Foundations and Analysis Questions

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MORSE Society
A Year 1 Christmas Present

Contents

Introduction

I was studying for my January exam over two years ago, and I was wondering: How was it going to be like? Will I be able to do the paper? Have I done enough practice? These questions were floating through my mind, and I had few seniors to help me resolve these questions. This year, I thought I would alleviate some questions for the year 1s doing their first ever University exam. And so, I wrote this compilation of past year exam paper questions, meant as a ‘easy reference guide’, and as a means to test yourself - the exam will consist of questions similar to this standard. Thanks goes to the numerous seniors who have loaned me their exam papers. Consider this as a Christmas present. Merry Christmas :)

Disclaimer: I do not guarantee that the format of the exam will remain the same, nor that successfully being able to do all the questions within will net you a First Class, but it will aid you in understanding the concepts covered during the Foundations and Analysis lectures.

PS. Feel free to distribute this to other Year 1s who want this - they don't need to be a member of the MORSE Society. However, do encourage them to join if they are not - firms look at society's memberships at the start of the year, not before socials / printing of revision guides - so how much everyone gets subsidized is really due to membership levels! However, I do not want this to be uploaded on TSR or any other forums. I am shy.

1 Foundations

This exam paper usually consists of four questions. You have to do three questions, being Question 1 (which is compulsory), and another two out of the remaining Questions 2, 3, and 4. The trend in the last three years focuses on the following three topics for the three non compulsory questions, namely: Polynomials, Equivalence Relations, and Groups/Isomorphisms. However, all (other) possible questions will be listed here, just in case.

1.1 The Compulsory Question 1

For the last several years, the first question usually comprised of many subquestions, each testing the general concepts and understanding of the entire module. Here are questions which have come out over the years. The same numerical questions will not be repeated, *but* they will be in the same format as those below. Some subquestions from other questions are also put in here, because it just, *just* might come out. Generally, definitions may come out here as well, and it is good to know them.

1.1.1 Basics

Question 1.1. State the Well Ordering Principle.

Question 1.2. State the Fundamental Theorem of Arithmetic

Question 1.3. If $a, b, c \in \mathbb{N}$, what does the notation $a|b$ mean? Show that if $a|b$ and $b|c$ then $a|c$.

Question 1.4. Let $h = \text{hcf}\{245, 52\}$. Find h and write it as an integer linear combination of 245 and 52.

Question 1.5. Find $h = \text{hcf}(750, 231)$ and then find $a, b \in \mathbb{Z}$ such that $h = 750a + 231b$.

Question 1.6. Find $h = \text{hcf}(1900, 2011)$, and then find $a, b \in \mathbb{Z}$ such that $h = 1900a+2011b$.

Question 1.7. Express the repeating decimal $0.11232323\ldots$ as a rational number in the form $\frac{m}{n}$ where $m, n \in \mathbb{N}$. (You do not have to simplify your answer.)

1.1.2 Subgroups

Question 1.8. For $n, m \in \mathbb{N}$ define the sets $n\mathbb{Z}$ and $n\mathbb{Z} + m\mathbb{Z}$. Which of the following three sets are subgroups of $\mathbb{Z}$? (i) $5\mathbb{Z} \cap 3\mathbb{Z}$; (ii) $5\mathbb{Z} + 3\mathbb{Z}$; (iii) $5\mathbb{Z} \cup 3\mathbb{Z}$. In each case where the set is a subgroup, find an $n \in \mathbb{Z}$ such that the subgroup is equal to $n\mathbb{Z}$.

Question 1.9. $7\mathbb{Z} \cap 3\mathbb{Z}$ is an additive subgroup of $\mathbb{Z}$. Which additive subgroup is it?

Question 1.10. $8\mathbb{Z} + 12\mathbb{Z}$ is an additive subgroup of $\mathbb{Z}$. Which additive subgroup is it? (Give a shorter answer than "$8\mathbb{Z} + 12\mathbb{Z}$"!)

Question 1.11. $9\mathbb{Z} + 15\mathbb{Z}$ is an additive subgroup of $\mathbb{Z}$. Which additive subgroup is it? (Give a shorter answer than "$9\mathbb{Z} + 15\mathbb{Z}$"!)

Question 1.12. What is the smallest positive integer (i) in $7\mathbb{Z} + 9\mathbb{Z}$? (ii) $7\mathbb{Z} \cap 9\mathbb{Z}$?

1.1.3 Sets and Venn Diagrams

Question 1.13. State (without proof) *De Morgan's Laws* for sets A, B and their complements.

Question 1.14. Draw a Venn diagram illustrating the fact that $A \backslash (B \backslash C) \neq (A \backslash B) \backslash C$ for some choices of sets A, B and C.

Question 1.15. The set $(A \backslash B) \backslash C$ is equal, for all sets A, B and C, to one of the following sets; which one? $A \backslash (B \cap C)$; $A \backslash (B \backslash C)$; $A \backslash (B \cup C)$.

Question 1.16. For all sets A, B and C, the set $(A \backslash B) \cap (A \backslash C)$ is equal to one of the following sets; which one? $A \backslash (B \cup C)$; $A \backslash (B \cap C)$; $A \cup (B \cap C)$; $A \cap (B \cup C)$.

Question 1.17. Draw a truth table for $(P \wedge Q) \vee (\neg P \wedge Q)$ and hence demonstrate that this proposition is logically equivalent to Q.

Question 1.18. Which if any, of the following three statements are logically equivalent to the statement $P \Rightarrow (P \Rightarrow Q)$?
a) $(P \Rightarrow P) \Rightarrow Q$
b) $Q \Rightarrow P$
c) $P \Rightarrow Q$

Question 1.19. Which of the propositions (i)-(iv) is logically equivalent to $P \Rightarrow Q$?
(i) $\neg(P \vee Q)$; (ii) $P \vee (\neg Q)$; (iii) $(\neg P) \vee Q$; (iv) $(\neg P) \vee (\neg Q)$.

Question 1.20. Which of the propositions (i)-(iv) is logically equivalent to $P \Rightarrow \neg Q$?
(i) $\neg(P \vee (\neg Q))$; (ii) $P \vee Q$; (iii) $(\neg P) \vee (\neg Q)$; (iv) $(\neg P) \vee Q$.
Show truth tables to prove that your answer is correct.

Question 1.21. Which of the propositions (i)-(iv) is logically equivalent to $\neg(\neg P \Rightarrow Q)$?
(i) $P \Rightarrow Q$; (ii) $P \vee Q$; (iii) $(\neg P) \vee (\neg Q)$; (iv) $(\neg P) \wedge (\neg Q)$.
Show truth tables to prove that the one you select is equivalent to $\neg(\neg P \Rightarrow Q)$.

Question 1.22. One of the following propositions (i)-(iv) is logically equivalent to $\neg(a \Rightarrow b)$.
(i) $a \wedge \neg b$ (ii) $b \Rightarrow a$ (iii) $\neg a \vee b$ (iv) $\neg b \Rightarrow a$.
Which one is it? Use a truth table to justify your answer.

Question 1.23. One of the following propositions (i) - (iv) is logically equivalent to $a \Rightarrow b$.
(i) $a \wedge b$; (ii) $(\neg a) \vee b$; (iii) $(\neg b) \vee a$; (iv) $(\neg a) \wedge b$.
Which one is it? Use a truth table to justify your answer.

1.1.4 Mathematical Induction

Question 1.24. Prove by induction that $\displaystyle\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$ for all $n \in \mathbb{N}$.

Question 1.25. Prove by induction $1 + 3 + 5 + \ldots + (2n-1) = n^2$.

1.1.5 Remainder Theorem

Question 1.26. State the Remainder Theorem for polynomials.

Question 1.27. Which of the factors $(x+4), (x-2)$ and $(x-1)$ divide the polynomial $x^4 - 10x^2 + 21x - 12$ without remainder?

Question 1.28. Which of the factors $(x+2)$, $(x-3)$ and $(x-1)$ divide the polynomial $x^4 - 5x^3 - 13x^2 + 77x - 60$ without remainder?

Question 1.29. Which of the factors $(x+2), (x+3)$, and $(x-2)$ divide the polynomial $x^4 - 3x^3 - 15x^2 + 19x + 30$ without remainder?

Question 1.30. Which of the factors $(2x+1), (2x-1)$ and $(x+4)$ divide the polynomial $2x^4 + 9x^3 + 2x^2 - 9x - 4$ without remainder?

