MA 131

THE UNIVERSITY OF WARWICK

FIRST YEAR EXAMINATION: January 2011

Analysis I

Time Allowed: **1.5 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

ANSWER 3 QUESTIONS.
If you have answered more than the required 3 questions in this examination, you will only be given credit for your 3 best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

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1. a) Define what it means for a sequence (a_n) to be bounded. **[2]**

b) For what values of x is the following inequality valid:

$$|x+2| \leq 5.$$

[2]

c) Let the sequence (a_n) be defined according to $a_1 = 1$ and $a_n = \sqrt{3a_{n-1} - a_{n-1}^2}$ for $n \geq 2$. Assume that (a_n) tends to $a > 0$. Find the numerical value of a. Motivate your answer. **[2]**

d) Assume that (a_n) is a decreasing sequence. Which of the following statements are true:

i) (a_n) has a limit.

ii) (a_n) is eventually negative.

iii) (a_n) tends to minus infinity.

iv) (a_n) tends to zero. **[2]**

e) State, without proof, whether the following sequences tends to A) infinity, B) minus infinity, C) zero, D) a constant $c \neq 0$ or or E) does not tend to either infinity, minus infinity or any real number.

 Question 1 continued overleaf

i) $(a_n) = \left(\frac{n^2}{2^n}\right)$,

ii) $(a_n) = \left(\frac{2n^2+\sin(n)}{5n^2+3n+17}\right)$,

iii) $(a_n) = \left(\left(10^{n^2} + \sqrt{n}\right)^{1/n}\right)$,

iv) $(a_n) = \left(e^n \cos(n)\right)$. **[4]**

f) Prove that $\inf\{a_n^2 : n \in \mathbb{N}\} = 0$ for any null sequence (a_n). **[2]**

g) An increasing sequence (a_n) satisfies $a_n < 4$ for every $n \in \mathbb{N}$. State, without proof, whether (a_n) must converge. **[1]**

h) State the Bolzano-Weierstrass Theorem and use it to decide whether the sequence $(\cos(2^n))$ has a convergent subsequence. **[3]**

i) Find a sequence (a_n) that satisfies $|a_{n+1} - a_n| < |a_n - a_{n-1}|$ for every $n > 1$ but (a_n) is not Cauchy. **[2]**

j) Prove that every Cauchy sequence is bounded. **[3]**

k) Give an example of a *diverging* series $\sum_{n=1}^{\infty} a_n$ such that $\frac{a_{n+1}}{a_n} \to 1$ as $n \to \infty$. **[2]**

l) Give an example of a *converging* series $\sum_{n=1}^{\infty} a_n$ such that $\frac{a_{n+1}}{a_n} \to 1$ as $n \to \infty$. **[2]**

m) Compute $\sum_{n=1}^{\infty} \frac{\pi}{3^n}$. **[2]**

n) Is $\sum_{n=1}^{\infty} \frac{n+\sin^2 n}{n^2+\cos n}$ finite or infinite? Motivate your answer. **[3]**

o) Is $\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}}$ finite or infinite? Motivate your answer. **[3]**

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2. a) State and prove the Sandwich Theorem for sequences. **[5]**

b) Prove Bernoulli's inequality: If $x > -1$ and n is a natural number then

$$(1+x)^n \geq 1 + nx.$$

[3]

c) Prove the following statement: If $x > 0$ then $(x^{1/n}) \to 1$.
HINT: Consider the sequence $(x^{1/n} - 1)$. **[7]**

d) Prove that the sequence

$$(a_n) = \left(\left(\frac{1+\cos^2(n)}{2+\cos(n)}\right)^{1/n}\right)$$

tends to 1 as n tends to infinity. **[5]**

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3. a) Suppose that A is a non-empty set of real numbers. Define what it means for a real number u to be

- an upper bound for A;
- a least upper bound for A. **[2]**

b) Suppose that (a_n) is any bounded increasing sequence. Prove that (a_n) tends to the least upper bound of the set $\{a_n : n \in \mathbb{N}\}$. **[3]**

c) Let

$$b_n = (-1)^n \left(\frac{n}{n+1} + \frac{(-1)^n}{2(n+1)} \right).$$

(i) Calculate the first five terms of (b_n). Is (b_n) monotone? **[2]**

(ii) Show that (b_n) is bounded. State, without proof, whether (b_n) converges. **[2]**

