Assignment 6

Due Monday 14 November 15:00 (in supervisor pigeon hole)

Problem 1. Find sup $A$ and inf $A$ where $A$ is the set defined by

(a) $A = \{x \in \mathbb{R} : x^4 < 16\}$
(b) $A = \{x \in \mathbb{R} : x^4 \leq 16\}$
(c) $A = \{x \in \mathbb{Q} : x = \frac{1}{n} + 2^{-n}, n \in \mathbb{N}\}$
(d) $A = \{x \in \mathbb{R} : |x| < 3 \text{ and } x^2 > 2\}$
(e) $A = \{x = \frac{m}{2n+1}, \ m, n \in \mathbb{N}\}$
(f) $A = \{x \in \mathbb{Q} : 0 < \sqrt{x} < 3\}$

Problem 2. Prove the following theorem, originally due to Cauchy. Suppose that $(a_n) \to a$. Then the sequence $(b_n)$ defined by

$$b_n = \frac{a_1 + a_2 + ... + a_n}{n}$$

is convergent and $(b_n) \to a$.

Problem 3. Find the limit of the sequence $(a_n)$ defined by

$$a_n = \frac{1 + \sqrt{2} + \sqrt[3]{3} + ... + \sqrt{n}}{n}.$$  

(The result of the previous exercise may help!)

Problem 4. Let $(a_n)$ be a decreasing sequence that is bounded below. Does it necessarily converge? If yes, prove it. If not, give a counter-example.

Problem 5. Consider the sequence defined by

$$a_{n+1} = \frac{c}{a_n} + \frac{a_n}{2}, \quad a_0 = 2c.$$  

Here, $c > 0$ is a fixed parameter. Show the following:

(i) $(a_n)$ is bounded below by $\sqrt{2c}$
(ii) $(a_n)$ is decreasing;
(iii) $(a_n)$ is convergent (and find the limit).

We studied the case $c = 1$ in class, and we noticed that the sequence converges very, very fast to its limit $\sqrt{2}$. We actually proved that

$$|a_{n+1} - \sqrt{2}| \leq \frac{1}{2^{2n}}.$$  

(iv) State and prove something similar for the sequence with parameter $c$.

Problem 6. Give an example of a sequence $(a_n)$ which is not convergent, but such that $a_{n+1} - a_n \to 0$. 