Question 1.31. Which of the factors $x-2$, $x+1$ and $x-3$ divides the polynomial $P(x) = x^3+12x^2+9x-2$ without remainder?

Question 1.32. For a certain value of the integer a, the polynomial $P = 2x^3-3x^2+ax-6$ is divisible (without remainder) by $x-1$. What is the remainder when P is divided by $x-2$? [Hint: find the value of a.]

1.1.6 Pascal's Triangle

Question 1.33. Prove, for integers $1 \le k \le n$, that

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

You may assume that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Question 1.34. For n a positive integer and $0 \le k \le n$, what is the coefficient of x^k in the expansion of the polynomial $(a+bx)^n$? Hence prove that the alternating sum of the numbers in the n^{th} row of Pascal's Triangle is zero. [*In other words, show that* $\sum_{k=0}^{n}(-1)^k \binom{n}{k} = 0$.]

Question 1.35. Prove that the sum of the n^{th} row of Pascal's triangle is 2^n. [*In other words, show that* $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.]

1.1.7 Injections and Surjections

Question 1.36. Is the following statement always true? Justify your answer. If $f : A \to B$ and $g : C \to D$ are surjections where $Im(f) \subseteq C$, then $g \circ f : A \to D$ is a surjection.

Question 1.37. For each of the following pairs of sets, give a bijection between them [Note: you do not need to show that the function you have constructed is a bijection]: (i) the interval $(0, 1)$ to $\mathbb{R}$ (ii) $\mathbb{N}$ to $\mathbb{Z}$.

Question 1.38. For a function $f : A \Rightarrow B$, define what it means for f to be *surjective*, *injective*, and hence *bijective*.

Question 1.39. Give examples of well-defined functions $f : \mathbb{Z} \to \mathbb{Z}$ which are:
(i) surjective but not injective;
(ii) injective but not surjective;
(iii) neither injective nor surjective;
(iv) bijective

Question 1.40. Give examples of well-defined functions $f : \mathbb{N} \to \mathbb{N}$ which are:
(i) surjective but not injective;
(ii) injective but not surjective;
(iii) neither injective nor surjective;

Question 1.41. Define $f : \mathbb{Q}^+ \to \mathbb{N}$ (where $\mathbb{Q}$ is the positive rationals) by $f(r) = 2^m 3^n$ where $r = \frac{m}{n}$ and $\text{hcf}\{m, n\} = 1$. Show that f is an injection.

Question 1.42. Give an example of an injection $h : \mathbb{N} \to \mathbb{Q}^+$.

Question 1.43. Give an example of an injection $\mathbb{Q} \to \mathbb{N}$.

Question 1.44. Give an example of an injection $\mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.

Question 1.45. Let A and B be sets. True or false: If there is an injection $f : A \to B$ and a surjection $g : A \to B$, then there exists a bijection $h : A \to B$. Explain your answer.

Question 1.46. Give examples of well-defined functions $f : \mathbb{R} \to \mathbb{R}$ which are:
(i) surjective but not injective;
(ii) injective but not surjective;
(iii) neither injective nor surjective;

1.1.8 Counting

Question 1.47. If $A = \{a, b, c\}$, list the elements of $\mathcal{P}(A)$.

Question 1.48. Let A and B be two (possibly infinite) sets. What precisely does it mean to say that $|A| < |B|$?

Question 1.49. What precisely does it mean to say that the cardinality of two infinite sets, A and B, is equal?

Question 1.50. What precisely does it mean to say that $|\mathbb{N}| = |\mathbb{Q}|$ but that $|\mathbb{N}| \neq |\mathbb{R}|$?

Question 1.51. If A and B are two (possibly infinite) sets, and there is a surjection $A \to B$, then is there an injection $B \to A$? Answer with a proof or a counterexample.

Question 1.52. For a set S, define what it means for a collection of subsets $\{A_i\}$ to be a *partition* of S.

Question 1.53. State Schroder-Bernstein's theorem.

Question 1.54. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$. How many elements does $\mathcal{P}(A)$ (the set of all subsets of A) have? How many elements of $\mathcal{P}(A)$ have 5 elements?

1.1.9 Samir

Question 1.55. State Fermat's Little Theorem.

Question 1.56. If $n, n+2, n+4$ are all prime, $\{n, n+2, n+4\}$ is called a prime triple. Prove that there is only one prime triple.

Question 1.57. Show that if x and y are integers and $x^3 + 6 = y^2$ then x is odd.

Question 1.58. Calculate $17^{18} \mod 19$.

Question 1.59. Calculate $15^{22} \mod 11$.

Question 1.60. Calculate $15^{17} \mod 17$. Briefly explain your method if it is clever.

Question 1.61. Calculate $10^{12} \mod 11$. Explain your answer.

1.1.10 Permutations

Question 1.62. If α is the permutation of $(16)(2435)$ and β is the permutation of $(136)(245)$ then write the composition $\alpha\beta$ in disjoint cycle notation.

Question 1.63. If α is the permutation (1234) and β is the permutation (125) then write the composition $\alpha\beta$ in disjoint cycle notation.

Question 1.64. If α is the permutation (2143) and β is the permutation (142) then write the composition $\alpha\beta$ in disjoint cycle notation.

Question 1.65. If α is the permutation (3241) and β is the permutation (321) then write the compositions $\alpha\beta$ and $\beta\alpha$ in disjoint cycle notation.

Question 1.66. Define a *permutation* of a finite set X, and define the permutation group S_n.

1.1.11 Groups

Question 1.67. What conditions must a set G with a binary operation satisfy in order to be a group?

Question 1.68. What properties are required of a subset H of a group G to be a *subgroup* of G?

Question 1.69. Let X be a nonempty set. In the group $\mathcal{P}(X)$ with operation $\triangle$ (the symmetric difference operator), what is the neutral (or identity) element? What is the inverse of $A \in \mathcal{P}(X)$.

1.1.12 Equivalence Relations and Isomorphisms

Question 1.70. Define an equivalence relation on a set S.

Question 1.71. State Lagrange's Theorem.

Question 1.72. Define what it means for two groups G_1, G_2 to be *isomorphic*.

1.2 Long Questions on Foundation Basics

If such a long question comes out in your year, count yourself extremely lucky, because this is standard A level material. Think of it like sitting an AEA paper or an 'S' paper.

*Anyway, I believe the January 2006 paper was an extraordinarily easy paper, and chances of it happening again are virtually zero. Sorry to disappoint!

1. *January 2000 Question 2 Resit*

   (a) The symbols $\neg, \wedge, \vee$, and $\Rightarrow$ denote the logical connectives 'not', 'and', 'or', and 'implies' respectively. The statement $P \Rightarrow Q$ is defined by just one of the following two expressions:
   $$(\neg P) \vee Q \text{ or } P \vee (\neg Q)$$
   State which of these two expressions is correct and briefly explain why.

      i. Write down the negation of $P \Rightarrow Q$.
      ii. Work out the truth table for $P \Rightarrow (P \Rightarrow Q)$.
      iii. Which, if any, of the following three statements are logically equivalent to the statement $P \Rightarrow (P \Rightarrow Q)$?
         A. $(P \Rightarrow P) \Rightarrow Q$
         B. $Q \Rightarrow P$
         C. $P \Rightarrow Q$

2. *January 2001 Question 3*

   (a) Use the Euclidean Algorithm to find the highest common factor c of 1925 and 189. Find integers p, q so that $c = 1925p + 189q$.

   (b) Let d denote the highest common factor of $a, b \in \mathbb{N}$. Show that:
   $$\{ra + sb \mid r, s \in \mathbb{Z}\} \supseteq \{md \mid m \in \mathbb{Z}\}$$

      i. Show that these two sets are in fact equal.
      ii. Find an example of r, s, a, b with $\text{hcf}\{ra + sb, b\} \neq \text{hcf}\{a, b\}$.

3. *January 2002 Question 2*

   (a) Define the highest common factor, $\text{hcf}(m, n)$, and lowest common multiple, $\text{lcm}(m, n)$, of two natural numbers $m, n \in \mathbb{N}$.

