theorem to state that this sequence *always* converges.

Theorem *Infinite decimal sequences version*
Every infinite decimal $\pm d_0.d_1d_2d_3\dots$ represents a real number.

Exercise 15 Check that the sequence of sums is monotonic and bounded. Use the bounded increasing sequence version of the Completeness Axiom to show that the infinite decimal represents a real number.

The Decimal Dream

We know that every time we write down a list of decimal digits

$$d_0.d_1d_2d_3d_4\dots$$

we succeed in defining a real number.

This is result 6 from earlier in the workbook. With this result, our analysis of completeness is complete.

*** Is $0.999\dots$ Equal to 1? ***

Although we have finished our examination of completeness, there are still some things we can do with decimals.

Example What is $0.12\dot{1}\dot{2}$?

For this decimal, we have $d_0 = 0$ and the sequence (d_n) is defined by $d_{2n} = 2$ and $d_{2n+1} = 1$. Then the sequence of sums is:

$$\left(\sum_{j=0}^{n} d_j \times 10^{-j}\right)$$

We know that this converges and thus to find the limit it is sufficient to find the limit of a subsequence of the sequence of sums. The subsequence we choose is that of the even terms. This is given by:

$$\begin{aligned}
\sum_{j=0}^{2n} d_j \times 10^{-j} &= \sum_{k=1}^{n} \left(d_{2k-1}10^{-2k+1} + d_{2k}10^{-2k}\right) \\
&= \sum_{k=1}^{n} (1 \times 10 + 2) \times 10^{-2k} \\
&= \sum_{k=1}^{n} \frac{12}{100^k} \\
&= \frac{12}{100} \sum_{k=0}^{n-1} \frac{1}{100^k} \\
&= \frac{12}{100} \left(\frac{1 - \left(\frac{1}{100}\right)^n}{1 - \frac{1}{100}} \right) \\
&= \frac{12}{99} \left(1 - \left(\frac{1}{100}\right)^n \right)
\end{aligned}$$

and we can see that this converges to $\frac{12}{99}$. Thus $0.12\dot{1}\dot{2} = \frac{12}{99}$.

Exercise 16 Prove that $0.999\dot{9} = 1$.

This last exercise already shows one of the annoying features of decimals. You can have two *different* decimal representations for the *same* number. Indeed, any number with a finite decimal representation also has a representation as a decimal with recurring 9's, for example 2.15 is the same as 2.1499999. . . .

Theorem
Suppose a positive real number has two different representations as an infinite decimal. Then one of these is finite and the other ends with a recurring string of nines.

Proof. Suppose a positive real number x has two decimal representations $a_0.a_1a_2a_3\ldots$ and $b_0.b_1b_2b_3\ldots$ and that the decimal places agree until the N^{th} place where $a_N < b_N$. Then:

$$
\begin{aligned}
x &= \lim_{n\to\infty} \sum_{k=0}^{n} a_k 10^{-k} \\
&= \sum_{k=0}^{N} a_k 10^{-k} + \lim_{n\to\infty} \sum_{k=N+1}^{n} a_k 10^{-k} \\
&\le \sum_{k=0}^{N} a_k 10^{-k} + \lim_{n\to\infty} \sum_{k=N+1}^{n} 9 \times 10^{-k} \\
&= \sum_{k=0}^{N} a_k 10^{-k} + \frac{9}{10^{N+1}} \lim_{n\to\infty} \frac{1-\left(\frac{1}{10^{n-N}}\right)}{1-\frac{1}{10}} \\
&= \sum_{k=0}^{N} a_k 10^{-k} + \frac{1}{10^N} \\
&= \sum_{k=0}^{N-1} a_k 10^{-k} + (a_N + 1)10^{-N} \\
&\le \sum_{k=0}^{N} b_k 10^{-k} && \text{as } a_N < b_N \\
&\le \lim_{n\to\infty} \sum_{k=0}^{n} b_k 10^{-k} && \text{as the sequence is increasing} \\
&= x
\end{aligned}
$$

■

Since we started and ended with the number x, the above inequalities must all be equalities. So it is possible to have two decimal representations for x provided all the decimal digits a_n are 9 when $n > N$ and all the digits b_n are 0 when $n > N$ and $b_N = a_N + 1$. If even a single one of these digits fails to be a 9 (respectively a 0) then the above chain of inequalities becomes a strict inequality and we reach the contradiction $x < x$.

Recurring Nines

The problem of non-uniqueness of decimal representations is annoying but not too bad. In many problems we can just agree to use one of the two representations - for instance by banning any representation that has recurring nines.

*** Classifying Decimals ***

We now classify decimals into three types.

Definition
An infinite decimal $\pm d_0.d_1d_2d_3d_4\ldots$ is

terminating	if it ends in repeated zeros i.e. there exists N such that $d_n = 0$ whenever $n > N$.
recurring	if it eventually repeats itself i.e. there exist N and r such that $d_n = d_{n+r}$ whenever $n > N$.
non-recurring	if it is neither terminating nor recurring

Examples

- 532.89764 is terminating.
- $0.333\dot{3}$ is recurring.
- $3.1415\ldots$, the decimal expansion of π, is nonrecurring.

You can see that a terminating decimal is really just a finite decimal in disguise. It is also an example of a recurring decimal, since it ends with a string of repeated zeros.

*** Terminating Decimals ***

Exercise 17 Suppose $x = p/q$ for integers p, q where the only prime factors of q are 2's and 5's. Show that x has a terminating decimal representation. [Hint: show that $x = p'/10^n$ for some integer p' and some $n \geq 0$.]

Exercise 18 Show that if x has a terminating decimal expansion then $x = p/q$ for integers p, q where the only prime factors of q are $2's$ and $5's$.

Together the last two exercises have shown the following theorem:

Theorem *Characterisation of terminating decimals*
A number x can be represented by a terminating decimal if and only if $x = p/q$ for integers p, q where the only prime factors of q are 2's and 5's.

*** Recurring Decimals ***

Exercise 19 Express the recurring decimal $1.23\dot{4}5\dot{6}$ as a fraction.

Working through this example should convince you of the following.

Theorem
Every recurring decimal represents a rational number.

Exercise 20 To show this, suppose that x has a decimal representation that has recurring blocks of length k. Explain why $10^k x - x$ must have a terminating decimal representation. Now use the characterisation of terminating decimals to show that $x = \frac{p}{q(10^k-1)}$ for some integers p, q where q has no prime factors except 2's and 5's.

Corollary
A recurring decimal x with repeating blocks of length k can be written as $x = \frac{p}{q(10^k - 1)}$ where the only prime factors of q are 2's or 5's.

Exercise 21

1. Express $\frac{333}{22}$ as a recurring decimal.

2. Use long division to express $\frac{1}{7}$ as a recurring decimal. Write out the long division sum explicitly (don't use a calculator). In your long division circle the remainders after each subtraction step. Are all the possible remainders 0,1,2,3,4,5,6 involved? How long is the repeating block?

3. Use long division to express $\frac{1}{13}$ as a recurring decimal. In your long division circle the remainders after each subtraction step. Are all the possible remainders $0, 1, 2, \ldots, 11, 12$ involved? How long is the repeating block?

The exercise above should convince you of the following result:

Theorem
Every rational number can be represented by a *recurring* infinite decimal or a terminating infinite decimal.

*** Complete Classification ***

We now have a complete understanding of recurring decimals. Recurring decimals represent rationals and rationals always have recurring decimal representations. What about non-recurring decimals? Since every number has a decimal representation, it follows that any irrational number must have a non-recurring infinite decimal representation.

Theorem
Every real number has a decimal representation and every decimal represents a real number.
The *rationals* are the set of terminating or recurring decimals.
The *irrationals* are the set of non-recurring decimals.
If a number has two distinct representations then one will terminate and the other will end with a recurring string of nines.

Check Your Progress
By the end of this chapter you should be able to:

- Prove and use the fact that every bounded increasing sequence is convergent.
- Prove and use the fact that every bounded decreasing sequence is convergent.
- Prove the Bolzano-Weierstrass Theorem: that every bounded sequence of real numbers has a convergent subsequence.
- State, use, and understand the definition of a Cauchy sequence.
- Prove that a sequence of real numbers is convergent if and only if it is Cauchy.
- Understand and be able to use the definition of a decimal expansion.

Chapter 7

Series I

7.1 Definitions

We saw in the last booklet that decimal expansions could be defined in terms of sequences of sums. Thus a decimal expansion is like an infinite sum. This is what we shall be looking at for the rest of the course.

Our first aim is to find a good definition for summing infinitely many numbers. Then we will investigate whether the rules for finite sums apply to infinite sums.

Exercise 1 What has gone wrong with the following argument? Try putting $x = 2$.

