Problem 1.
(a) Show that the norm induced by an inner product satisfies the parallelogram identity.
(b) Let $\| \cdot \|$ be a norm that satisfies the parallelogram identity. Show that the polarization identity defines an inner product. (Hint: You may want to establish the following identity:
$$\|x + y + z\|^2 - \|x - y - z\|^2 = \|x + y\|^2 - \|x - y\|^2 + \|x + z\|^2 - \|x - z\|^2.$$ 
This may help to prove linearity. As far as I know, this exercise is not easy.)

Problem 2. Show that $L^p(\mathbb{R})$ can be turned into a Hilbert space if and only if $p = 2$, in which case the inner product is 
$$(f, g) = \int_{-\infty}^{\infty} f(x)g(x)dx.$$ 
(Hint: Consider two functions with disjoint supports. Then the parallelogram identity reduces to showing that $(a^p + b^p)^{2/p} = a^2 + b^2$ for any positive numbers $a, b$.)

Problem 3. Let $X$ be a separable infinite-dimensional Hilbert space. Show that:
(a) Every orthonormal sequence converges weakly to 0.
(b) The unit sphere $S = \{x : \|x\| = 1\}$ is weakly dense in the unit ball $B = \{x : \|x\| \leq 1\}$.
(Note: These properties also hold for nonseparable Hilbert spaces.)

In Problems 4 and 5 we prove that the Fourier functions form an orthonormal basis of $L^2([0, 2\pi])$. Fourier series are then a special case of Proposition 4.7. Problem 6 is a simple, cute, nontrivial, and rather surprising application of Fourier series to the Riemann zeta function.

Problem 4. Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ denote the circle, where operations $x + y$ and $x - y$ are taken modulo $2\pi$. Recall that the convolution of two functions is defined by 
$$(f * g)(x) = \int_{\mathbb{T}} f(x - y)g(y)dy.$$
Let \( \varphi_n \geq 0 \) be a function on \( T \) that satisfies
\[
\int_T \varphi_n(x) \, dx = 1, \quad \text{and} \quad \lim_{n \to \infty} \int_{|x| > \delta} \varphi_n(x) \, dx = 0
\]
for any \( \delta > 0 \). Show that, for any \( f \in C(T) \), \( \varphi_n * f \) converges uniformly to \( f \), i.e.
that
\[
\lim_{n \to \infty} \sup_{x \in T} |(\varphi_n * f)(x) - f(x)| = 0.
\]

**Problem 5.**

(a) Show that the function \( \varphi_n(x) = c_n (1 + \cos x)^n \), where \( c_n \) is chosen so that \( \int_T \varphi_n = 1 \), satisfies the properties of Problem 4.

(b) Check that \( \{ \frac{1}{\sqrt{2\pi}} e^{ikx} \}_{k \in \mathbb{Z}} \) is an orthonormal set in \( L^2(T) \).

(c) Show that \( \{ \frac{1}{\sqrt{2\pi}} e^{ikx} \}_{k \in \mathbb{Z}} \) is an orthonormal basis for \( L^2(T) \). For this, show that
\[
\varphi_n(x) = \sum_{k=-n}^{n} a_{nk} e^{ikx}, \quad \text{with} \quad a_{nk} = 2^{-n} c_n \left( \frac{2n}{n+k} \right).
\]
Then show that for any \( f \in C(T) \),
\[
(\varphi_n * f)(x) = \sum_{k=-n}^{n} b_{nk} e^{ikx}, \quad \text{with} \quad b_{nk} = a_{nk} \int_T e^{-ikx} f(y) \, dy.
\]
Then use Problem 4 and Proposition 4.7 of the course to conclude that \( \{ \frac{1}{\sqrt{2\pi}} e^{ikx} \}_{k \in \mathbb{Z}} \) is an orthonormal basis.

**Problem 6.** Show that \( \zeta(2) \equiv \sum_{n \geq 1} n^{-2} = \pi^2/6 \). Hint: Use the fact that \( \{ \frac{1}{\sqrt{2\pi}} e^{ikx} \}_{k \in \mathbb{Z}} \) is an orthonormal basis for \( L^2(T) \), and that the function \( f(x) = x \) has Fourier coefficients proportional to \( 1/n \).

Can you get another identity with \( f(x) = x^2 \)?