(b) i. Use Euclid's algorithm to determine:

$$d = \text{hcf}(585, 364)$$

and to find integers r and s such that:

$$585r + 364s = d$$

ii. Find non-zero integers k and l such that:

$$585k + 364l = 0$$

iii. Hence find an infinite set of integer solutions to the equation

$$585r + 364s = d$$

(c) Assuming the Fundamental Theorem of Arithmetic, prove that:

$$\text{hcf}(m, n) \cdot \text{lcm}(m, n) = mn$$

Hence find lcm(585, 364).

4. *January 2003 Question 1*

(a) Define the Fibonacci sequence by $F_1 = 1$, $F_2 = 1$ and

$$F_{n+2} = F_{n+1} + F_n \qquad \forall n \in \mathbb{N}$$

Prove, by induction, that $\forall n \in \mathbb{N}$

i. $F_1 + F_2 + F_3 + ... + F_n = F_{n+2} - 1$
ii. $F_1 + F_3 + F_5 + ... + F_{2n-1} = F_{2n}$

(b) Hence find and prove a formula for the sum S of the first n even Fibonacci numbers $S = F_2 + F_4 + ... + F_{2n}$.

(c) Calculate F_6 and F_7. Illustrate the use of Euclid's algorithm by calculating the highest common factor $hcf\{F_6, F_7\}$.

(d) Prove that for all $n \in \mathbb{N}$ we have $hcf\{F_n, F_{n+1}) = 1$.

5. *January 2006 Question 2*

(a) Prove by induction:

$$1^2 + 3^2 + 5^2 + ... + (2n+1)^2 = \frac{1}{3}(n+1)(2n+1)(2n+3)$$

(b) Find the highest common factor $d = hcf\{720, 5670\}$. Find integers a and b such that $d = 720a+5670b$. Show your working clearly (for both parts of this question).

6. *January 2006 Question 3*

   (a) What is the remainder on division of $(x^3 + 3x^2 + 2x + 1)$ by $(x^2 + 2x - 3)$?

   (b) $(x^3 + 2x^2 + 3x + 1)$ is exactly divisible by $(x^2 + bx + c)$ if and only if two equations in b and c are satisfied. Find the equations. [You do not need to find values of b and c that satisfy them.]

7. *January 2007 Question 2*

   (a) Let n and m be natural numbers. Define the *lowest common multiple*, $\mathrm{lcm}(m, n)$, and *highest common factor*, $\mathrm{hcf}(m, n)$, of m and n.

   (b) State the Well Ordering Principle (for $\mathbb{N}$) and use it to show that if $m_1, ..., m_n \in \mathbb{N}$ then there is a *least common multiple* of $m_1, ..., m_n$ in $\mathbb{N}$.

   (c) Let $q_1 = \frac{m_1}{n_1}$ and $q_2 = \frac{m_2}{n_2}$ be positive rational numbers. We define the *lowest common multiple* of q_1 and q_2, $\mathrm{lcm}(q_1, q_2)$, to be the smallest positive rational number q such that $q = Mq_1 = Nq_2$ for some positive integers M and N. [*Note: This definition was not used in the course; you will need to think about it.*]

      i. Find $\mathrm{lcm}(\frac{1}{7}, \frac{2}{9})$ and $\mathrm{lcm}\ (\frac{3}{4}, \frac{5}{6})$.

      ii. Find a formula for $\mathrm{lcm}\ (q_1, q_2)$. Your answer can include expressions of the form $\mathrm{lcm}(m, n)$, or $\mathrm{hcf}(m, n)$ where m and n are integers.

8. *January 2011 Question 3*

   (a) Show that the number of primes is infinite.

   (b) Use *either* induction *or* the Well-Ordering Principle to show that every integer $n \geq 2$ is a product of (one or more) primes.

   (c) Let S be the set of all primes. Define a relation $\sim$ on S as follows:

   $$p \sim q \Leftrightarrow p \text{ and } q \text{ end in the same decimal digit;}$$

   so, for example, $3 \sim 13$ but $3 \nsim 17$. Show that $\sim$ is an equivalence relation on S.

   (d) With S and $\sim$ as in part (c), how many equivalence classes are there? How many of these have more than one element?

   (e) For each $i \in \mathbb{N}$, let X_i be a countable set.

      i. Show that $\displaystyle\bigcup_{i=0}^{n} X_i$ is countable for all $n \in \mathbb{N}$.

ii. Is $\bigcup_{i=0}^{\infty} X_i$ countable? Prove or give a counterexample.

1.3 Long Questions on Polynomials

Questions of these kind have come out in the last three years. It would be good to know how to do these questions.

1. *January 2007 Question 3*

   (a) Let $P_1(x) = x^4 - 3x^3 + 3x - 4$ and $P_2(x) = x^4 - 2x^3 + 4x^2 + 2x - 5$. Find $\mathrm{hcf}(P_1, P_2)$.

   (b) Let $P \in \mathbb{R}[x]$. What does it mean to say that P is *irreducible* in $\mathbb{R}[x]$?

   (c) Is the polynomial $x^2 + 4x + 36$ irreducible in $\mathbb{R}[x]$? In $\mathbb{C}[x]$? Explain.

   (d) Prove by induction that every polynomial in $\mathbb{R}[x]$ can be written as a product of irreducible polynomials in $\mathbb{R}[x]$.

2. *January 2008 Question 2*

   (a) Let $P_1(x) = x^3 - 3x - 2$ and $P_2(x) = x^4 + x^3 + 2x^2 + 5x + 3$. Find $\mathrm{hcf}(P_1, P_2)$.

   (b) Let $P \in \mathbb{R}[x]$. What does it mean to say that P is *irreducible* in $\mathbb{R}[x]$?

   (c) Is the polynomial $x^2 - 2$ irreducible in $\mathbb{R}[x]$? In $\mathbb{Q}[x]$? Explain.

   (d) Is the polynomial $x^4 - 92x^3 + \pi x^2 + \sqrt{3}$ irreducible in $\mathbb{C}[x]$? Explain.

   (e) Find the lcm of the two polynomials P_1, and P_2 of part (a).

3. *January 2009 Question 2*

   (a) Let $P_1(x) = x^4 + x^3 + 5x^2 + 2x + 6$ and $P_2(x) = x^3 - 4x^2 + 2x - 8$. Find $\mathrm{hcf}(P_1, P_2)$.

   (b) Let $P \in \mathbb{R}[x]$. What does it mean to say that P is *irreducible* in $\mathbb{R}[x]$?

   (c) Let $P(x) = x^4 + 1$.

      i. Write $P(x)$ as a product of irreducible polynomials in $\mathbb{C}[x]$.

      ii. Write $P(x)$ as a product of irreducible polynomials in $\mathbb{R}[x]$. Hint: use your answer to (a).

   (d) Can a polynomial of degree 3 be irreducible in $\mathbb{Q}[x]$? How about in $\mathbb{R}[x]$? Explain.

4. *January 2010 Question 2*

   (a) State the Binomial theorem.

(b) For each $k \in \mathbb{N}\{0\}$, define a polynomial with rational coefficients P_k by the formula:

$$P_k(x) = \frac{x(x-1)\dots(x-k+1)}{k!}$$

What is the degree of P_k?

(c) Prove the formula:

$$P_k(x) + P_{k+1}(x) = P_{k+1}(x+1)$$

(d) Prove that, for each $k \geq 1$, $P_k(x) \in \mathbb{N}$ for all $x \in \mathbb{N}$. (Hint: use induction and the previous statement)

(e) Let R be a polynomial with integer coefficients with the leading coefficient one. Prove that if $q \in \mathbb{Q}$ is a root of R, then q is an integer.

5. *January 2011 Question 2*

(a) Let $P_1, P_2 \in \mathbb{R}[X]$ with $P_2 \neq 0$. Show that there exist unique polynomials $Q, R \in \mathbb{R}[X]$ such that $P_1 = QP_2 + R$, and either $R = 0$ or $\deg(R) < \deg(P_2)$.

(b) Explain how to use the result of (a) repeatedly to find a highest common factor (hcf) of P_1 and P_2.

(c) Find a highest common factor of $X^{15} - 1$ and $X^{21} - 1$. Make a conjecture for the highest common factor of $X^m - 1$ and $X^n - 1$ for all $m, n \geq 1$.

(d) Let $P \in \mathbb{R}[X]$ with $\deg(P) \geq 3$. Show that P is reducible. [You are allowed to use the Fundamental Theorem of Algebra.]