(iii) Find an increasing subsequence of (b_n). Show that your subsequence is strictly increasing and identify the limit. **[3]**

(iv) Write down a subsequence of (b_n) which converges to a limit different to that found in (iii). Write the limit explicitly. **[2]**

d) Define what it means for a sequence (a_n) to be Cauchy. **[2]**

e) Suppose that (a_n) satisfies $|a_{n+1} - a_n| \leq 2^{-n}$. Using the fact that $\sum_{k=n}^{\infty} 2^{-k} \to 0$ as $n \to \infty$, show that (a_n) is Cauchy. **[4]**

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4. a) (i) State the Alternating Test for the convergence of series. **[2]**

(ii) Suppose that (a_n) is decreasing and null. Show that

$$0 \leq \sum_{k=N}^{\infty} (-1)^k a_k \leq a_N$$

for all even numbers N. **[2]**

(iii) Let $b_n = \frac{(-1)^n}{(n+1)^{1/3}}$. Find an N such that the difference between the sum of the first N terms, $\sum_{k=1}^{N} b_k$, and the whole series, is guaranteed to be less than 10^{-2}. (Give justifications.) **[2]**

b) Either prove the statements below, or give a counter-example.

(i) "If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$." **[2]**

(ii) "If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges." **[2]**

(iii) "If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^2$ converges." **[2]**

c) Consider $\sum_{n=1}^{\infty} \frac{1}{n} x^n$, where x is a real number (positive or negative). Find all numbers x such that (i) the series converges; (ii) the series converges absolutely; (iii) the series diverges to $+\infty$; (iv) the series does not converge. (No justifications needed.) **[4]**

 Question 4 continued overleaf

d) Consider the series $\sum_{n=1}^{\infty}\left(\sqrt{n+2}-\sqrt{n}\right)$.

(i) State whether it converges or diverges. [**1**]

(ii) Give a full proof of whether it converges or diverges. [**3**]

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MATHEMATICS DEPARTMENT
FIRST YEAR UNDERGRADUATE EXAMS

Course Title: Analysis I

Model Solution No: 1

a) A sequence (a_n) is bounded if there exists a real number C such that $|a_n| \leq C$ for all $n \in \mathbb{N}$. **[2]**

b) We have two cases.

Case 1: If $x \geq -2$. Then $|x+2| = x+2$ which implies that if $x \geq -2$ then $|x+2| \leq 5$ if and only if $x \leq 3$. So $-2 \leq x \leq 3$ are all the solutions when $x \geq -2$.

Case 2: If $x < -2$. Then $|x+2| = -x-2$ which implies that if $x < -2$ then $|x+2| \leq 5$ if and only if $-x \leq 7$. That is, the only solutions when $x < -2$ are $-7 \leq x < -2$.

Putting the two cases together gives the answer $|x+2| < 5$ if and only if $-7 \leq x \leq 3$. **[2]**

c) Since $a_n \to a$ we can conclude that $a = \sqrt{3a - a^2}$. Squaring both sides and rearranging terms implies that $a(2a-3) = 0$ which has the solutions $a = 0$ and $a = 3/2$. But $a > 0$ by assumption so we can conclude that the limit is $a = 3/2$. **[2]**

d) None of the statements are true. Two marks for the right answer. One mark for three correct answers. No marks for two or less correct answers.

e) i) C)

ii) D)

iii) A)

iv) E)

One mark for each correct answer.

f) Firstly, 0 is a lower bound since $a_n^2 \geq 0$. Now, suppose $\epsilon > 0$. Since (a_n) is null, there exists $N \in \mathbb{N}$ such that $|a_n| < \sqrt{\epsilon}$ for $n > N$. Hence, for such n, $a_n^2 = |a_n|^2 < \epsilon$. So ϵ is not a lower bound. **[2]**

g) Yes, it does since any increasing sequence that is bounded above converges by the completeness axiom. **[1]**

h) **Bolzano-Weierstrass** Any bounded sequence has a convergent subsequence. (1 mark)