> If $S = 1 + x + x^2 + \dots,$
> then $xS = x + x^2 + x^3 + \dots,$
> so $S - xS = 1,$
> and therefore $S = \frac{1}{1-x}$.

If the argument were correct then we could put $x = -1$ to obtain the sum of the series

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - \dots$$

as $1/2$. But the same series could also be be thought of as

$$(1 - 1) + (1 - 1) + (1 - 1) + \dots$$

with a sum of 0, or as

$$1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + \dots$$

with a sum of 1. This shows us that great care must be exercised when dealing with infinite sums.

We shall repeatedly use the following convenient notation for finite sums: given integers $0 \le m \le n$ and numbers $(a_n : n = 0, 1, \dots)$ we define

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \dots + a_n$$

Example $1+4+9+\cdots+100=\sum_{k=1}^{10} k^2$

Exercise 2 Express the following sums using the $\sum$ notation:

$$1.\ \frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\cdots+\frac{1}{3628800} \quad 2.\ \frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\ldots\frac{7}{128}$$

Exercise 3 Show that $\sum_{k=1}^{n} a_{k-1}=\sum_{k=0}^{n-1} a_k$.

Exercise 4

1. By decomposing $1/r(r+1)$ into partial fractions, or by induction, prove that
$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=1-\frac{1}{n+1}.$$
Write this result using $\sum$ notation.

2. If
$$s_n=\sum_{r=1}^{n}\frac{1}{r(r+1)},$$
prove that $(s_n)\to 1$ as $n\to\infty$. This result could also be written as
$$\sum_{r=1}^{\infty}\frac{1}{r(r+1)}=1.$$

Serious Sums

The problem of how to deal with infinite sums vexed the analysis of the early 19th century. Some said there wasn't a problem, some pretended there wasn't until inconsistencies in their own work began to unnerve them, and some said there was a terrible problem and why wouldn't anyone listen? Eventually, everyone did.

A *series* is an expression of the form $\sum_{n=1}^{\infty} a_n=a_1+a_2+a_3+a_4+\ldots$. As yet, we have not defined what we mean by such an infinite sum. To get the ball rolling, we consider the "partial sums" of the series:

$$\begin{aligned}
s_1 &= a_1\\
s_2 &= a_1+a_2\\
s_3 &= a_1+a_2+a_3\\
\vdots\ & \quad \vdots\\
s_n &= a_1+a_2+a_3+\cdots+a_n
\end{aligned}$$

To have any hope of computing the infinite sum $a_1+a_2+a_3+\ldots$, then the partial sums s_n should represent closer and closer approximations as n is chosen larger and larger. This is just an informal way of saying that the infinite sum $a_1+a_2+a_3+\ldots$ ought to be the limit of the sequence of partial sums.

Definition
Consider the series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$ with partial sums (s_n), where

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i.$$

We say:

1. $\sum_{n=1}^{\infty} a_n$ *converges* if (s_n) converges. If $s_n \to S$ then we call S the sum of the series and we write $\sum_{n=1}^{\infty} a_n = S$.
2. $\sum_{n=1}^{\infty} a_n$ *diverges* if (s_n) does not converge.
3. $\sum_{n=1}^{\infty} a_n$ *diverges to infinity* if (s_n) tends to infinity.
4. $\sum_{n=1}^{\infty} a_n$ *diverges to minus infinity* if (s_n) tends to minus infinity.

We sometimes write the series $\sum_{n=1}^{\infty} a_n$ simply as $\sum a_n$.

Example Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$. The sequence of partial sums is given by

$$s_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \frac{1}{2}\left(\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}\right) = 1 - \left(\frac{1}{2}\right)^n$$

Clearly $s_n \to 1$. It follows from the definition that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$$

We could express the argument more succinctly by writing

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2^k} = \lim_{n\to\infty} \left(1 - \left(\frac{1}{2}\right)^n\right) = 1$$

Example Consider the series $\sum_{n=1}^{\infty} (-1)^n = -1 + 1 - 1 + 1 - 1 + \dots$. Here we have the partial sums:

$$\begin{aligned} s_1 &= a_1 = -1 \\ s_2 &= a_1 + a_2 = 0 \\ s_3 &= a_1 + a_2 + a_3 = -1 \\ s_4 &= a_1 + a_2 + a_3 + a_4 = 0 \\ &\quad \dots\dots\dots \end{aligned}$$

and we can see at once that the sequence $(s_n) = -1, 0, -1, 0, \dots$ does not converge.

Exercise 5 Look again at the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$. Plot on two small separate graphs both the sequences $(a_n) = \left(\frac{1}{2^n}\right)$ and $(s_n) = \left(\sum_{k=1}^{n} \frac{1}{2^k}\right)$.

Double Trouble
There are two sequences associated with every series $\sum_{n=1}^{\infty} a_n$: the sequence (a_n) and the sequence of partial sums $(s_n) = \left(\sum_{i=1}^{n} a_i\right)$. Do not get these sequences confused!

Series Need Sequences
Notice that series convergence is defined entirely in terms of sequence convergence. We haven't spent six weeks working on sequences for nothing!

Dummy Variables
Make careful note of the way the variables k and n appear in this example. They are dummies - they can be replaced by any letter you like.

Frog Hopping
Heard about that frog who hops halfway across his pond, and then half the rest of the way, and the half that, and half that, and half that . . . ?
Is he ever going to make it to the other side?

Exercise 6 Find the sum of the series $\sum_{n=1}^{\infty} \left(\frac{1}{10^n}\right)$.

Exercise 7 Reread your answer to exercise 4 and then write out a full proof that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \cdots = 1$$

Exercise 8 Show that the series $1+\frac{1}{2}+\frac{1}{2}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{5}+\ldots$ diverges to $+\infty$. [Hint: Calculate the partial sums $s_1, s_3, s_6, s_{10}, \ldots$.]

7.2 Geometric Series

Theorem *Geometric Series*
The series $\sum_{n=0}^{\infty} x^n$ is convergent if $|x| < 1$ and the sum is $\frac{1}{1-x}$. It is divergent if $|x| \geq 1$.

Exercise 9 $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \ldots$ converges.
$\sum_{n=0}^{\infty}(2.1)^n$, $\sum_{n=0}^{\infty}(-1)^n$ and $\sum_{n=0}^{\infty}(-3)^n = 1 - 3 + 9 - 27 + 81 - \ldots$ all diverge.

GP Consultation
How could you ever forget that $a + ax + ax^2 + \cdots + ax^{n-1} = a\left(\frac{x^n-1}{x-1}\right)$ when $x \neq 1$?

Exercise 10 Prove the theorem [Hint: Use the GP formula to get a formula for s_n].

7.3 The Harmonic Series

The series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ is called the Harmonic Series. The following grouping of its terms is rather cunning:

$$1 + \underbrace{\frac{1}{2}}_{\geq 1/2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq 1/2} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq 1/2} + \underbrace{\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}}_{\geq 1/2} + \ldots$$

Harmonic History
There are other proofs that the Harmonic Series is divergent, but this is the original. It was contributed by the English mediaeval mathematician Nicholas Oresme (1323-1382) who also gave us the laws of exponents: $x^m.x^n = x^{m+n}$ and $(x^m)^n = x^{mn}$.

Exercise 11 Prove that the Harmonic Series diverges. Structure your proof as follows:

1. Let $s_n = \sum_{k=1}^{n} \frac{1}{k}$ be the partial sum. Show that $s_{2n} \geq s_n + \frac{1}{2}$ for all n. (Use the idea in the cunning grouping above).

2. Show by induction that $s_{2^n} \geq 1 + \frac{n}{2}$ for all n.

3. Conclude that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Conflicting Convergence
You can see from this example that the convergence of (a_n) does not imply the convergence of $\sum_{n=1}^{\infty} a_n$.

Exercise 12 Give, with reasons, a value of N for which $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{N} \geq 10$.

7.4 Basic Properties of Convergent Series

Some properties of finite sums are easy to prove for infinite sums:

Theorem *Sum Rule for Series*
Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series. Then, for all real numbers c and d, $\sum_{n=1}^{\infty}(ca_n + db_n)$ is a convergent series and

$$\sum_{n=1}^{\infty}(ca_n + db_n) = c\sum_{n=1}^{\infty} a_n + d\sum_{n=1}^{\infty} b_n$$

Proof. $\sum_{i=1}^{n}(ca_n + db_n) = c\left(\sum_{i=1}^{n} a_i\right) + d\left(\sum_{i=1}^{n} b_i\right) \to c\sum_{n=1}^{\infty} a_n + d\sum_{n=1}^{\infty} b_n$ ■

Theorem *Shift Rule for Series*
Let N be a natural number. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{n=1}^{\infty} a_{N+n}$ converges.

Example We showed that $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. It follows that $\sum_{n=1}^{\infty} \frac{1}{n+1}$ is divergent.

Exercise 13 Prove the shift rule.