1.4 Long Questions on Injections, Surjections, and Counting

Questions on this usually come out in the first question, but not necessarily - like in 2010!

1. *January 2001 Question 1*
   (a) Let X and Y be sets with $|X| = 2,\ |Y| = 3$. How many maps are there from X to Y, and how many from Y to X? In each case say how many of the maps are injective, surjective, neither of these, bijective.
   (b) Let A and B be sets. What is meant by the power set $\mathscr{P}(A)$? Consider the map $f : \mathscr{P}(A) \times \mathscr{P}(B) \to \mathscr{P}(A \cup B)$ given by $f(C, D) = C \cup D$. Show that f is surjective.
      i. If A and B are disjoint, show that f is injective.
      ii. If f is injective, show that A and B are disjoint.
2. *January 2001 Question 2*
   (a) Show that $P \Rightarrow Q$ is logically equivalent to $(\neg P) \vee Q)$.
   (b) To what simpler sentence is $(P \Rightarrow Q) \wedge (P \vee Q)$ logically equivalent?
   (c) What is meant by the term uncountable? Prove that $\mathbb{R}$ is uncountable.
3. *January 2002 Question 1*
   (a) What does it mean to say that $f : D \to T$ is a function?
   (b) Define the terms *injection*, *surjection* and *bijection*. Give examples of functions $f_i : \mathbb{N} \to \mathbb{N}(i = 1, 2, 3, 4)$ such that:
      i. $f_1 : \mathbb{N} \to \mathbb{N}$ is an injection but not a surjection.
      ii. $f_2 : \mathbb{N} \to \mathbb{N}$ is a surjection but not an injection.
      iii. $f_3 : \mathbb{N} \to \mathbb{N}$ is a bijection.
      iv. $f_4 : \mathbb{N} \to \mathbb{N}$ is neither an injection nor a surjection.
   (c) For a subset $A \subset D$ define:
   $$f(A) = \{t \in T \mid f(a) = t \text{ for some } a \in A\}$$
   For each of the following two statements about subsets $A, B \subset D$, determine whether it is always true or sometimes false. In each case give a proof or counterexample as appropriate.

   i. $f(A \cup B) = f(A) \cup f(B)$.
   ii. $f(A \cap B) = f(A) \cap f(B)$.

4. *January 2003 Question 2*

   (a) State precisely what it means to say that a function $f : D \to T$ is (i) an *injection*, (ii) a *surjection*, (iii) a *bijection*.

   (b) Given two functions $f : A \to B$ and $g : B \to C$, state precisely what is meant by the *composition* $g \circ f$ of f and g.

   (c) Show that if $f : A \to B$ and $g : B \to C$ are both injections, then $g \circ f : A \to C$ is also an injection.

   (d) Explain what is meant to say that a set A is finite.

   (e) Suppose now that $f : A \to B$ and $g : B \to A$ are both injections. If A is finite, show that f and g must be bijections and that B is finite. [You may assume that the only injections from a finite set to itself are bijections.]

   (f) Give an example where A and B, $A \neq B$, are infinite sets and where $f : A \to B$ and $g : B \to A$ are both injections and not surjections.

5. *January 2005 Question 2*

   (a) Show that $|\mathbb{P}(x)| = 2^{|x|}$ for all finite sets X.

   (b) Define $f : \mathbb{Q}^+ \to \mathbb{N}$ (where $\mathbb{Q}^+$ is the positive rationals) by $f(r) = 2^m 3^n$ where $r = \frac{m}{n}$ and $\text{hcf}\{m, n\} = 1$. Show that f is an injection.

   (c) Give an example of an injection $h : \mathbb{N} \to \mathbb{Q}^+$.

   (d) State (but do not prove) Schroder-Bernstein's theorem.

   (e) Using the previous results, show that $|\mathbb{Q}| = |\mathbb{Z}|$.

6. *January 2006 Question 4*

   (a) Define the composition $g \circ f$ of two functions $f : A \to B$ and $g : B \to C$.

   (b) Prove that if f and g are injective, then $g \circ f$ is also injective.

   (c) Prove that if $g \circ f$ is injective then f is injective, but give an example (of functions f, g and sets A, B, C) to show that $g \circ f$ can be injective without g being injective.

   (d) Outline the proof (without assuming anything about the cardinality of the two sets involved) that there is no surjection f from the natural numbers $\mathbb{N}$ to the unit interval $(0, 1) \subset \mathbb{R}$.

7. *January 2010 Question 3*

   (a) Give definitions of injective, surjective and bijective maps.

   (b) Find an injective map $f : \mathbb{Q} \to \mathbb{N}$.

   (c) Find a left inverse to the map f defined in the previous item.

   (d) What does it mean for two (possibly infinite) sets to have the same cardinality?

   (e) State the Schroeder-Bernstein theorem.

   (f) Show that the interval $[0, 1]$ and the square $[0, 1]^2$ have the same cardinality.

   (g) Prove that there is no surjection from a non-empty set X to its power set $\mathscr{P}(X)$.

1.5 Long Questions on Groups and Permutations

These questions have not come out in a very long time, because again, the material is covered in the compulsory Question 1. But they are fun to do.

1. *January 2000 Question 3 Resit*

   (a) Define a *permutation* of a set X. Write down the axioms of a group and show that they are satisfied by the set S_n of all permutations of $\{1, 2, \ldots, n\}$ when the binary operation is a composition of maps.

   (b) What properties are required of a subset H of a group G for H to be a *subgroup* of G? Which, if any, of the following subsets of S_3 are subgroups?

      i. $\{\imath, (12)\}$
      ii. $\{(123), (132)\}$
      iii. $\{\imath, (12), (13), (23)\}$

      Give brief reasons for your answers.

2. *January 2001 Question 4*

   (a) What is meant by the *order* of an element of a group? Explain how to deduce the order of a permutation in S_n from its expression in disjoint cycle notation.

   (b) Find the orders of the elements (12)(34) and (12)(24) of S_4.

   (c) Find the smallest value of n for which S_n contains an element of order 6.

   (d) Give an example of a cyclic subgroup of S_4 of order 4 and an example of a non-cyclic subgroup of S_4 of order 4.

3. *January 2003 Question 4*
   Let S_n be the group of permutations of $\{1, 2, ..., n\}$ under composition.

   (a) Explain what is meant by *disjoint cycle notation* for a permutation $\rho \in S_n$.

   (b) Express the elements $\tau = (23)(24)(25)$ and $\mu = (145)(235)$ of S_5 in disjoint cycle notation.

   (c) Define the order of an element $\rho \in S_n$.

   (d) What is the order k of the element (123)(4567) in S_7?

   (e) Prove that no other element in S_7 can have order larger than k.

4. *January 2005 Question 3*

   (a) Define a *permutation* of a finite set X, and define the permutation group S_n.

   (b) Write down the axioms for a set G to be a *group*.

   (c) What properties are required of a subset H of a group G to be a *subgroup* of G?

   (d) Which, if any, of the following subsets of S_4 are subgroups? Give brief reasons for your answers.

      i. $\{\imath, (12)\}$;

      ii. $\{(123), (132)\}$;

      iii. $\{\imath, (12)(34), (14)(23)\}$;

      iv. $\{\imath, (12), (14), (24), (142), (124)\}$.

1.6 Long Questions on Equivalence Relations

This has been a popular choice of question as well.

1. *January 2002 Question 3*

   (a) Define each of the following properties of a relation $\sim$ on a set S:

      i. reflexivity,

      ii. symmetry,

      iii. transitivity.

   (b) Show that $\sim$ on $\mathbb{R}^2 \setminus \{(0,0)\}$ is an equivalence relation, where $\sim$ is defined by $(a,b) \sim (x,y)$ if and only if $ay = bx$.
   Explain why $\sim$ is not an equivalence relation on $\mathbb{R}^2$.

   (c) If $\sim$ is an equivalent relation on S, show that the set of equivalence classes forms a partition of S.

2. *January 2003 Question 3*

   (a) State carefully the three defining properties that must be satisfied by an equivalence relation $\sim$ on a set S.

