Assignment 5

Due Monday 8 November 15:00 (in supervisor pigeon hole)

1. Find a rational number which lies between $\frac{57}{65}$ and $\frac{64}{73}$ and may be written in the form $\frac{m}{2^n}$, where m is an integer and n is non-negative integer.

2. Let $a < b$. Prove that there is an infinite number of irrational numbers in the interval (a, b).

3. Prove that a set A can have at most one least upper bound (supremum).

4. Consider the sequence (a_n) defined by

$$a_1 = \tfrac{5}{2}, \quad a_{n+1} = \tfrac{1}{5}(a_n^2 + 6).$$

Show by induction that $2 < a_n < 3$. Show that (a_n) is decreasing. Finally, show that (a_n) is convergent and find its limit.

5. Consider the sequence (a_n) defined by

$$a_1 = \sqrt{3}, \quad a_{n+1} = \sqrt{3 + a_n}.$$

Prove that this sequence is convergent and find its limit.

6. Let $x \geq 0$. Consider the sequence (a_n) defined by

$$a_1 = x, \quad a_{n+1} = \sqrt{2a_n}.$$

Prove that this sequence is convergent and find all possible limits (the limit may depend on x).

7. Let A be a non-empty set of real numbers. Define $-A = \{x : -x \in A\}$. Show that

$$\sup(-A) = -\inf A$$
$$\inf(-A) = -\sup A$$

8. Find
(a) $\sup\{x \in R : x^2 + 4x + 1 < 0\}$
(b) $\inf\{z = x + x^{-1} : x > 0\}$

9. If (a_n) is an increasing sequence that is not bounded above, show that $(a_n) \to \infty$.

10. Prove that

$$\sqrt{3} = \inf\{x \in \mathbb{Q} : x > 0 \text{ and } x^2 > 3\}.$$