7.5 Boundedness Condition

If the terms of a series are all non-negative, then we shall show that the boundedness of its partial sums is enough to ensure convergence.

Theorem *Boundedness Condition*
Suppose $a_n \geq 0$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence of partial sums $(s_n) = \left(\sum_{j=1}^{n} a_j\right)$ is bounded.

Proof. The sequence $\left(\sum_{k=1}^{n} a_k\right)$ is increasing. We saw in Section 6.3 that the sequence either converges, or it diverges to infinity. If it is bounded, it must converge. ■

7.6 Null Sequence Test

Exercise 14

1. Prove that if $\sum_{n=1}^{\infty} a_n$ converges then the sequence (a_n) tends to zero.
   (Hint: Notice that $a_{n+1} = s_{n+1} - s_n$ and use the Shift Rule for sequences.)
2. Is the converse true: If $(a_n) \to 0$ then $\sum_{n=1}^{\infty} a_n$ converges?

We have proved that if the series $\sum_{n=1}^{\infty} a_n$ converges then it must be the case that (a_n) tends to zero. The contrapositive of this statement gives us a test for *divergence*:

Red Alert
The Null Sequence Test is a test for *divergence* only. You can't use it to prove series convergence.

Theorem *Null Sequence Test*
If (a_n) does not tend to zero, then $\sum_{n=1}^{\infty} a_n$ diverges.

Example The sequence (n^2) does not converge to zero, therefore the series $\sum_{n=1}^{\infty} n^2$ diverges.

7.7 Comparison Test

The next test allows you to test the convergence of a series by comparing its terms with those of a series whose behaviour you already know.

Theorem *Comparison Test*
Suppose $0 \leq a_n \leq b_n$ for all natural numbers n. If $\sum b_n$ converges then $\sum a_n$ converges and $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$.

Like the Boundedness Condition, you can only apply the Comparison Test (and the other tests in this section) if the terms of the series are non-negative.

Example You showed in assignment 7 that $\sum \frac{1}{n(n+1)}$ converges. Now $0 \leq \frac{1}{(n+1)^2} \leq \frac{1}{n(n+1)}$. It follows from the Comparison Test that $\sum \frac{1}{(n+1)^2}$ also converges and via the Shift Rule that the series $\sum \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$ converges.

Exercise 15 Give an example to show that the test fails if we allow the terms of the series to be negative, i.e. if we only demand that $a_n \leq b_n$.

Exercise 16 Prove the Comparison Test [Hint: Consider the partial sums of both $\sum b_n$ and $\sum a_n$ and show that the latter is increasing and bounded].

Exercise 17 Check that the *contrapositive* of the statement: *"If $\sum b_n$ converges then $\sum a_n$ converges."* gives you the following additional comparison test:

Corollary *Comparison Test*
Suppose $0 \le a_n \le b_n$. If $\sum a_n$ diverges then $\sum b_n$ diverges.

Examples

1. Note $0 \le \frac{1}{n} \le \frac{1}{\sqrt{n}}$. We know $\sum \frac{1}{n}$ diverges, so $\sum \frac{1}{\sqrt{n}}$ diverges too.
2. To show that $\sum \frac{n+1}{n^2+1}$ diverges, notice that $\frac{n+1}{n^2+1} \ge \frac{n}{n^2+n^2} = \frac{1}{2n}$. We know that $\sum \frac{1}{2n}$ diverges, therefore $\sum \frac{n+1}{n^2+1}$ diverges.

Exercise 18 Use the Comparison Test to determine whether each of the following series converges or diverges. In each case you will have to think of a suitable series with which to compare it.

$$\text{(i)} \quad \sum \frac{2n^2+15n}{n^3+7} \qquad \text{(ii)} \quad \sum \frac{\sin^2 nx}{n^2} \qquad \text{(iii)} \quad \sum \frac{3^n+7^n}{3^n+8^n}$$

Way To Go
Stare deeply at each series and try to find a simpler series whose terms are very close for large n. This gives you a good idea which series you might hope to compare it with, and whether it is likely to be convergent or divergent. For instance the terms of the series $\sum \frac{n+1}{n^2+1}$ are like those of the series $\sum \frac{1}{n}$ for large values of n, so we would expect it to diverge.

7.8 * Application - What is e? *

Over the years you have no doubt formed a working relationship with the number e, and you can say with confidence (and the aid of your calculator) that $e \approx 2.718$. But that is not the end of the story.

Just what is this e number?

To answer this question, we start by investigating the mysterious series $\sum_{n=0}^{\infty} \frac{1}{n!}$. Note that we adopt the convention that $0! = 1$.

Exercise 19 Consider the series $\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$ and its partial sums $s_n = \sum_{k=0}^{n} \frac{1}{k!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots + \frac{1}{n!}$.

1. Show that the sequence (s_n) is increasing.
2. Prove by induction that $\frac{1}{n!} \le \frac{1}{2^{n-1}}$ for $n > 0$.
3. Use the comparison test to conclude that $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges.

Now here comes the Big Definition we've all been waiting for...!!!

Definition
$e := \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$

Binomial Theorem
For all real values x and y and integer $n = 1, 2, \dots$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Note here we use $0! = 1$.

Recall that in the last chapter we showed that $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$ exists. We can now show, with some rather delicate work, that this limit equals e.

First, we show that $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \le e$. Using the Binomial Theorem,

$$\left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k = \sum_{k=0}^{n} \frac{1}{k!} \underbrace{\frac{n(n-1)\dots(n-k+1)}{n \cdot n \cdot \dots \cdot n}}_{\le 1} \le \sum_{k=0}^{n} \frac{1}{k!}.$$

As $n \to \infty$, we get $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \leq \mathrm{e}$.

Second, we show that $\mathrm{e} \leq \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. This is more difficult! The first step is to show that for all m and n

$$\left(1+\frac{1}{n}\right)^{m+n} \geq \sum_{k=0}^{m}\frac{1}{k!}$$

By the Binomial theorem,

$$\begin{aligned}\left(1+\frac{1}{n}\right)^{m+n} &= \sum_{k=0}^{n+m}\frac{1}{k!}\frac{(n+m)!}{(n+m-k)!}\frac{1}{n^k}\\ &\geq \sum_{k=0}^{m}\frac{1}{k!}\frac{(n+m)!}{(n+m-k)!}\frac{1}{n^k}\end{aligned}$$

where we have thrown away the last n terms of the sum. So

$$\begin{aligned}\left(1+\frac{1}{n}\right)^{m+n} &\geq \sum_{k=0}^{m}\frac{1}{k!}\frac{(n+m)(n+m-1)\dots(n+m-k+1)}{n^k}\\ &\geq \sum_{k=0}^{m}\frac{1}{k!}.\end{aligned}$$

We have for all $m, n \geq 1$:

$$\left(1+\frac{1}{n}\right)^{m}\left(1+\frac{1}{n}\right)^{n} \geq \sum_{k=0}^{m}\frac{1}{k!}.$$

Taking the limit $n \to \infty$, we get

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} \geq \sum_{k=0}^{m}\frac{1}{k!}.$$

Taking now the limit $m \to \infty$, we get $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \geq e$.

We have proved:

Theorem
$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

Exercise 20

1. Show that $\left(1-\frac{1}{n+1}\right) = \frac{1}{(1+1/n)}$ and hence find $\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^n$.
2. Use the shift rule to find $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n$.

The last exercise in this chapter is the proof that e is an irrational number. The proof uses the fact that the series for e converges very rapidly and this same idea can be used to show that many other series also converge to irrational numbers.

Theorem
e is irrational.

Exercise 21 (Not easy) Prove this result by contradiction. Structure your proof as follows:

1. Suppose $e = \frac{p}{q}$ and show that $e - \sum_{i=0}^{q} \frac{1}{i!} = \frac{p}{q} - \sum_{i=0}^{q} \frac{1}{i!} = \frac{k}{q!}$ for some positive integer k.
2. Show that $e - \sum_{i=0}^{q} \frac{1}{i!} = \frac{1}{(q+1)!} + \frac{1}{(q+2)!} + \frac{1}{(q+3)!} + \cdots < \frac{1}{q!}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\right)$ and derive a contradiction to part 1.

Check Your Progress
By the end of this chapter you should be able to:

- Understand that a series converges if and only if its partial sums converge, in which case $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \left(\sum_{i=1}^{n} a_i\right)$.
- Write down a list of examples of convergent and divergent series and justify your choice.
- Prove that the *Harmonic Series* is divergent.
- State, prove, and use the *Sum* and *Shift Rules* for series.
- State, prove, and use the *Boundedness Condition*.
- Use and justify the *Null Sequence Test*.
- Describe the behaviour of the *Geometric Series* $\sum_{n=1}^{\infty} x^n$.
- State, prove and use the *Comparison Theorem* for series.
- Justify the limit $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$ starting from the definition $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.
- Prove that e is irrational.