   (b) Let $S = \{(m,n) : m,n \in \mathbb{Z}, n \neq 0\}$ and let $\sim$ be defined by

   $$(m,n) \sim (a,b) \text{ iff } mb = na$$

      i. Show that $\sim$ is an equivalence relation on S.

      ii. Define the equivalence class $[(a,b)]$ of (a,b) by

      $$[(a,b)] = \{(m,n) : (m,n) \sim (a,b)\}.$$

      Briefly describe the set $[(3,4)]$.

      iii. Define the operator $\triangle$ on the set of equivalence classes of S by

      $$[(a,b)]\triangle[(c,d)] = [(ad+bc, bd)].$$

      Show that $[(2,3)]\triangle[(5,6)] = [(3,2)]$.

   (c) Let $S^\star = \{(m,n) : m,n \in \mathbb{Z}\}$ and again let $\sim$ be defined by

   $$(m,n) \sim (a,b) \text{ iff } mb = na$$

   Show that $\sim$ is not an equivalence relation on $S^\star$.

3. *January 2005 Question 4*

   (a) Define an *equivalence relation* on a set S.

   (b) Prove that an equivalence relation on a set S uniquely defines a partition of that set.

   (c) Let n be a fixed non-negative integer, and define a relation $\sim$ on the set of integers by setting $a \sim b$ if and only if $a - b$ is an integer multiple of n. Show that $\sim$ is an equivalence relation and describe the corresponding equivalence classes when i) $n = 2$, (ii) $n = 1$ and (iii) $n = 0$.

   (d) A relation $\sim$ is defined on the three element set $S = \{x, y, z\}$ as follows:

   $$x \sim x,\ y \sim y,\ x \sim y \text{ and } y \sim x, \text{ and no other relations hold.}$$

   Prove that $\sim$ satisfies two out of three axioms for an equivalence relation but fails to satisfy the other axiom.

4. *January 2008 Question 3*

   (a) What is an *equivalence relation*?

   (b) For each of the following relations, state whether it is an equivalence relation, and if it is not, say which of the defining properties fail to hold:

      i. In the set of all people, $a \sim b$ if a and b have the same father.

      ii. In the set of all people, $a \sim b$ if a and b share a grandfather.

      iii. In $\mathbb{R}[x]$, $P \sim Q$ if deg P - deg Q is even.

      iv. In the power set $\mathscr{P}(X)$ of a finite set X, $A \sim B$ if $|A \cap B|$ is even.

   (c) Let $n \in \mathbb{Z}$. Show that the relation in $\mathbb{Z}$

   $$a \sim b \text{ if } b - a \text{ is divisible by } n$$

   is an equivalence relation. How many equivalence classes are there? What is the usual name for the set of equivalence classes?

   (d) In $\mathbb{C}[X]$, define a relation $\sim$ by

   $$P_1 \sim P_2 \text{ if } P_1 - P_2 \text{ is divisible by } X - 1.$$

   Show that $\sim$ is an equivalence relation. Let Q denote the set of equivalence classes of $\sim$ in $\mathbb{C}[X]$. Find an explicit bijection $Q \to \mathbb{C}$. Hint: What if the relation were "$P_1 \sim P_2$ if $P_1 - P_2$ is divisible by X"?

5. *January 2009 Question 3*

   (a) What is an *equivalence relation* on a set A?

   (b) For each of the follow relations, state, without proof, whether or not it is an equivalence relation. If it is not, say which of the defining properties fail to hold.

      i. In the set of all people, $a \sim b$ if a and b have the same father.

      ii. In the set of all people, $a \sim b$ if a and b have a parent in common.

   (c) For each of the following relations, either show that it is an equivalence relation, or explain which of the defining properties it fails to have. (We decree that the denominator of the rational number 0 in its lowest form is 1).

      i. In $\mathbb{Q}$, $a \sim b$ if the denominator of $a - b$ (in its lowest form) is not divisible by 2.

      ii. In $\mathbb{Q}$, $a \sim b$ if the denominator of $a - b$ (in its lowest form) is divisible by 2.

      iii. in $\mathbb{R}\backslash\{0\}$, $a \sim b$ if $\frac{a}{b} \in \mathbb{Q}$.

   (d) In $\mathbb{R}[X]$, define a relation $\sim$ by

$$P_1 \sim P_2 \text{ if } P_1 - P_2 \text{ is divisible by } X.$$

      i. Show that $\sim$ is an equivalence relation.

      ii. Let Q denote the set of equivalence classes of $\sim$ in $\mathbb{R}[X]$. Find an explicit bijection $Q \to \mathbb{R}$.

1.7 Long Questions on Cosets, Groups and Isomorphisms

This is the last kind of topic on which questions may come out on.

1. *January 2002 Question 4*

   (a) Define a permutation of an arbitrary set X and show that composition is a binary operation on the set S_X of such permutations.
   Show that $(S_X, \circ)$ is a group.

   (b) Let S_n be the group of permutations of $\{1, 2, \ldots, n\}$ under composition.

      i. Show that $V = \{\imath, (12)(34), (13)(24), (14)(23)\}$ is a non-cyclic subgroup of S_4.

      ii. List the elements of two different left cosets of V in S_4.

      iii. For each left coset given in your answer for b), show it is also a right coset of V in S_4.

2. *January 2007 Question 4*

   (a) Let X be a non-empty set. What is the neutral element for the binary operation of intersection, $\cap$, in the set $\mathscr{P}(X)$ of subsets of X? Is $\mathscr{P}(X)$, with this operation, a group? Explain.

   (b) Let $\mathbb{Z}/n$ be the set of integers modulo n, and denote the elements of the set by $[0], [1], ..., [n-1]$. Show that $[m]$ has a *multiplicative* inverse in $\mathbb{Z}/n$ if and only if $\text{hcf}(m, n) = 1$.

   (c) Find 100^{99} mod 101. You will get full marks only if you do this without a lengthy calculation.

   (d) Suppose that G is a group and H is a subgroup of G. Define the left coset gH of H.

   (e) Suppose that G is a group and H is a subgroup of G, and that g_1H and g_2H are left cosets of H. Show that $|g_1H| = |g_2H|$. [*Note: you should not assume that G is finite.*]

3. *January 2008 Question 4*

   (a) In each of the following examples of binary operation, say whether there is a neutral element. If there is, say what it is. If there is not, explain why not.

      i. $\mathbb{N}$ with the binary operation max.

      ii. $\mathbb{N}$ with the binary operation min.

iii. $\mathbb{N}\backslash\{0\}$ with the binary operation lcm.

(b) Suppose that G is a group, H is a subgroup of G, and $g_1, g_2 \in G$. Define the *left coset* g_1H and the *right coset* Hg_2.

(c) If $G = \mathbb{Z}$, $H = 2\mathbb{Z}$ and $g = 1$, what is a simple description of the left coset $g + H$? If $G = \mathbb{Z}$, $H = 4\mathbb{Z}$, and $g = 2$, give a simple description of $g + H$.

(d) If G is a group and H a subgroup, state and prove a simple criterion for equality of the two left cosets g_1H and g_2H.

(e) Suppose that G is a group, H is a subgroup of G, and $g_1, g_2 \in G$. Show that $|g_1H| = |Hg_2|$. [*Note: you should not assume that G is finite.*]

4. *January 2009 Question 4*

(a) For which subsets $S \subset \mathbb{Z}$ does the binary operation max on S have a neutral element? And the binary operation min? When there is a neutral element, what is it?

(b) State *Lagrange's Theorem*.

(c) If G is a group and $g \in G$, what is the *order* of g? Show that if G is a finite group and if $g \in G$, then the order of g must divide $|G|$.

(d) Suppose that G is a group in which $gg = e$ (= neutral element) for all $g \in G$. Show that G is abelian (i.e. that $g_1g_2 = g_2g_1$ for all $g_1, g_2 \in G$).

(e) Let X be a finite set, let $\triangle$ denote the binary operation of *symmetric difference*, $A\triangle B = (A\backslash B) \cup (B\backslash A)$ on subsets of X, and let $\mathscr{P}(X)$ be the power set of X. Then $\mathscr{P}(X)$, with the operation $\triangle$, is a group (you are not asked to show this). Find an isomorphism

$$\mathscr{P}(X) \to (\mathbb{Z}/2)^n,$$

where $n = |X|$, explaining briefly *why* it is an isomorphism.

5. *January 2010 Question 4*

(a) What is a binary operation on a set X?

(b) Define what it means for a set G with a binary operation defined on it to be a group.

(c) Let G be a group. What does it mean for a subset $H \subset G$ to be a subgroup of G?

(d) Define what it means for two groups G_1 and G_2 to be isomorphic.

