Assignment 8

Due Monday 29 November 15:00 (in supervisor pigeon hole)

Problem 1. Use the Comparison Test to determine whether each of the following series converges or diverges.

(a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2+1}}$

(b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^7+1}}$

(c) $\sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n})$

Problem 2. Determine whether each of the following series converges or diverges. Make your reasoning clear.

(a) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$

(b) $\sum_{n=1}^{\infty} \frac{5^n+4^n}{7^n-2^n}$

Problem 3. Let $a_n = \frac{n^2}{2^n}$. Prove that if $n \geq 3$, then

$$\frac{a_{n+1}}{a_n} \leq \frac{8}{9}.$$

By using this inequality for $n = 3, 4, 5, ...$, prove that

$$a_{n+3} \leq \left(\frac{8}{9}\right)^n a_3.$$

Using the Comparison Test and results concerning the convergence of the Geometric Series, show that $\sum_{n=1}^{\infty} a_{n+3}$ is convergent. Now use the Shift Rule to show that $\sum_{n=1}^{\infty} a_n$ is convergent.

Problem 4. Write down an example of a convergent series and a divergent series both of which satisfy the condition $\frac{a_{n+1}}{a_n} \to 1$.

Problem 5. Use the Ratio Test to determine whether each of the following series converges or diverges. Make your reasoning clear.

(a) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ (b) $\sum_{n=1}^{\infty} \frac{3^n}{n^n}$ (c) $\sum_{n=1}^{\infty} \frac{n!}{n^{n/2}}$

Problem 6. Show that $\sum_{n=101}^{200} \frac{1}{k} = \frac{1}{101} + \frac{1}{102} + ... + \frac{1}{200} \in [0.688, 0.694]$.

Problem 7. Show that (a) $\sum_{n=1}^{\infty} \frac{1}{(n+1)\log(n+1)}$ is divergent;
(b) $\sum_{n=1}^{\infty} \frac{1}{(n+1)(\log(n+1))^2}$ is convergent.

Problem ♣ (1pt extra credit; thanks to Tony Xu for suggesting it!)
Prove or disprove (e.g. with a counter-example) the following statement: “If (a_n) is a bounded sequence such that $a_{n+1} - a_n \to 0$ as $n \to \infty$, then (a_n) converges.”