Now, $|\cos(x)| \leq 1$ for every $x \in \mathbb{R}$, so $\cos(2^n)$ is bounded. (1 mark) It follows that there is a convergent subsequence from the Bolzano-Weierstrass theorem. (1 mark) **[3]**

i) Any sequence whose terms get closer together but does not converge will do. For example take $a_n = \sqrt{n}$, which tends to ∞. Then, using $(x-y)(x+y) = x^2 - y^2$,

$$a_{n+1} - a_n = \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

But,

$$\frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n} + \sqrt{n-1}} = a_n - a_{n-1}.$$

[2]

j) Suppose that (a_n) is Cauchy. Then, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon$ for any $n, m > N$. Choosing $m = N + 1$, it follows that $a_{N+1} - \epsilon < a_n < a_{N+1} + \epsilon$ for $n > N$. (2 marks for this)

Now, take $L = \min\{a_1, \ldots, a_N, a_{N+1} - \epsilon\}$, $U = \max\{a_1, \ldots, a_N, a_{N+1} + \epsilon\}$. Then $L \leq a_n \leq U$ for every $n \in \mathbb{N}$. (Final mark for this). **[3]**

k) $a_n = 1$. **[2]**

l) $a_n = 1/n^2$. **[2]**

m) $\sum_{n=1}^{\infty} \frac{\pi}{3^n} = \frac{\pi}{2}$. **[2]**

n) It is infinite. Indeed, the series is bigger than $\sum_{n=1}^{\infty} \frac{n}{n^2+1} \geq \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$ and we know that the harmonic series diverges. **[3]**

o) It is finite. Indeed, Stirling formula gives $(2n)! \leq (2n)^{2n} e^{-2n} (2n)$, so the series is less than $2 \sum_{n=1}^{\infty} n (\frac{2}{e})^{2n}$. This series is easily shown to converge, e.g. by the ratio test, since $2/e < 1$. **[3]**

MATHEMATICS DEPARTMENT
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Course Title: Analysis I

Model Solution No: 2

a) **Sandwich Theorem for sequences:** *Suppose that* $(a_n) \to l$ *and* $(b_n) \to l$*. If* $a_n \leq c_n \leq b_n$ *then* $(c_n) \to l$. **[2]**

Proof: Since $(b_n) \to l$ there exist for each $\epsilon > 0$ a natural number $N_{b,\epsilon}$ such that

$$l - \epsilon < b_n < l + \epsilon \qquad (*)$$

for all $n > N_{b,\epsilon}$.

Similarly, there exists for each $\epsilon > 0$ a natural number $N_{a,\epsilon}$ such that

$$l - \epsilon < a_n < l + \epsilon \qquad (**)$$

for all $n > N_{a,\epsilon}$.

Let $\epsilon > 0$ and $N_\epsilon = \max(N_{a,\epsilon}, N_{b,\epsilon})$ then $a_n \leq c_n \leq b_n$, $(*)$ and $(**)$ implies that $l - \epsilon < c_n < l + \epsilon$ for all $n > N_\epsilon$. That is $|c_n - l| < \epsilon$ for all $n > N_\epsilon$.

We have proved that for each $\epsilon > 0$ there exist an N_ϵ such that $|c_n - l| < \epsilon$ for all $n > N_\epsilon$. This is exactly the definition of $(c_n) \to l$. **[3]**

b) We will prove this by induction. The inequality is trivial for $n = 1$. Assume that the inequality holds for $n = k$. That is

$$(1 + x)^k \geq 1 + kx \text{ for all } x > -1. \quad (*)$$

For any $x > -1$ we can multiply both sides of $(*)$ by $1 + x > 0$ without changing the inequality:

$$(1+x)^{k+1} = (1+x)^k \cdot (1+x) \geq (1+kx) \cdot (1+x) = 1 + (k+1)x + kx^2 \geq 1 + (1+k)x.$$

The last inequality is valis since $kx^2 > 0$.