Chapter 8

Series II

Usually we are doomed to failure if we seek a formula for the sum of a series. Nevertheless we can often tell whether the series converges or diverges without explicitly finding the sum. To do this we shall establish a variety of convergence tests that allow us in many cases to work out from the formula for the terms a_n whether the series converges or not.

Explicit Sums

For most convergent series there is no simple formula for the sum $\sum_{n=1}^{\infty} a_n$ in terms of standard mathematical objects. Only in very lucky cases can we sum the series explicitly, for instance geometric series, telescoping series, various series found by contour integration or by Fourier expansions. But these cases are so useful and so much fun that we mention them often.

8.1 Series with positive terms

Series with positive terms are easier than general series since the partial sums (s_n) form an increasing sequence and we have already seen that monotonic sequences are easier to cope with than general sequences.

All our convergence tests are based on the most useful test - the comparison test - which you have already proven.

Sometimes the series of which we want to find the sum looks quite complicated. Often the best way to find a series to compare it with is to look at which terms dominate in the original series.

Example Consider the series $\sum \frac{\sqrt{n}+2}{n^{3/2}+1}$. We can rearrange the n^{th} term in this series as follows:

$$\frac{\sqrt{n}+2}{n^{3/2}+1} = \frac{1+\frac{2}{\sqrt{n}}}{n+\frac{1}{\sqrt{n}}}.$$

As n gets large then $\frac{1}{\sqrt{n}}$ gets small so the dominant term in the numerator is the 1 and in the denominator is the n. Thus a possible series to compare it with is $\sum \frac{1}{n}$. Since this diverges, we want to show that our series is greater than some multiple of $\sum \frac{1}{n}$:

$$\begin{aligned}\frac{\sqrt{n}+2}{n^{3/2}+1} &= \frac{1+\frac{2}{\sqrt{n}}}{n+\frac{1}{\sqrt{n}}} \\ &> \frac{1+\frac{2}{\sqrt{n}}}{2n} \\ &> \frac{1}{2n}.\end{aligned}$$

hence by the comparision test, $\sum \frac{\sqrt{n}+2}{n^{3/2}+1}$ diverges.

Look Where You're Going!

As with many of the results in this course, the Comparison Test requires you to know in advance whether you are trying to prove convergence or divergence. Otherwise you may end up with a comparison that is no use!

Exercise 1 Use the Comparison Test to show that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p \in [2, \infty)$ and diverges if $p \in (0, 1]$. [Hint - you already know the answer for $p = 1$ or 2.]

Exercise 2 Use this technique with the Comparison Test to determine whether each of the following series converges or diverges. Make your reasoning clear.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$ 2. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n + 4^n}{7^n - 2^n}$

Exercise 3 Use the Comparison Test to determine whether each of the following series converges or diverges.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2+1}}$ 2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^7+1}}$ 3. $\displaystyle\sum_{n=1}^{\infty} \left(\sqrt{n+1} - \sqrt{n}\right)$

8.2 Ratio Test

The previous tests operate by comparing two series. Choosing a Geometric Series for such a comparison gives rise to yet another test which is simple and easy but unsophisticated.

The Missing Case

The case $\ell = 1$ is omitted from the statement of the Ratio Test. This is because there exist both convergent *and* divergent series that satisfy this condition.

A.K.A.

This test is also called D'Alembert's Ratio Test, after the French mathematician Jean Le Rond D'Alembert (1717 - 1783). He developed it in a 1768 publication in which he established the convergence of the Binomial Series by comparing it with the Geometric Series.

Theorem *Ratio Test*
Suppose $a_n > 0$ for all $n \geq 1$ and $\frac{a_{n+1}}{a_n} \to \ell$. Then $\sum a_n$ converges if $0 \leq \ell < 1$ and diverges if $\ell > 1$.

Examples

1. Consider the series $\sum \frac{1}{n!}$. Letting $a_n = \frac{1}{n!}$ we obtain $\frac{a_{n+1}}{a_n} = \frac{n!}{(n+1)!} = \frac{1}{n+1} \to 0$. Therefore $\sum \frac{1}{n!}$ converges.

2. Consider the series $\sum \frac{n^2}{2^n}$. Letting $a_n = \frac{n^2}{2^n}$ we obtain $\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{n^2} \cdot \frac{2^n}{2^{n+1}} = \frac{1}{2}\left(1 + \frac{1}{n}\right)^2 \to \frac{1}{2}$. Therefore $\sum \frac{n^2}{2^n}$ converges.

Proof.
(a) Convergence if $\ell < 1$. Since $\frac{a_{n+1}}{a_n} \to \ell < 1$, there exists N such that $\frac{a_{n+1}}{a_n} < \frac{\ell+1}{2} < 1$ for all $n > N$. (There is nothing special about the number $\frac{\ell+1}{2}$, we only need a number that is strictly greater than ℓ and strictly less than 1.) Then

$$a_{n+1} < \frac{\ell+1}{2} a_n < \left(\frac{\ell+1}{2}\right)^2 a_{n-1} < \ldots < \left(\frac{\ell+1}{2}\right)^{n-N+1} a_N.$$

By the comparison test,

$$\sum_{n=1}^{\infty} a_n < \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} \left(\frac{\ell+1}{2}\right)^{n-N+1} a_N$$
$$= \sum_{n=1}^{N} a_n + \left(\frac{\ell+1}{2}\right)^{-N+1} a_N \sum_{n=N+1}^{\infty} \left(\frac{\ell+1}{2}\right)^{n} < \infty.$$

The expression in the last line is finite indeed, because the first sum involves finitely many terms, and the second sum is a geometric series with number less than 1. Then $\sum a_n$ converges.

(b) Divergence if $\ell > 1$. This is similar as above — a bit simpler, actually. Since $\frac{a_{n+1}}{a_n} \to \ell < 1$, there exists N such that $\frac{a_{n+1}}{a_n} \geq 1$ for all $n > N$. (It is important here that ℓ be strictly greater than 1, this would not be true in general if $\ell = 1$.) Then

$$a_{n+1} \geq a_n \geq a_{n-1} \geq ... \geq a_N.$$

By the comparison test,

$$\sum_{n=1}^{\infty} a_n \geq \sum_{n=N}^{\infty} \geq a_N \sum_{n=N}^{\infty} 1 = \infty.$$

Then $\sum a_n$ diverges to infinity. ■

Exercise 4 Write down an example of a convergent series and a divergent series both of which satisfy the condition $\ell = 1$. [This shows why the Ratio Test cannot be used in this case.]

Exercise 5 Use the Ratio Test to determine whether each of the following series converges or diverges. Make your reasoning clear.

$$1. \sum \frac{2^n}{n!} \qquad 2. \sum \frac{3^n}{n} \qquad 3. \sum \frac{n!}{n^n}$$

8.3 Integral Test

We can use our integration skills to get hugely useful approximations to sums. Consider a real-valued function $f(x)$ which is non-negative and decreasing for $x \geq 1$. We have sketched such a function in Figure 8.1 (actually we sketched $f(x) = 1/x$).

The shaded blocks lie under the graph of the function so that the total area of all the blocks is less than the area under the graph between $x = 1$ and $x = n$. So:

$$\sum_{k=2}^{n} f(k) \leq \int_1^n f(x)dx$$

Forward and Back

In later Analysis courses you will formally define both the integral and the logarithm function. Using what you know from A-level for the moment gives us access to more interesting examples.

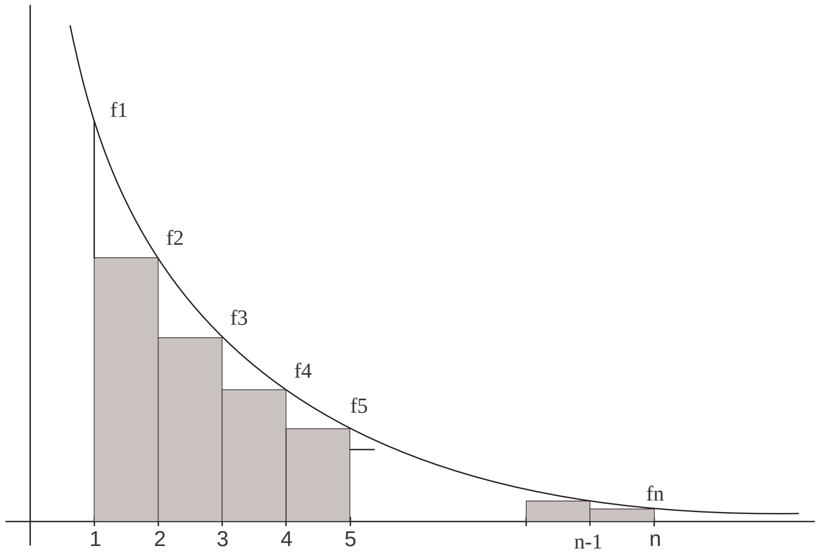


Figure 8.1: Calculating a lower bound of an integral.

