Assignment 8

Due Monday 28 November 15:00 (in supervisor pigeon hole)

Problem 1. Use the Comparison Test to determine whether each of the following series converges or diverges.
   (a) \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}} \)
   (b) \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}} \)
   (c) \( \sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n}) \)

Problem 2. Determine whether each of the following series converges or diverges. Make your reasoning clear.
   (a) \( \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} \)
   (b) \( \sum_{n=1}^{\infty} \frac{5^n + 4^n}{7^n - 2^n} \)

Problem 3. Let \( a_n = \frac{n^2}{2\pi} \). Prove that if \( n \geq 3 \), then
   \[ \frac{a_{n+1}}{a_n} \leq \frac{8}{9}. \]

By using this inequality for \( n = 3, 4, 5, \ldots \), prove that
   \[ a_{n+3} \leq \left( \frac{8}{9} \right)^n a_3. \]

Using the Comparison Test and results concerning the convergence of the Geometric Series, show that \( \sum_{n=1}^{\infty} a_{n+3} \) is convergent. Now use the Shift Rule to show that \( \sum_{n=1}^{\infty} a_n \) is convergent.

Problem 4. Write down an example of a convergent series and a divergent series both of which satisfy the condition \( \frac{a_{n+1}}{a_n} \to 1 \).

Problem 5. Use the Ratio Test to determine whether each of the following series converges or diverges. Make your reasoning clear.
   (a) \( \sum_{n=1}^{\infty} \frac{2^n}{n!} \)
   (b) \( \sum_{n=1}^{\infty} \frac{3^n}{n^n} \)
   (c) \( \sum_{n=1}^{\infty} \frac{n!}{n^{n/2}} \)

Problem 6. Show that \( \sum_{n=101}^{200} \frac{1}{k} = \frac{1}{101} + \frac{1}{102} + \ldots + \frac{1}{200} \in [0.688, 0.694] \).

Problem 7. Show that (a) \( \sum_{n=1}^{\infty} \frac{1}{(n+1)\log(n+1)} \) is divergent;
   (b) \( \sum_{n=1}^{\infty} \frac{1}{(n+1)(\log(n+1))^2} \) is convergent.

Problem ♣ (1pt extra credit; thanks to Tony Xu for suggesting it!)
Prove or disprove (e.g. with a counter-example) the following statement: “If \( (a_n) \) is a bounded sequence such that \( a_{n+1} - a_n \to 0 \) as \( n \to \infty \), then \( (a_n) \) converges.”