Assignment 2

Problem 1. (Quotient space) Let $X$ be a normed vector space and $Z$ be a subspace of $X$. It induces an equivalence relation, $x \sim y$ iff $x - y \in Z$.

(a) Check that the set of equivalence classes of $X$, $X/Z$, is a linear space when equipped with the following linear operations:
$$\alpha [x] + \beta [y] = [\alpha x + \beta y].$$

(b) Suppose that $Z$ is closed, and define the quotient norm on $X/Z$ by
$$\| [x] \|_q = \inf \{ \| y \| : y \sim x \} = \inf \{ \| x + z \| : z \in Z \}.$$ Check that $\| \cdot \|_q$ is a norm. Why do we need $Z$ to be closed?

Problem 2.

(a) Check that the operator norm is a norm indeed.
(b) Prove Theorem 1.4 (a linear map is continuous iff it is bounded).
(c) Prove that $\| Ax \| \leq \| A \| \| x \|$, and that $\| x_n \| \to \| x \|$ if $x_n \to x$.

Problem 3. In the linear space $C^\infty([0,1])$ with the sup norm, we consider the two operators $M$ (multiplication) and $D$ (differentiation):
$$(Mf)(x) = xf(x), \quad (Df)(x) = f'(x).$$ Show that $\| M \| = 1$ and $\| D \| = \infty$.

Problem 4. Prove that the integral operator $K$ on $C([0,1])$ defined by
$$(Kf)(x) = \int_0^1 k(x,y)f(y)dy,$$ where $k \in C([0,1] \times [0,1])$, has norm
$$\| K \| = \max_{0 \leq x \leq 1} \int_0^1 |k(x,y)|dy.$$ 

Problem 5. Prove the claims of Proposition 1.7 about finite-dimensional Banach spaces.

Problem 6. Consider the functional $\delta : C([0,1]) \to \mathbb{R}$ such that $\delta(f) = f(0)$. Find the norm of $\delta$, when $C([0,1])$ is equipped with the $L^p$ norm, $1 \leq p \leq \infty$. 