The same argument gives more general upper bounds. It also gives lower bounds, when the blocks are chosen so that their area *contains* the area below the curve. Precisely, one can prove the following claims.

Theorem *Integral bounds*
Suppose that $f(x)$ is a non-negative and decreasing function. Then for all $m \leq n$,

$$\int_{m}^{n+1} f(x)dx \leq \sum_{k=m}^{n} f(k) \leq \int_{m-1}^{n} f(x)dx.$$

We can use this bound to help us with error estimates. Let us consider our favorite series $\sum_{k=1}^{\infty} \frac{1}{k^2}$, which turns out to be equal to $\frac{\pi^2}{6}$. If we sum only the first N terms of this series we will reach a total less than $\frac{\pi^2}{6}$. Can we estimate the size of the error?

The error is precisely $\sum_{k=N+1}^{\infty} \frac{1}{k^2}$. Using the theorem above, we obtain the bound:

$$\sum_{k=N+1}^{n} \frac{1}{k^2} \leq \int_{N}^{n} \frac{1}{x^2}dx = -\frac{1}{x}\bigg|_{N}^{n} = \frac{1}{N} - \frac{1}{n} \leq \frac{1}{N}$$

Since this is true for any value of n we see that $\sum_{k=N+1}^{\infty} \frac{1}{k^2} = \lim_{n\to\infty} \sum_{k=N+1}^{n} \frac{1}{k^2} \leq \frac{1}{N}$.

So if we sum the first 1,000,000 terms we will reach a total that is within 10^{-6} of $\pi^2/6$.

Exercise 6 Fourier analysis methods also lead to the formula:

$$\frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \dots$$

Find a value of N so that $\sum_{k=1}^{N} \frac{1}{k^4}$ is within 10^{-6} of $\pi^4/90$.

Exercise 7 Use the upper and lower bounds in the theorem above to show

$$\sum_{k=101}^{200} \frac{1}{k} = \frac{1}{101} + \frac{1}{102} + \cdots + \frac{1}{200} \in [0.688, 0.694]$$

We now use these upper and lower bounds to establish a beautiful test for convergence.

Corollary *Integral Test*
Suppose that the function $f(x)$ is non-negative and decreasing for $x \geq 1$.

(a) If $\int_1^\infty f(x)dx < \infty$, then $\sum_{n=1}^\infty f(n)$ converges.

(b) If $\int_1^\infty f(x)dx = \infty$, then $\sum_{n=1}^\infty f(n)$ diverges.

Example The Integral Test gives us another proof of the fact that $\sum_{n=1}^\infty \frac{1}{n^2}$ converges. Let $f(x) = \frac{1}{x^2}$. We know that this function is non-negative and decreasing when $x \geq 1$. Observe that $\int_1^n f(x)dx = \int_1^n \frac{1}{x^2} = -\frac{1}{x}\Big|_1^n = 1 - \frac{1}{n} \to 1$. Since $f(n) = \frac{1}{n^2}$, the Integral Test assures us that $\sum_{n=1}^\infty \frac{1}{n^2}$ converges.

Example If you are familiar with the behaviour of the log function, the Integral Test gives you a neat proof that the Harmonic Series $\sum_{n=1}^\infty \frac{1}{n}$ diverges. Suppose $f(x) = \frac{1}{x}$. Again, this function is non-negative and decreasing when $x \geq 1$. Observe that $\int_1^n f(x)dx = \int_1^n \frac{1}{x}dx = \log x\Big|_1^n = \log n \to \infty$. Therefore $\sum_{n=1}^\infty \frac{1}{n}$ diverges to infinity.

Exercise 8 Use the Integral Test to investigate the convergence of $\sum_{n=1}^\infty \frac{1}{n^p}$ for values of $p \in (1, 2)$.

Combining this with the result of exercise 1, you have shown:

Corollary
The series $\sum_{n=1}^\infty \frac{1}{n^p}$ converges for $p > 1$ and diverges for $0 < p \leq 1$.

We now examine some series right on the borderline of convergence.

Exercise 9 Show that $\sum_{n=2}^\infty \frac{1}{n \log n}$ is divergent and that $\sum_{n=2}^\infty \frac{1}{n(\log n)^2}$ is convergent.

8.4 * Application - Error Bounds *

If we have established that a series $\sum a_n$ converges then the next question is to calculate the total sum $\sum_{n=1}^{\infty} a_n$. Usually there is no explicit formula for the sum and we must be content with an approximate answer - for example, correct to 10 decimal places.

The obvious solution is to calculate $\sum_{n=1}^{N} a_n$ for a really large N. But how large must N be to ensure the error is small - say less than 10^{-10}? The error is the sum of all the terms we have ignored $\sum_{n=N+1}^{\infty} a_n$ and again there is usually no explicit answer. But by a comparison with a series for which we *can* calculate the sum (i.e. geometric or telescoping series) we can get a useful upper bound on the error.

Example Show how to calculate the value of e to within an error of 10^{-100}.

Solution We shall sum the series $\sum_{n=0}^{N} \frac{1}{n!}$ for a large value of N. Then the error is:

$$\begin{aligned} e - \sum_{n=0}^{N} \frac{1}{n!} &= \sum_{n=N+1}^{\infty} \frac{1}{n!} \\ &= \frac{1}{(N+1)!}\left(1 + \frac{1}{N+2} + \frac{1}{(N+2)(N+3)} \cdots\right) \\ &\leq \frac{1}{(N+1)!}\left(1 + \frac{1}{2} + \frac{1}{4} + \ldots\right) \\ &= \frac{2}{(N+1)!} \end{aligned}$$

Then the error is less than 10^{-100} provided that $\frac{2}{(N+1)!} \leq 10^{-100}$ which occurs when $N \geq 70$.

Exercise 10 The following formula for $\sqrt{e}$ is true, although it will not be proved in this course.

$$\sqrt{e} = \sum_{n=0}^{\infty} \frac{1}{2^n n!} = 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} + \frac{1}{384} + \ldots$$

Show that the error $\sqrt{e} - \sum_{n=0}^{N} \frac{1}{2^n n!}$ is less than $\frac{1}{2^N (N+1)!}$. Hence find a value of N that makes the error less than 10^{-100}.

8.5 * Euler's product formula *

In this section we discuss the facinating formula of Euler that involves a product over all prime numbers, and its relation with Riemann's zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.

Theorem *Euler's product formula*
For all $s > 1$, we have

$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right) = \frac{1}{\zeta(s)},$$

where the product is over all prime numbers $p = 2, 3, 5, 7, ...$

Since $(1 - \frac{1}{p^s})^{-1} = \frac{1}{1-p^{-s}}$, the formula can also be written

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

Proof. We establish the latter formula. Recall the formula for geometric series $\frac{1}{1-a} = 1 + a + a^2 + ...$ We have

$$\begin{aligned}
\prod_{p \text{ prime}} \frac{1}{1-p^{-s}} &= \frac{1}{1-2^{-s}} \cdot \frac{1}{1-3^{-s}} \cdot \frac{1}{1-5^{-s}} \cdot ... \\
&= \left(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \dots\right)\left(1 + \frac{1}{3^s} + \frac{1}{3^{2s}} + \dots\right)\left(1 + \frac{1}{5^s} + \frac{1}{5^{2s}} + \dots\right) ... \\
&= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{2^{2s}} + \frac{1}{5^s} + \frac{1}{2^s \cdot 3^s} + ... \\
&= \zeta(s).
\end{aligned}$$

Indeed, all combinations of products of prime numbers appear when we expand the product of sums. We then obtain all natural numbers, since each natural number has a unique decomposition into prime numbers. ■

Check Your Progress
By the end of this chapter you should be able to:

- Use and justify the following tests for sequence convergence:
- *Comparison Test:* If $0 \le a_n \le b_n$ and $\sum b_n$ is convergent then $\sum a_n$ is convergent.
- *Ratio Test:* If $a_n > 0$ and $\frac{a_{n+1}}{a_n} \to \ell$ then $\sum a_n$ converges if $0 \le \ell < 1$ and diverges if $\ell > 1$.
- *Integral Test:* If $f(x)$ is non-negative and decreasing for $x \ge 1$ then $\sum f(n)$ converges if and only if $\int_1^\infty f(x)dx < \infty$, and $\sum f(n)$ diverges to infinity if and only if $\int_1^\infty f(x)dx = \infty$.
- You should also be able to use comparisons to establish error bounds when evalutating infinite sums.

Chapter 9

Series III

With the exception of the Null Sequence Test, all the tests for series convergence and divergence that we have considered so far have dealt only with series of non-negative terms. Series with *both* positive and negative terms are harder to deal with.