(e) Show that $\mathbb{Z}$ with the binary operation $m \star n = m + (-1)^m$ is a group. Denote this group by $(\mathbb{Z}, \star)$.

(f) Denote by $(\mathbb{Z}, +)$ the group of integers with the usual binary operation of addition. Show that $(\mathbb{Z}, \star)$ has a subgroup which is isomorphic to $(\mathbb{Z}, +)$.

(g) Define S_n, the group of permutations of $\{1, 2, \ldots, n\}$.

(h) Let G be a finite group with $|G| = n$. Give an example of an isomorphism between G and some subgroup of S_n. (You are not required to prove that the example you have given is an isomorphism.)

6. *January 2010 Question 4*

(a) Let G be a set with a binary operation written multiplicatively as $(g_1, g_2) \mapsto g_1 g_2$. Define what it means for G to be a *group* under this operation, and what it means for a subset H of G to be a *subgroup* of G.

Let G be the group of 2×2 matrices with integer entries and determinant 1:

$$G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z};\ ad - bc = 1 \right\}.$$

You may assume without proof that G is a group under matrix multiplication.

(b) Let $H = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{Z} \right\}$. Show that H is a subgroup of G. Show further that H is isomorphic to the group $\mathbb{Z}$ (under addition).

(c) What is the order of $g = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ in G? Show that every element of H which is not the identity has infinite order.

(d) Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. Show that the coset Hg consists of all matrices in G whose second row is $(c\ d)$.

2 Analysis

This exam paper usually consists of four questions. You have to do three questions, being Question 1 (which is compulsory), and another two out of the remaining Questions 2, 3, and 4. Unlike Foundations, the optional three questions (to choose two from) all follow the same style, one on series, one on sequences, and the last on boundedness, infinum, suprenums and so on.

2.1 The Compulsory Question 1

This question is comprised out of many subquestions in order to test how well you know the Analysis syllabus. Be prepared to answer simple T/F questions, and give simple examples, work out 'toy' problems. You actually should be able to answer most of them instantly at this point in time without having to refer to any textbooks at all.

2.1.1 True / False Questions

Say whether each of the following statements is true or false. Give a brief explanation or counter example for those that are false.

Question 2.1. If (a_n) has a convergent subsequence then (a_n) is bounded.

Question 2.2. (a_n) is strictly increasing $\Rightarrow (a_n) \to \infty$.

Question 2.3. There exists a sequence (a_n) for which $(|a_n|)$ doesn't converge but (a_n) does.

Question 2.4. Every convergent series is either absolutely convergent or conditionally convergent.

Question 2.5. The following is correct as a definition of (a_n) is **not** null:

$$\exists\ \epsilon > 0\ s.t.\ \forall\ N \in \mathbb{N},\ \exists\ n > N\ s.t.\ |a_n| > \epsilon$$

Question 2.6. If a sequence (a_n) converges to a then we must have:

$$|a_{n+1} - a| < |a_n - a|\ \forall\ n \in \mathbb{N}$$

Question 2.7. If $a_n < 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} a_n$ is convergent then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

Question 2.8. If $(|a_n|)$ is a Cauchy sequence then (a_n) is bounded.

Question 2.9. If (a_n) has two different subsequences which both converge to a, then (a_n) converges to a.

Question 2.10. If (s_n) is the sequence of partial sums of $\sum_{n=1}^{\infty} \frac{1}{n}$, then $(s_n) \to 0$ as $n \to \infty$.

Question 2.11. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge, and $b_n > 0$ for all $n \in \mathbb{N}$ then $\sum_{n=1}^{\infty} \frac{a_n}{b_n}$ converges.

Question 2.12. Suppose $a_n \leq b_n$ for all $n \geq 1$. If $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$.

Question 2.13. If (a_n) is increasing and bounded above then $(-a_n)$ converges.

Question 2.14. (a_n) converges if and only if $(|a_n|)$ converges.

Question 2.15. If $x \in \mathbb{Q}$ and $x > 0$ then $(2 - \sqrt{x})(2 + \sqrt{x}) \in \mathbb{Q}$.

Question 2.16. There exists a Cauchy sequence which doesn't converge.

Question 2.17. $\left(\frac{5}{\sqrt{n}} + 1 \right)$ is a Cauchy sequence.

Question 2.18. The series $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}} + ...$ can be rearranged to converge to any real number.

Question 2.19. If (a_n) is a sequence such that $\frac{a_{n+1}}{a_n} < 1$ for every $n \in \mathbb{N}$ then $(a_n) \to 0$.

Question 2.20. $((3^n + 5^n)^{\frac{1}{n}})$ tends to 3 as n tends to infinity.

Question 2.21. $((2^n + 5^n)^{\frac{1}{n}}) \to 1$

Question 2.22. If a sequence (a_n) converges to a then $|a_{n+1} - a| \leq |a_n - a|$ for all natural numbers n.

Question 2.23. If x and y are real numbers such that $x > y - \epsilon$ for all $\epsilon > 0$ then $x > y$.

Question 2.24. If (a_n) is a sequence such that $|a_{n+2} - a_{n+1}| \leq \frac{1}{2}|a_{n+1} - a_n|$ for all natural numbers n then (a_n) converges.

Question 2.25. If (a_n) is a sequence such that $a_{2n} = 0$ for all natural numbers n then (a_n) converges if and only if (a_{2n-1}) tends to 0.

Question 2.26. If both $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ diverge to infinity then $\displaystyle\sum_{n=1}^{\infty} a_n b_n$ diverges to infinity.

Question 2.27. If $\displaystyle\sum_{n=1}^{\infty} a_n = 0$ and $\displaystyle\sum_{n=1}^{\infty} b_n = 0$ then $\displaystyle\sum_{n=1}^{\infty}(a_n + b_n) = 0$.

Question 2.28. If (a_n) is an increasing sequence which is bounded above then (a_n) tends to $\sup\{a_n \mid n \in \mathbb{N}\}$ as n tends to infinity.

Question 2.29. The total of the sum:

$$\sum_{i=1}^{n} \frac{(-1)^{i+1}}{i}$$

depends on the order in which the terms are added.

Question 2.30. The sequence (s_n) of partial sums of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is Cauchy.

Question 2.31. If (a_n) is an increasing sequence such that for all $C > 0$ there exists a natural number, n, such that $a_n > C - 1$, then (a_n) tends to infinity.

Question 2.32. For every real number a there exists a sequence (a_n) of irrational numbers such that $\displaystyle\lim_{n\to\infty} a_n = a$.

Question 2.33. Let (a_n) be a positive decreasing null sequence. The series $\displaystyle\sum_{n=1}^{\infty}(-1)^n a_n$ can always be rearranged so that it converges to 1.

Question 2.34. If a sequence (a_n) does not converge to 0, then there exists $\epsilon > 0$ such that $|a_n| > \epsilon$ for every n.

Question 2.35. Let f be a positive decreasing function such that $\displaystyle\int_0^{\infty} f(x)\, dx$ is finite and let (a_n) be a sequence such that $a_n \leq f(n)$ for every n. Then, the series $\sum a_n$ converges.

Question 2.36. The inequality $|x-y| \leq |x|+|y|$ holds for every two real numbers x and y.

Question 2.37. Given any sequence (a_n), the sequence $\sin(a_n)$ contains a convergent subsequence.

Question 2.38. If $(1/a_n)^2 \to 0$, then $a_n \to \infty$.

Question 2.39. A non-monotonic sequence is always bounded.

Question 2.40. The limit of the sequence $(0.9, 0.99, 0.999, \ldots)$ is the largest number strictly less than 1.

Question 2.41. If $a_n < 1$ and $a_n \to a$ then $a < 1$.

Question 2.42. Every bounded monotonic sequence has a convergent subsequence.

Question 2.43. Every sequence contains a monotonic subsequence.

Question 2.44. e has a non-recurring decimal expansion

Question 2.45. If for each $\epsilon > 0$ there exists an N such that $|a_N - a| < \epsilon$, then $(a_n) \to a$.