We have thus shown that the Bernoulli inequality for $n = k$ implies the Bernoulli inequality for $n = k + 1$. The inequality follows by induction. **[3]**

c) Let us assume that $x \geq 1$. Then $a_n = x^{1/n} - 1 \geq 0$ since $x^{1/n} \geq 1$ for all $x \geq 1$. By Bernoulli's inequality, which is applicable since $a_n \geq 0 > -1$, we may conclude that

$$(1 + a_n)^n \geq 1 + na_n. \qquad (*)$$

But $(1 + a_n)^n = (x^{1/n})^n = x$ so we may conclude from $(*)$ that

$$x \geq 1 + na_n \implies \frac{x-1}{n} \geq a_n.$$

So $0 \leq a_n \leq \frac{x-1}{n}$. By the product rule we can conclude that $\left(\frac{x-1}{n}\right) \to 0$ since $1/n \to 0$ and $(x-1)$ is a constant sequence.

It follows from the sandwich rule that $(a_n) \to 0$. That is $(x^{1/n} - 1) \to 0$ which implies that $(x^{1/n}) \to 1$ by the sum rule. This proves the statement for all $x \geq 1$. **[5]**

If $0 < x < 1$ then $1/x = y \geq 1$ which implies that $(y^{1/n}) \to 1$ by the previous reasoning. But

$$x^{1/n} = \frac{1}{y^{1/n}} \to 1$$

where we used the quotient rule in the last step. It follows that $(x^{1/n}) \to 1$ if $x > 0$. **[2]**

d) Notice that, since $-1 \leq \cos(n) \leq 1$,

$$\frac{1}{3} \leq \frac{1 + \cos^2(n)}{2 + \cos(n)} \leq 2.$$

Since $x \leq y$ is equivalent to $x^n \leq y^n$ for positive numbers we can conclude that

$$\left(\frac{1}{3}\right)^{1/n} \leq a_n \leq 2^{1/n}.$$

From c) we know that $(1/3)^{1/n} \to 1$ and $2^{1/n} \to 1$ and therefore by the sandwich rule for sequences $a_n \to 1$. **[5]**

MATHEMATICS DEPARTMENT
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Course Title: Analysis I

Model Solution No: 3

a) The real number u is an upper bound of A if $a \leq u$ for every $a \in A$. (1 mark)

It is a least upper bound of A, if for any upper bound U, we have $u \leq U$. (1 mark) **[2]**

b) Suppose $\epsilon > 0$ and let $s = \sup\{a_n : n \in \mathbb{N}\}$. Then, there exists $N \in \mathbb{N}$ such that $a_N > s - \epsilon$ (otherwise, $s - \epsilon$ is an upper bound for $\{a_n : n \in \mathbb{N}\}$ which is smaller than s). (1 mark)

Since (a_n) is increasing, we have $a_n \geq a_{n-1} \geq \ldots \geq a_{N+1} \geq a_N > s - \epsilon$ for any $n > N$. (1 mark)

Hence, using the fact that s is an upper bound for a_n, we have $s - \epsilon < a_n \leq s$ for $n > N$. But $s - \epsilon < a_n \leq s$ implies that $|a_n - s| < \epsilon$ so we're done. (final mark) **[3]**

c) (i) $b_n = \frac{1+2n(-1)^n}{2(n+1)}$, so

$$b_1 = -1/4, b_2 = 5/6, b_3 = -5/8, b_4 = 9/10, b_5 = -9/12.$$

Not monotone (signs of terms alternate). **[2]**

(ii) The sequence is bounded above by 1 since

$$b_n = \frac{1+2n(-1)^n}{2(n+1)} \leq \frac{1+2n}{2n+2} < 1.$$

It is bounded below by -1,

$$b_n = \frac{1+2n(-1)^n}{2(n+1)} \geq \frac{1-2n}{2n+2} = \frac{-(2+2n)+3}{2n+2} \geq -1.$$

It does not converge. **[2]**

(iii) The only increasing subsequence is a subsequence of the even numbered terms (and possibly the first term is an odd numbered term). The even terms can be written,

$$b_{2n} = \frac{1+4n}{4n+2} = 1 - \frac{1}{4n+2}.$$

so

$$b_{2(n+1)} - b_{2n} = \frac{1}{4n+2} - \frac{1}{4(n+1)+2} > 0.$$

Furthermore, $b_{2n} > 0 > b_{2k-1}$ for any k. So any increasing subsequence of (b_n) is strictly increasing. **[3]**