9.1 Alternating Series

One very special case is a series whose terms alternate in sign from positive to negative. That is, series of the form $\sum(-1)^{n+1}a_n$ where $a_n \geq 0$.
Example $\sum \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$ is an alternating series.

Theorem *Alternating Series Test*
Suppose (a_n) is decreasing and null. Then the alternating series $\sum(-1)^{n+1}a_n$ is convergent.

Proof. We first observe that the subsequence (s_{2n}) converges:

$$s_{2n} = \sum_{k=1}^{2n}(-1)^{k+1}a_k = (a_1-a_2)+(a_3-a_4)+\ldots+(a_{2n-1}-a_{2n}) = \sum_{k=1}^{n}(a_{2k-1}-a_{2k}).$$

This is a sum of nonnegative terms. It does not tend to infinity, because it is bounded:

$$s_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \ldots - a_{2n} \leq a_1.$$

Then (s_{2n}) converges to a number ℓ. There remains to show that the whole sequence (s_n) converges to ℓ. For every $\varepsilon > 0$, there exists N such that for all even $m > N$,

$$|s_m - \ell| < \frac{\varepsilon}{2}.$$

If m is odd, we have

$$|s_m - \ell| \leq |s_{m+1} - \ell| + |s_{m+1} - s_m| < \frac{\varepsilon}{2} + |a_{m+1}|.$$

This is less than ε if m is large enough, since the sequence (a_m) tends to 0. ■

Example Since $\left(\frac{1}{n}\right)$ is a decreasing null sequence of this test tells us right away that $\sum \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$ is convergent.

Similarly, $\left(\frac{1}{\sqrt{n}}\right)$ is a decreasing null sequence, therefore $\sum \frac{(-1)^{n+1}}{\sqrt{n}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{4} + \frac{1}{\sqrt{5}} - \ldots$ is convergent.

Pairing terms in a suitable fashion, as in the proof above, one can get the following error bounds.

Exercise 1 Show that, if (a_n) is a decreasing and null sequence, then

$$\left|\sum_{k=N}^{\infty}(-1)^{k+1}a_k\right| \leq a_N.$$

Exercise 2 Let $s = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. Find a value of N so that $|\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n} - s| \leq 10^{-6}$.

The Alternating Series test requires that the sequence be decreasing and null, hence it must be non-negative. The next exercise shows that if we relax either the decreasing or null condition then the alternating series may not converge - even if we still insist on the terms being non-negative.

Exercise 3 Find a sequence (a_n) which is non-negative and decreasing but where $\sum(-1)^{n+1}a_n$ is divergent and a sequence (b_n) which is non-negative and null but where $\sum(-1)^{n+1}b_n$ is divergent.

Exercise 4 The series $1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \frac{1}{9!} - \ldots$ converges to $\sin 1$. Explain how to use the series to calculate $\sin 1$ to within an error of 10^{-10}.

Exercise 5 Using the Alternating Series Test where appropriate, show that each of the following series is convergent.

1. $\sum \frac{(-1)^{n+1}n^2}{n^3+1}$
2. $\sum \left(-\frac{1}{2}\right)^n$
3. $\sum \frac{2|\cos\frac{n\pi}{2}|+(-1)^n n}{\sqrt{(n+1)^3}}$
4. $\sum \frac{1}{n}\sin\frac{n\pi}{2}$

9.2 General Series

Series with positive terms are easier because we can attempt to prove that the partial sums (s_n) converge by exploiting the fact that (s_n) is increasing. For a general series $\sum a_n$, we get some information by studying the series of absolute values, $\sum |a_n|$, which involves only positive terms.

Definition
The series $\sum a_n$ is *absolutely convergent* if $\sum |a_n|$ is convergent.

Example The alternating series $\sum \frac{(-1)^{n+1}}{n^2}$ is absolutely convergent because $\sum |\frac{(-1)^{n+1}}{n^2}| = \sum \frac{1}{n^2}$ is convergent.

The series $\sum \frac{(-1)^n}{n}$ is not absolutely convergent because $\sum \frac{1}{n}$ diverges.

The series $\sum \left(-\frac{1}{2}\right)^n$ is absolutely convergent because $\sum \left(\frac{1}{2}\right)^n$ converges.

Exercise 6 Is the series $\sum \frac{(-1)^{n+1}}{\sqrt{n}}$ absolutely convergent?

Exercise 7 For what values of x is the Geometric Series $\sum x^n$ absolutely convergent?

Absolutely convergent series are important for the following reason.

Theorem *Absolute Convergence*
Every absolutely convergent series is convergent.

Proof. Let $s_n = \sum_{i=1}^n a_i$ and $t_n = \sum_{i=1}^n |a_i|$. We know that (t_n) is convergent, hence Cauchy: for every $\varepsilon > 0$, there exists N such that $|t_m - t_n| < \varepsilon$ for all $m, n > N$. We now show that (s_n) is also Cauchy. Let $n > m$.

$$|s_n - s_m| = \Big|\sum_{k=m+1}^{n} a_k\Big| \leq \sum_{k=m+1}^{n} |a_k| = t_n - t_m < \varepsilon.$$

Then (s_n) is Cauchy, and it converges. ■

Exercise 8 Is the converse of the theorem true: "Every convergent series is absolutely convergent"?

The Absolute Convergence Theorem breathes new life into all the tests we developed for series with non-negative terms: if we can show that $\sum |a_n|$ is convergent, using one of these tests, then we are guarenteed that $\sum a_n$ converges as well.

Exercise 9 Show that the series $\sum \frac{\sin n}{n^2}$ is convergent.

We see that $0 \leq \frac{|\sin n|}{n^2} \leq \frac{1}{n^2}$. Therefore $\sum \frac{|\sin n|}{n^2}$ is convergent by the Comparison Test. It follows that $\sum \frac{\sin n}{n^2}$ is convergent by the Absolute Convergence Theorem.

The Ratio Test can be modified to cope directly with series of mixed terms.

Theorem *Ratio Test*
Suppose $a_n \neq 0$ for all n and $|\frac{a_{n+1}}{a_n}| \to \ell$. Then $\sum a_n$ converges absolutely (and hence converges) if $0 \leq \ell < 1$ and diverges if $\ell > 1$.

Proof. If $0 \leq \ell < 1$, then $\sum |a_n|$ converges by the "old" Ratio Test. Therefore $\sum a_n$ converges by the Absolute Convergence Theorem.

If $\ell > 1$, we are guaranteed that $\sum |a_n|$ diverges, but this does not, in itself, prove that $\sum a_n$ diverges (why not?). We have to go back and modify our original proof.

We know that there exists N such that $\frac{|a_{n+1}|}{|a_n|}| \geq 1$ when $n > N$. Then

$$|a_{n+1}| \geq |a_n| \geq |a_{n-1}| \geq ... \geq |a_N| > 0.$$

Therefore the sequence (a_n) does not tend to 0, and $\sum a_n$ diverges by the null sequence test. ■

Example Consider the series $\sum \frac{x^n}{n}$. When $x = 0$ the series is convergent. (Notice that we cannot use the Ratio Test in this case.)

Now let $a_n = \frac{x^n}{n}$. When $x \neq 0$ then $|\frac{a_{n+1}}{a_n}| = |\frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n}| = \frac{n}{n+1}|x| \to |x|$. Therefore $\sum \frac{x^n}{n}$ is convergent when $|x| < 1$ and divergent when $|x| > 1$, by the Ratio Test.

What if $|x| = 1$? When $x = 1$ then $\sum \frac{x^n}{n} = \sum \frac{1}{n}$ which is divergent. When $x = -1$ then $\sum \frac{x^n}{n} = \sum -\frac{(-1)^{n+1}}{n}$ which is convergent.

Theorem *Ratio Test Variant*
Suppose $a_n \neq 0$ for all n and $|\frac{a_{n+1}}{a_n}| \to \infty$, then $\sum a_n$ diverges.

Exercise 10 Prove this theorem.

Exercise 11 In the next question you will need to use the fact that if a non-negative sequence $(a_n) \to a$ and $a > 0$, then $\left(\sqrt{a_n}\right) \to \sqrt{a}$. Prove this, by first showing that

$$\sqrt{a_n} - \sqrt{a} = \frac{a_n - a}{\sqrt{a_n} + \sqrt{a}}.$$

Exercise 12 Determine for which values of x the following series converge and diverge. [Make sure you don't neglect those values for which the Ratio Test doesn't apply.]

1. $\sum \frac{x^n}{n!}$
2. $\sum \frac{n}{x^n}$
3. $\sum n!x^n$
4. $\sum \frac{(2x)^n}{n}$
5. $\sum \frac{(4x)^{3n}}{\sqrt{n+1}}$
6. $\sum (-nx)^n$

9.3 Euler's Constant

Our last aim in this booklet is to find an explicit formula for the sum of the alternating series:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

On the way we shall meet Euler's constant, usually denoted by γ, which occurs in several places in mathematics, especially in number theory.