Question 2.46. Three of the sequences defined as follows tend to infinity:

$$a_n = \begin{cases} 0 & n \text{ odd} \\ n & n \text{ even} \end{cases} \quad a_n = \begin{cases} n & n \text{ odd} \\ n+1 & n \text{ even} \end{cases} \quad a_n = \begin{cases} n & n \text{ odd} \\ \sqrt{n} & n \text{ even} \end{cases} \quad a_n = \begin{cases} \frac{n}{2} & n \text{ odd} \\ \frac{n}{4} & n \text{ even} \end{cases}$$

Question 2.47. If for each $\epsilon > 0$ there exists an N such that $|a_n - a_{n+1}| < \epsilon$ for all $n > N$ then (a_n) converges.

Question 2.48. Every Cauchy sequence is bounded.

Question 2.49. Every convergent sequence is a Cauchy sequence.

Question 2.50. Given a non-empty bounded set of real numbers A, $\inf A < a$ for all $a \in A$.

Question 2.51. The sequence $\left(\dfrac{8^n + n^4}{4n^4 + 3^{2n}}\right) \to 0$.

Question 2.52. Every real number has exactly two distinct decimal expansions.

Question 2.53. If $\sum a_n$ converges then $\sum |a_n|$ converges.

Question 2.54. If the sequence (a_n) converges then $\sum a_n$ also converges.

Question 2.55. If $\frac{a_{n+1}}{a_n} \to 0$ then $\sum a_n$ converges.

Question 2.56. The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ can be rearranged to sum to 42.

Question 2.57. If $\sum_n a_n$ diverges, then $\sum_n (-1)^n a_n$ also diverges.

Question 2.58. Every decreasing sequence is bounded above.

Question 2.59. If $(a_n) \to 0$ then $(\frac{1}{a_n}) \to \infty$.

Question 2.60. If $a_n^2 \geq 1$ for all n and $a_n \to a$ then $a \geq 1$.

Question 2.61. If (a_n) satisfies $a_n = 3$ for infinitely many n then $(a_n) \to 3$.

Question 2.62. Every bounded sequence is Cauchy.

Question 2.63. If (a_n) is unbounded then $|a_n| \to \infty$.

Question 2.64. The sequence $\sin(n!)$ has a Cauchy subsequence.

Question 2.65. Every increasing sequence which is bounded above is Cauchy.

Question 2.66. If $\lim_{n\to\infty}(a_{2n}) = \lim_{n\to\infty}(a_{2n+7})$ then (a_n) converges.

Question 2.67. If $\limsup_{n\to\infty}(a_n) = 10$ then $a_n \leq 10$ eventually.

Question 2.68. If A is a bounded non-empty set of real numbers such that $a < 1$ for all $a \in A$ then $\inf A < 1$.

Question 2.69. If $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} a_n^2$ converges.

Question 2.70. If $\sum_{n=1}^{\infty} a_n$ converges then $a_n \to 0$.

Question 2.71. If $\frac{a_{n+1}}{a_n} \to \frac{1}{2}$ then $\sum_{n=1}^{\infty} a_n$ converges.

Question 2.72. If $\liminf(a_n) \neq \limsup(a_n)$ then $\sum_{n=1}^{\infty} a_n$ diverges.

Question 2.73. If (a_n) is strictly contracting then $\sum_{n=1}^{\infty} |a_{n+1} - a_n|$ converges.

Question 2.74. There can exist a Cauchy sequence (a_n) in $\mathbb{R}$, which does not converge in $\mathbb{R}$.

Question 2.75. A bounded and monotonic subsequence is always convergent.

Question 2.76. The series $\sum_{n=1}^{\infty} \frac{\sin(n!)}{10^n}$ is convergent.

Question 2.77. The series $\sum_{n=1}^{\infty} \frac{n!}{10^n}$ is convergent.

Question 2.78. The series $1 - 1 + \frac{1}{2} - \frac{1}{2} + \ldots + \frac{1}{n} - \frac{1}{n} + \ldots$ can be rearranged to converge to any real number.

Question 2.79. The convergence of $\sum_{n=1}^{\infty} |a_n|$ implies that $\sum_{n=1}^{\infty} a_n$ is convergent.

Question 2.80. The sequence $\cos n$ has a convergent subsequence.

Question 2.81. The sequence $(\sin(n) + \cos(n))$ has a convergent subsequence.

Question 2.82. If $(a_n) > 0$ for all $n \in \mathbb{N}$ and $a_n \to a$ then $a > 0$.

Question 2.83. If $\sum_{n=1}^{\infty} a_n = 1$, then $\sum_{n=2}^{\infty} a_n = 1$.

Question 2.84. There exists a sequence which has no bounded subsequence.

Question 2.85. The sequence $a_n \to 0$ if, and only if, the series $\sum_{n=1}^{\infty} a_n$ converges.

Question 2.86. The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ converges.

Question 2.87. The series $\sum_{n=1}^{\infty} a_n$ converges if its sequence of partial sums (s_n) is bounded.

Question 2.88. If A is a non-empty set of real numbers such that $a < 3$ for all $a \in A$ then $\sup A < 3$.

Question 2.89. There exists a series $\sum_{n=1}^{\infty} a_n$ which converges but for which $\sum_{n=1}^{\infty} |a_n|$ diverges.

2.1.2 "Give an Example" Questions

Give an example of each of the following (no proofs needed):

Question 2.90. A bounded sequence which doesn't converge.

Question 2.91. A Cauchy sequence.

Question 2.92. A series $\sum_{n=1}^{\infty} a_n$ which converges, but such that $\sum_{n=1}^{\infty} \sqrt[3]{a_n}$ diverges.

Question 2.93. A series $\sum_{n=1}^{\infty} a_n$ with sequence of partial sums (s_n) such that $s_n = a_n$ for all natural numbers n.

Question 2.94. An unbounded sequence that has a convergent subsequence.

Question 2.95. An unbounded sequence that has no convergent subsequence.

Question 2.96. A non-monotonic, unbounded sequence.

Question 2.97. A null sequence (a_n) such that the series $\sum a_n$ does not converge.

Question 2.98. A sequence which is not a Cauchy sequence but has the property that for every $\epsilon > 0$ and every $N > 0$ there exists $n > N$ and $m > 2n$ such that $|a_n - a_m| \leq \epsilon$.

Question 2.99. Two sequences (a_n) and (b_n) such that the sequence (c_n) defined by $c_n = a_n + b_n$ converges to 1, but nether (a_n) nor (b_n) converge.

Question 2.100. A sequence (a_n) that tends to $+\infty$ but is neither increasing nor eventually increasing.

Question 2.101. A sequence (a_n) such that $\frac{a_{n+1}}{a_n}$ tends to 1 but (a_n) doesn't converge.

Question 2.102. A bounded sequence (a_n) such that $\frac{a_{n+1}}{a_n}$ tends to 1 but (a_n) doesn't converge.

Question 2.103. A sequence of the form $(x^n + y^n)^{\frac{1}{n}}$ which converges to 6 as $n \to \infty$.

Question 2.104. A sequence of irrationals which has a subsequence converging to 2 and a subsequence converging to 3.

Question 2.105. A sequence which is not increasing, but tends to infinity.

Question 2.106. A sequence bounded above which does not have a convergent subsequence.

Question 2.107. A sequence which is not bounded above but which doesn't tend to infinity.

Question 2.108. A number with only one decimal expansion.

Question 2.109. A sequence (a_n) of non-zero terms such that $\left(\frac{a_{n+2}}{a_n}\right) \to 1$ but (a_n) does not converge.

Question 2.110. A sequence (a_n) of non-zero terms which is bounded above but such that the sequence $\left(\frac{1}{a_n}\right)$ is not bounded below.

Question 2.111. A sequence (a_n) which tends to infinity but has $a_{n+1} < a_n$ for infinitely many n.

Question 2.112. A natural number N such that for all $n > N$, $\left|\frac{3n+10}{n} - 3\right| < \frac{1}{100}$.

Question 2.113. An integer N so that $\left|\frac{n+10}{n} - 1\right| < \frac{1}{100}$ whenever $n > N$.

Question 2.114. A convergent series $\sum_{n=1}^{\infty} a_n$ with all terms non-zero and $\sum_{n=1}^{\infty} a_n \neq 7$ can be rearranged so that its sum is 7.

Question 2.115. A convergent series $\sum_{n=1}^{\infty} a_n$ such that $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ does not converge.

Question 2.116. A sequence a_n such that, for all $m \in \mathbb{N}$, $a_n = m$ for infinitely many natural numbers n.

Question 2.117. A series $\sum a_n$, with non-zero terms, which can be rearranged to form $3\frac{1}{2}$.