(iv) The sequence must eventually involve only odd terms. They can be written

$$b_{2n-1} = \frac{3-4n}{4n},$$

which converges to -1 as $n \to \infty$. **[2]**

d) A sequence (a_n) is Cauchy if, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon$ for any $n, m > N$. **[2]**

e) First of all observe that, for $n > m$,

$$|a_n - a_m| = |a_n - a_{n-1} + a_{n-1} - a_{n-2} + \ldots + a_{m+1} - a_m|.$$

Hence, applying the triangle inequality to the right hand side,

$$|a_n - a_m| \leq \sum_{k=m}^{n-1} |a_{k+1} - a_k|.$$

(1 mark)

Next use the fact that $0 \leq |a_{k+1} - a_k| < 2^{-k}$ to get

$$|a_n - a_m| \leq \sum_{k=m}^{n-1} 2^{-k} \leq \sum_{k=m}^{\infty} 2^{-k}.$$

(1 mark)

Now suppose $\epsilon > 0$. Since $\sum_{k=m}^{\infty} 2^{-k} \to 0$, there exists $N \in \mathbb{N}$ such that $|\sum_{k=m}^{\infty} 2^{-k}| < \epsilon$ for every $m > N$. (1 mark)

It follows that, for $n > m > N$,

$$|a_n - a_m| \leq \sum_{k=m}^{\infty} 2^{-k} < \epsilon.$$

If $n = m$ then the difference $|a_n - a_m|$ vanishes, and if $m > n > N$ we just swap m, n in the argument above.

(final mark) **[4]**

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Course Title: Analysis I

Model Solution No: 4

a) (i) If (a_n) is decreasing and null, then the series $\sum(-1)^{n+1}a_n$ is convergent.

(ii) We have

$$\sum_{k=N}^{\infty}(-1)^k a_k = (a_N - a_{N+1}) + (a_{N+2} - a_{N+3}) + ... \geq 0,$$

since all brackets are positive. Also,

$$\sum_{k=N}^{\infty}(-1)^k a_k = a_N - (a_{N+1} - a_{N+2}) - (a_{N+3} - a_{N+4}) - ... \leq a_N,$$

since all brackets are positive. Then $0 \leq \sum_{k=N}^{\infty}(-1)^k a_k \leq a_N$.

(iii) Without the $(-1)^n$, the sequence is decreasing and null. By similar considerations as above, we have

$$\Big|\sum_{k=1}^{N} b_k - \sum_{k=1}^{\infty} b_k\Big| \leq |b_{N+1}|.$$

It is then enough that $|b_{N+1}| \leq 10^{-2}$, which certainly holds for $N \geq 10^6$.

b) Either prove the statements below, or give a counter-example.

(i) True statement. If $(b_N) = (\sum_{k=1}^{N} a_k)$ converges, then $b_N - b_{N-1} \to 0$, i.e., $a_N \to 0$.

(ii) Wrong statement. A counter-example is $a_n = \frac{1}{n}$.

(iii) Wrong statement. A counter-example is $a_n = \frac{(-1)^n}{\sqrt{n}}$.

c) (i) The series converges for $|x| < 1$ and for $x = -1$. (ii) The series converges absolutely for $|x| < 1$. (iii) The series diverges to $+\infty$ for $x \geq 1$. (iv) The series does not converge for $|x| > 1$ and for $x = 1$.

d) (i) The series $\sum_{n=1}^{\infty}\left(\sqrt{n+2} - \sqrt{n}\right)$ diverges.

(ii) Since $(\sqrt{n+2} - \sqrt{n})(\sqrt{n+2} + \sqrt{n}) = 2$, we have $\sqrt{n+2} - \sqrt{n} = \frac{2}{\sqrt{n+2}+\sqrt{n}} \geq \frac{1}{\sqrt{n+2}}$. The latter series is the shifted series of $1/\sqrt{n}$, which diverges, as can be proved e.g. by integral bounds.