Exercise 13 Let $D_n = \sum_{i=1}^n \frac{1}{i} - \int_1^{n+1} \frac{1}{x} dx = \sum_{i=1}^n \frac{1}{i} - \log(n+1)$.

1. In your solution book, draw on the copy of Figure 9.1 the areas represented by D_n.

2. Show that (D_n) is increasing.

3. Show that (D_n) is bounded - and hence convergent.

Euler's Constant

The limit of the sequence

$$D_n = \sum_{i=1}^n \frac{1}{i} - \log(n+1)$$

is called Euler's Constant and is usually denoted by γ (gamma). Its value has been computed to over 200 decimal places. To 14 decimal places, it is 0.57721566490153. No-one knows whether γ is rational or irrational.

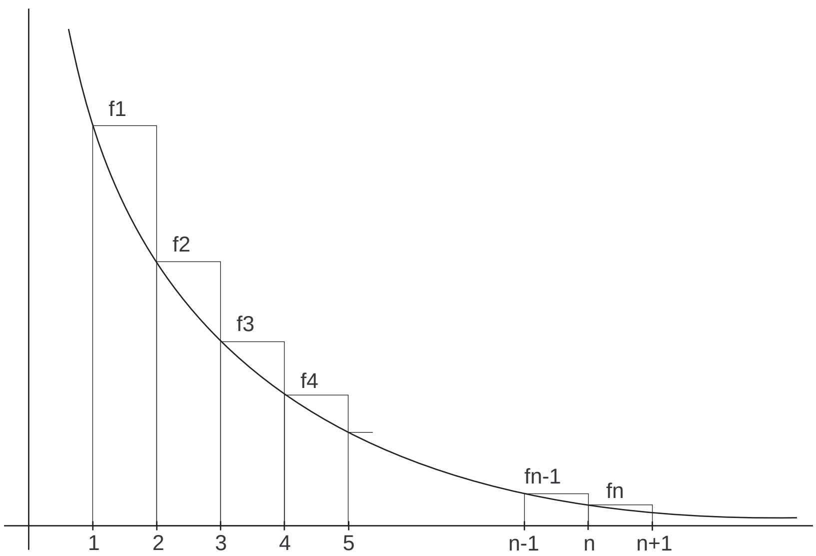


Figure 9.1: Calculating an upper bound of an integral.

The limit of the sequence $D_n = \sum_{i=1}^n \frac{1}{i} - \log(n+1)$ is called Euler's Constant and is usually denoted by γ.

Exercise 14 Show that $\sum_{i=1}^{2n-1} \frac{(-1)^{i+1}}{i} = \log 2 + D_{2n-1} - D_{n-1}$. Hence evaluate $\sum \frac{(-1)^{n+1}}{n}$.

Hint: Use the following identity:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n-1}$$
$$=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n-1}-2\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots\frac{1}{2n-2}\right)$$

9.4 * Application - Stirling's Formula *

Using the alternating series test we can improve the approximations to $n!$ that we stated in workbook 4. Take a look at what we did there: we obtained upper and lower bounds to $\log(n!)$ by using block approximations to the integral of $\int_1^n \log x dx$. To get a better approximation we use the approximation in Figure 9.2.

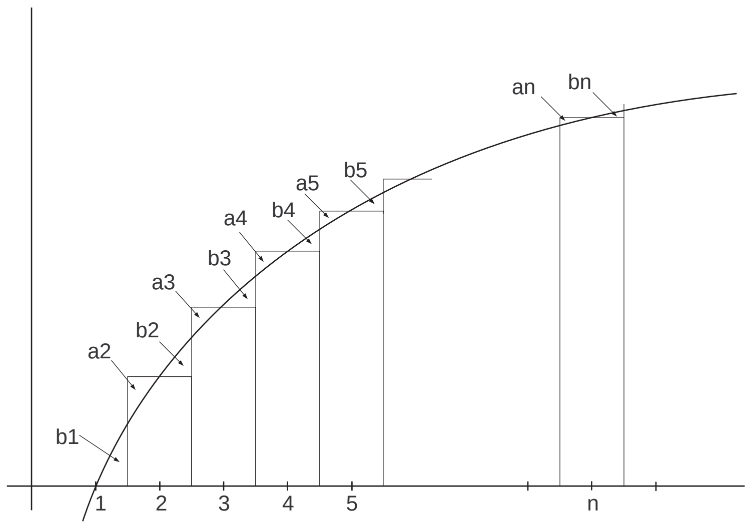


Figure 9.2: Approximating the integral by the mid point.

Now the area of the blocks approximates $\int_1^n \log x dx$ except that there are small triangular errors below the graph (marked as $b_1, b_2, b_3, \dots$) and small triangular errors above the graph (marked as $a_2, a_3, a_4, \dots$).

Note that $\log n! = \log 2 + \log 3 + \cdots + \log n =$ area of the blocks.

Exercise 15 Use the above diagram to show:

$$\log n! - \left(n+\tfrac{1}{2}\right)\log n + n = 1 - b_1 + a_2 - b_2 + a_3 - b_3 + \cdots - b_{n-1} + a_n$$

[Hint: $\int_1^n \log x dx = n \log n - n + 1$]

The curve $\log x$ is concave and it seems reasonable (and can be easily proved - try for yourselves), that $a_n \geq b_n \geq a_{n+1}$ and $(a_n) \to 0$.

Exercise 16 Assuming that these claims are true, explain why $(s_n) = (1 - b_1 + a_2 - b_2 + \cdots + a_n - b_n)$ converges.

This proves that $\log n! = (n+\frac{1}{2}) \log n - n + \Sigma_n$ where Σ_n tends to a constant as $n \to \infty$. Taking exponentials we obtain:

$$n! \simeq \text{constant} \cdot n^n e^{-n} \sqrt{n}$$

What is the constant? This was identified with only a little more work by the mathematician James Stirling. Indeed, he proved that:

$$\frac{n!}{n^n e^{-n} \sqrt{2\pi n}} \to 1 \quad \text{as } n \to \infty$$

a result known as Stirling's formula.

Check Your Progress
By the end of this chapter you should be able to:

- Use and justify the Alternating Series Test: If (a_n) is a decreasing null sequence then $\sum (-1)^{n+1} a_n$ is convergent.
- Use the proof of Alternating Series Test to establish error bounds.
- Say what it means for a series to be *absolutely convergent* and give examples of such series.
- Prove that an absolutely convergent series is convergent, but that the converse is not true.
- Use the modified Ratio Test to determine the convergence or divergence of series with positive and negative terms.
- Prove that $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \log 2$.

Chapter 10

Series IV

10.1 Rearrangements of Series

If you take any *finite* set of numbers and rearrange their order, their sum remains the same. But the truly weird and mind-bending fact about *infinite* sums is that, in some cases, you can rearrange the terms to get a totally different sum. We look at one example in detail.

The sequence

$$(b_n) = 1, -\frac{1}{2}, -\frac{1}{4}, \frac{1}{3}, -\frac{1}{6}, -\frac{1}{8}, \frac{1}{5}, -\frac{1}{10}, -\frac{1}{12}, \frac{1}{7}, -\frac{1}{14}, -\frac{1}{16}, \frac{1}{9}, \dots$$

contains all the numbers in the sequence

$$(a_n) = 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \frac{1}{7}, -\frac{1}{8}, \frac{1}{9}, -\frac{1}{10}, \frac{1}{11}, -\frac{1}{12}, \frac{1}{13}, -\frac{1}{14}, \dots$$

but rearranged in a different order: each of the positive terms is followed by not one but *two* of the negative terms. You can also see that each number in (b_n) is contained in (a_n). So this rearrangement effectively shuffles, or permutes, the indices of the original sequence. This leads to the following definition.

Definition
The sequence (b_n) is a *rearrangement* of (a_n) if there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ (i.e. a permutation on $\mathbb{N}$) such that $b_n = a_{\sigma(n)}$ for all n.

Shuffling the (Infinite) Pack

The permutation σ simply shuffles about the terms of the old sequence (a_n) to give the new sequence $(a_{\sigma(n)})$.

Exercise 1 What permutation σ has been applied to the indices of the sequence (a_n) to produce (b_n) in the example above? Answer this question by writing down an explicit formula for $\sigma(3n), \sigma(3n-1), \sigma(3n-2)$.

Reciprocal Rearrangements

If (b_n) is a rearrangement of (a_n) then (a_n) must be a rearrangement of (b_n). Specifically, if $b_n = a_{\sigma(n)}$ then $a_n = b_{\sigma^{-1}(n)}$.

Don't get hung up on this exercise if you're finding it tricky, because the really interesting part comes next.

