Assignment 1

Problem 1. Here is a counterexample to Proposition 1.2 ($C([a, b])$ is complete w.r.t. $\| \cdot \|_\infty$ norm). Let $f_n(x) = x^n$ on the interval $[0, 1]$. The sequence $(f_n)$ converges to

$$f(x) = \begin{cases} 
0 & \text{if } 0 \leq x < 1; \\
1 & \text{if } x = 1.
\end{cases}$$

Then $f$ must be continuous. What is wrong here?

Problem 2. Prove that the Banach space $C([0, 1], \mathbb{R})$, with the $\| \cdot \|_\infty$ norm, is separable.

Problem 3. Let $Y \subset \ell^p(\mathbb{C})$ be the set of sequences (of complex numbers) with finitely many non-zero entries, i.e.

$$Y = \{(x_1, x_2, \ldots) : \exists k < \infty \text{ such that } x_i = 0 \ \forall \ i > k\}$$

We equip $Y$ with the usual $\ell^p$ norm, $1 \leq p \leq \infty$. Explain why $Y$ is not a Banach space. (Prove your assertions!) Show that $Y$ is dense in $\ell^p(\mathbb{C})$.

Problem 4.

(a) Show that $\ell^\infty(\mathbb{C})$ is not separable. (Hint: This is similar to proving that it is not countable.)

(b) Let $Z \subset \ell^\infty(\mathbb{C})$ be the set of sequences $x = (x_1, x_2, \ldots)$ such that $\lim_{k \to \infty} x_k = 0$. Show that $Z$ is a closed subspace.

Problem 5.

(a) Show that $\ell^p(\mathbb{C}) \subset \ell^q(\mathbb{C})$ when $1 \leq p < q \leq \infty$.

(b) Find a sequence of numbers converging to 0, which is not in any $\ell^p$ space with $1 \leq p < \infty$.

(c) If $x \in \ell^p$ for some finite $p$, show that

$$\lim_{p \to \infty} \|x\|_p = \|x\|_\infty.$$

Problem 6. Consider two arbitrary norms $\| \cdot \|_1$ and $\| \cdot \|_2$ in $\mathbb{R}^n$. Show that a sequence $(x_k)$ converges with respect to $\| \cdot \|_1$ if it converges with respect to $\| \cdot \|_2$. Are the limits necessarily the same?

Hint: Show that, if $\|x_k\| \to 0$, then each component of $x_k$ goes to 0. This holds with respect to any norm.