Assignment 5

Due Monday 8 November 15:00 (in supervisor pigeon hole)

1. Find a rational number which lies between $\frac{57}{65}$ and $\frac{64}{73}$ and may be written in the form $\frac{m}{n}$, where $m$ is an integer and $n$ is non-negative integer.

2. Let $a < b$. Prove that there is an infinite number of irrational numbers in the interval $(a, b)$.

3. Prove that a set $A$ can have at most one least upper bound (supremum).

4. Consider the sequence $(a_n)$ defined by
   \[ a_1 = \frac{5}{2}, \quad a_{n+1} = \frac{1}{5}(a_n^2 + 6). \]
   Show by induction that $2 < a_n < 3$. Show that $(a_n)$ is decreasing. Finally, show that $(a_n)$ is convergent and find its limit.

5. Consider the sequence $(a_n)$ defined by
   \[ a_1 = \sqrt{3}, \quad a_{n+1} = \sqrt{3 + a_n}. \]
   Prove that this sequence is convergent and find its limit.

6. Let $x \geq 0$. Consider the sequence $(a_n)$ defined by
   \[ a_1 = x, \quad a_{n+1} = \sqrt{2a_n}. \]
   Prove that this sequence is convergent and find all possible limits (the limit may depend on $x$).

7. Let $A$ be a non-empty set of real numbers. Define $-A = \{ x : -x \in A \}$. Show that
   \[ \sup(-A) = -\inf A \]
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8. Find
   \begin{align*}
   & (a) \sup\{ x \in \mathbb{R} : x^2 + 4x + 1 < 0 \} \\
   & (b) \inf\{ z = x + x^{-1} : x > 0 \}
   \end{align*}

9. If $(a_n)$ is an increasing sequence that is not bounded above, show that $(a_n) \to \infty$.

10. Prove that
    \[ \sqrt{3} = \inf\{ x \in \mathbb{Q} : x > 0 \text{ and } x^2 > 3 \}. \]