We have defined the rearrangement of a sequence. Using this definition, we say that the *series* $\sum b_n$ is a rearrangement of the *series* $\sum a_n$ if the sequence (b_n) is a rearrangement of the sequence (a_n).

We know already that

$$\sum \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \log(2)$$

We now show that our rearrangement of this series has a different sum.

Exercise 2 Show that:

$$\sum b_n = 1 + -\frac{1}{2} + -\frac{1}{4} + \frac{1}{3} + -\frac{1}{6} + -\frac{1}{8} + \frac{1}{5} + -\frac{1}{10} + \\ -\frac{1}{12} + \frac{1}{7} + -\frac{1}{14} + -\frac{1}{16} + \frac{1}{9} + \cdots = \frac{\log 2}{2}$$

Hint: Let $s_n = \sum_{k=1}^n a_k$ and let $t_n = \sum_{k=1}^n b_k$. Show that $t_{3n} = \frac{s_{2n}}{2}$ by using the following grouping of the series $\sum b_n$:

$$\left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \left(\frac{1}{7} - \frac{1}{14}\right) - \ldots$$

This example is rather scary. However, for series with all positive terms it does not matter in what order you add the terms.

Theorem
Suppose that $a_n \geq 0$ for all n. Then, if (b_n) is a rearrangement of (a_n), we have $\sum b_n = \sum a_n$.

Remark: The theorem holds both if $\sum a_n$ is finite *or* infinite (in which case $\sum b_n = \infty$).

Proof. Let $s_n = \sum_{k=1}^n a_k$ and $t_n = \sum_{k=1}^n b_k$. Let $s = \lim s_n$ and $t = \lim t_n$ (either these limits exist, or s or t are equal to infinity). For all n, we have

$$s_n \leq t, \qquad t_n \leq s.$$

Indeed, t involves the sum over all b_k, so it involves the sum over $a_1, \ldots, a_n$. Same for s. Taking the limit $n \to \infty$, we find that $s \leq t$ and $t \leq s$. Then $s = t$. ∎

Nor does it matter what order you add the terms of an absolutely convergent series.

Theorem
Suppose that $\sum a_n$ is an absolutely convergent series. If (b_n) is a rearrangement of (a_n) then $\sum b_n$ is absolutely convergent and $\sum b_n = \sum a_n$.

Proof. We know from the previous theorem that $\sum |b_k| = \sum |a_k| < \infty$. Consider

$$\sum_{k=1}^n b_k = \sum_{k=1}^n |b_k| - \sum_{k=1}^n (|b_k| - b_k).$$

Both sums of the right side involve nonnegative terms, and they are absolutely convergent. Then we can rearrange their terms, so that they converge to the limit of

$$\sum_{k=1}^{n} |a_k| - \sum_{k=1}^{n} (|a_k| - a_k) = \sum_{k=1}^{n} a_k.$$

This proves that $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} a_k$. ■

In 1837 the mathematician Dirichlet discovered which type of series could be rearranged to give a different sum and the result was displayed in a startling form in 1854 by Riemann. To describe their results we have one final definition.

Definition
The series $\sum a_n$ is said to be *conditionally convergent* if $\sum a_n$ is convergent but $\sum |a_n|$ is not.

Example Back to our familiar example: $\sum \frac{(-1)^{n+1}}{n}$ is conditionally convergent, because $\sum \frac{(-1)^{n+1}}{n}$ is convergent, but $\sum |\frac{(-1)^{n+1}}{n}| = \sum \frac{1}{n}$ is not.

Exercise 3 Check from the definitions that every convergent series is either absolutely convergent or is conditionally convergent.

Exercise 4 State with reasons which of the following series are conditionally convergent.

$$1. \sum \frac{(-1)^{n+1}}{n^2} \qquad 2. \sum \frac{\cos(n\pi)}{n} \qquad 3. \sum \frac{(-1)^{n+1} n}{1+n^2}$$

Conditionally convergent series are the hardest to deal with and can behave very strangely. The key to understanding them is the following lemma.

Lemma
If a series is conditionally convergent, then the series formed from just its positive terms diverges to infinity, and the series formed from just its negative terms diverges to minus infinity.

Exercise 5 Prove this Lemma using the following steps.

1. Suppose $\sum a_n$ is conditionally convergent. What can you say about the sign of the sequences

$$u_n = \tfrac{1}{2}(|a_n| + a_n) \quad \text{and} \quad v_n = \tfrac{1}{2}(|a_n| - a_n)$$

in relation to the original sequence a_n.

2. Show that $a_n = u_n - v_n$ and $|a_n| = u_n + v_n$. We will prove by contradiction that neither $\sum u_n$ nor $\sum v_n$ converges.

3. Suppose that that $\sum u_n$ is convergent and show that $\sum |a_n|$ is convergent. Why is this a contradiction?

4. Suppose that $\sum v_n$ is convergent and use a similar argument to above to derive a contradiction.

5. You have shown that $\sum u_n$ and $\sum v_n$ diverge. Prove that they tend to $+\infty$. Use your answer to part 1. to finish the proof.

Theorem *Riemann's Rearrangement Theorem*
Suppose $\sum a_n$ is a conditionally convergent series. Then for every real number s there is a rearrangement (b_n) of (a_n) such that $\sum b_n = s$.

The last lemma allows us to construct a proof of the theorem along the following lines: We sum enough positive values to get us just above s. Then we add enough negative values to take us back down just below s. Then we add enough positive terms to get back just above s again, and then enough negative terms to get back down just below s. We repeat this indefinitely, in the process producing a rearrangement of $\sum a_n$ which converges to s.

The Infinite Case

We can also rearrange any conditionally convergent series to produce a series that tends to infinity or minus infinity.

How would you modify the proof to show this?

All Wrapped Up

Each convergent series is either conditionally convergent or absolutely convergent. Given the definition of these terms, there are no other possibilities.
This theorem makes it clear that conditionally convergent series are the *only* convergent series whose sum can be perturbed by rearrangement.

Proof. Let (p_n) be the subsequence of (a_n) containing all its positive terms, and let (q_n) be the subsequence of negative terms. First suppose that $s \geq 0$. Since $\sum p_n$ tends to infinity, there exists N such that $\sum_{i=1}^{N} p_i > s$. Let N_1 be the *smallest* such N and let $S_1 = \sum_{i=1}^{N_1} p_i$. Then $S_1 = \sum_{i=1}^{N_1} p_i > s$ and $\sum_{i=1}^{N_1-1} p_i \leq s$. Thus $S_1 = \sum_{i=1}^{N_1-1} p_i + p_{N_1} \leq s + p_{N_1}$, therefore $0 \leq S_1 - s \leq p_{N_1}$.

To the sum S_1 we now add just enough negative terms to obtain a new sum T_1 which is less than s. In other words, we choose the smallest integer M_1 for which $T_1 = S_1 + \sum_{i=1}^{M_1} q_i < s$. This time we find that $0 \leq s - T_1 \leq -q_{M_1}$.

We continue this process indefinitely, obtaining sums alternately smaller and larger than s, each time choosing the smallest N_i or M_i possible. The sequence:

$$p_1, \ldots, p_{N_1}, q_1, \ldots q_{M_1}, p_{N_1+1}, \ldots p_{N_2}, q_{M_1+1}, \ldots, q_{M_2}, \ldots$$

is a rearrangement of (a_n). Its partial sums increase to S_1, then decrease to T_1, then increase to S_2, then decrease to T_2, and so on.

To complete the proof we note that for all i, $|S_i - s| \leq p_{N_i}$ and $|T_i - s| \leq -q_{M_i}$. Since $\sum a_n$ is convergent, we know that (a_n) is null. It follows that subsequences (p_{N_i}) and (q_{M_i}) also tend to zero. This in turn ensures that the partial sums of the rearrangement converge to s, as required.

In the case $s < 0$ the proof looks almost identical, except we start off by summing enough negative terms to get us just below l. ■

Exercise 6 Draw a diagram which illustrates this proof. Make sure you include the limit s and some points $S_1, T_1, S_2, T_2, \ldots$.

Check Your Progress
By the end of this chapter you should be able to:

- Define what is meant by the *rearrangement* of a sequence or a series.
- Give an example of a rearrangement of the series $\sum \frac{(-1)^{n+1}}{n} = \log 2$ which sums to a different value.
- Prove that if $\sum a_n$ is a series with positive terms, and (b_n) is a rearrangement of (a_n) then $\sum b_n = \sum a_n$.
- Prove that if $\sum a_n$ is an absolutely convergent series, and (b_n) is a rearrangement of (a_n) then $\sum b_n = \sum a_n$.
- Conclude that conditionally convergent series are the *only* convergent series whose sum can be altered by rearrangement.
- Know that if $\sum a_n$ is a conditionally convergent series, then for every real number s there is a rearrangement (a_n) of (a_n) such that $\sum b_n = s$.

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