Assignment 8

**Problem 1.** Prove Theorem 6.7. That is, if $T$ is a densely defined operator, prove that

(a) $T^*$ is closed.
(b) If $T^*$ is densely defined, then $T$ is closable.
(c) If $T$ is symmetric, then $T$ is closable and $(T^*)^* = T^*$.
(d) If $T : D(T) \to X$ is bijective and closed, then $T^{-1}$ is bounded. (Hint: show that $T^{-1}$ is closed and use the closed graph theorem).

**Problem 2.** Let $T : D(T) \to X$ be densely-defined, and let $\sigma'(T)$ denote the set of *approximate eigenvalues*; precisely,

$$\sigma'(T) = \{ \lambda \in \mathbb{C} : \inf_{x \in D(T) \atop \|x\|=1} \| (T - \lambda 1 I)x \| = 0 \}.$$

(a) Show that, if $T$ is self-adjoint, $\sigma'(T) = \sigma(T)$. (Compare with Prop. 5.12 for bounded operators.)
(b) More generally, show that

$$\sigma_p(T) \cup \sigma_c(T) \subset \sigma'(T) \subset \sigma(T).$$

**Problem 3.** An example where the domain affects the spectrum. Let $X = \ell^2(\mathbb{N})$, $x = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$, and $M = \text{span}\{x\}$. Let $P$ be the orthogonal projection onto $M$.

(a) Suppose that $D(P) = \ell^2$. Show that $1 \in \sigma_p(P)$.
(b) Suppose that $D(P)$ is the space of elements of $\ell^2$ with finitely many non-zero entries. Show that $1 \in \sigma_r(P)$.

**Problem 4.** Generalisation of the above example. Let $S \subset T$ be densely-defined operators, with $T$ an extension of $S$. Prove that

$$\sigma_p(S) \subset \sigma_p(T);$$
$$\sigma_r(S) \supset \sigma_r(T);$$
$$\sigma_c(S) \subset \sigma_p(T) \cup \sigma_c(T).$$

**Problem 5.** This question is optional, but it is puzzling.

Let $\ell_0$ be the space of all sequences of complex numbers $(x_1, x_2, \ldots)$ with finitely many nonzero entries. Can you find a norm such that $\ell_0$ is complete? If yes, give it. If not, prove there exists none.

(I heard that Baire category theorem might help.)