Question 2.118. A series $\sum a_n$ with $a_n \to 0$ and with $s_n \to \infty$ where (s_n) is the sequence of partial sums.

Question 2.119. Two series $\sum a_n$ and $\sum b_n$ which diverge to infinity but where $a_n b_n$ converges.

Question 2.120. A sequence (a_n) which satisfies the following but does not diverge to infinity:

$$\forall\ C > 0,\ \exists\ N\ s.t.\ \exists\ n > N \text{ with } a_n > C$$

Question 2.121. A sequence (a_n) with $\frac{1}{a_n} \to 0$, but $a_n \nrightarrow \infty$.

Question 2.122. A sequence (a_n) which satisfies the following but does not converge to 1:

$$\forall\ \epsilon > 0,\ \forall\ N,\ \exists\ n > N \text{ with } |a_n - 1| < \epsilon$$

Question 2.123. A geometric series that neither converges nor diverges to infinity.

Question 2.124. A Cauchy sequence (a_n) such that a_n is rational for all $n \geq 1$, and such that its limit is not rational.

Question 2.125. A sequence that satisfies the conditions of the Ratio Lemma, and explain your answer.

Question 2.126. A sequence (a_n) such that $\frac{a_{n+1}}{a_n} < 1$ for all $n \geq 1$, and such that a_n does not tend to 0.

Question 2.127. A set A which has no supremum. Explain why it does not violate the completeness axiom.

Question 2.128. Give an example of a *diverging* series $\sum_{n=1}^{\infty} a_n$ such that $\frac{a_{n+1}}{a_n} \to 1$ as $n \to \infty$.

Question 2.129. Give an example of a *converging* series $\sum_{n=1}^{\infty} a_n$ such that $\frac{a_{n+1}}{a_n} \to 1$ as $n \to \infty$.

2.1.3 Toy Questions

Question 2.130. Define what it means for (a_n) to be a *null sequence.*

Question 2.131. Define what it means for a sequence (a_n) to be *bounded.*

Question 2.132. For what values of x is the following inequality valid:

$$|x + 2| \leq 5$$

Question 2.133. Let (a_n) be a sequence such that $|a_n| \leq 2^{-n}$ for every $n \geq 0$. Find a value N such that $|\sum_{n=N}^{\infty} a_n| \leq 10^{-3}$.

Question 2.134. State, without proof, whether the sequence $\tan(n)$ is convergent, bounded or monotone.

Question 2.135. State without proof whether the sequence $\dfrac{\cos n}{2-\sin n}$ is convergent, bounded, or monotone.

Question 2.136. Show that every sequence which is not bounded above has an increasing subsequence.

Question 2.137. Show that if $\lim_{n\to\infty} a_n = l$ and $l > 0$, then a_n is eventually positive.

Question 2.138. If $(a_n) \to a$ and $a_n > 0$ for all $n > 0$, then is it true that $a > 0$? Justify your answer.

Question 2.139. Show that if $(a_n) \to 0$ then either there is a subsequence of $(\frac{1}{a_n})$ which tends to $+\infty$, or there is a subsequence of $(\frac{1}{a_n})$ which tends to $-\infty$.

Question 2.140. If a is irrational and $b \neq 0$ is an integer, is $\frac{a}{b}$ rational or irrational? Justify your answer.

Question 2.141. What is the value of $\lim_{n\to\infty} (3^n + 5^n + 2^n)^{\frac{1}{n}}$?

Question 2.142. Find the value of $\inf\{\cos x \mid 0 < x < \pi\}$.

Question 2.143. Compute the limit of the sequence:

$$\left(\frac{2(n!) + 3^n}{4^n + n!}\right)$$

using any rules of algebra of limits and standard limits.

Question 2.144. Find real numbers a, b such that:

$$\{x \in \mathbb{R} \mid x > -3,\ x < 5\} = \{y \in \mathbb{R} \mid |y - a| < b\}$$

Question 2.145. Two of the following sequences eventually sandwich the third:

$$\left(\frac{1}{n^3}\right),\ \left(\frac{1}{36n}\right),\ \left(\frac{1}{(2n)^2}\right)$$

Write down the corresponding inequality in the form $a_n \le b_n \le c_n$ for all $n > N$ and find the smallest value of N for which it is true.

Question 2.146. If (a_n) converges to a, and $a_n > 1$, for all $n \ge 1$, then show that $a \ge 1$.

Question 2.147. State without proof the possible limits of the sequence (x^n) for $x > 0$.

Question 2.148. Consider the recursive sequence $a_{n+1} = \sqrt{1 + a_n},\ a_1 = 1$. Assuming that the sequence converges, what is the limit?

Question 2.149. Let the sequence (a_n) be defined according to $a_1 = 1$ and $a_n = \sqrt{3a_{n-1} - a_{n-1}^2}$ for $n \geq 2$. Assume that (a_n) tends to $a > 0$. Find the numerical value of a. Justify your answer.

Question 2.150. Find $\inf \left\{ \dfrac{1}{\tan x} \;\middle|\; \dfrac{\pi}{4} < x < \dfrac{\pi}{2} \right\}$

Question 2.151. State without proof for which $a \in \mathbb{R}$ the series $\sum_{n=1}^{\infty} a^n$ converges.

Question 2.152. Does there exist a rearrangement (b_n) of $(a_n) = \left(\dfrac{\cos n\pi}{n}\right)$ such that $\sum_{n=1}^{\infty} b_n = \pi$?

Question 2.153. Assume that one knows that $0 \leq e - \sum_{n=0}^{M} \dfrac{1}{n!} \leq \dfrac{2}{(M+1)!}$, for $M \geq 1$. Find the smallest $N \geq 1$ such that the error in approximation $e \approx \sum_{n=0}^{N} \dfrac{1}{n!}$ is less than $\dfrac{1}{50}$.

Question 2.154. Assume that (a_n) is a decreasing sequence. Which of the following statements are true:

1. (a_n) has a limit.
2. (a_n) is eventually negative.
3. (a_n) tends to minus infinity.
4. (a_n) tends to zero.

Question 2.155. State, without proof, whether the following sequences tends to A) infinity, B) minus infinity, C) zero, D) a constant $c \neq 0$ or E) does not tend to either infinity, minus infinity, or any real number.

1. $(a_n) = \left(\frac{n^2}{2^n}\right)$,

2. $(a_n) = \left(\frac{2n^2+\sin(n)}{5n^2+3n+17}\right)$,

3. $(a_n) = \left(\left(10^{n^2} + \sqrt{n}\right)^{\frac{1}{n}}\right)$,

4. $(a_n) = (e^n \cos(n))$.

Question 2.156. Prove that $\inf\{a_n^2 : n \in \mathbb{N}\} = 0$ for any null sequence (a_n).

Question 2.157. An increasing sequence (a_n) satisfies $a_n < 4$ for every $n \in \mathbb{N}$. State, without proof, whether (a_n) must converge.

Question 2.158. State the Bolzano - Weierstrass Theorem and use it to decide whether sequence $(\cos(2^n))$ has a convergent subsequence.

Question 2.159. Find a value N such that $\left|\frac{n^2}{n^2+1} - 1\right| \leq 10^{-3}$ for every value of $n \geq N$.

Question 2.160. Find a series $\sum_{n=1}^{\infty} a_n$ with every $a_n > 0$ which converges to 3.

Question 2.161. Find a sequence (a_n) that satisfies $|a_{n+1} - a_n| < |a_n - a_{n-1}|$ for every $n > 1$ but (a_n) is not Cauchy.

Question 2.162. Prove that every Cauchy sequence is bounded.

Question 2.163. Compute $\sum_{n=1}^{\infty} \frac{\pi}{3^n}$.

Question 2.164. Is $\sum_{n=1}^{\infty} \frac{n + \sin^2 n}{n^2 + \cos n}$ finite or infinite? Justify your answer.

Question 2.165. Is $\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}}$ finite or infinite? Justify your answer.

2.2 Questions on Sequences

There will invariably be a long question on sequences. Expect the question to be split into three parts: One part asking for basic definitions, another part for simple proofs, and the last part on applications. You should at least know the definitions and how to apply the theorems learnt in Analysis I by now.

1. *January 2000 Question 2*
   Let (a_n) be a sequence of real numbers.
   (a) Give the definition of $(a_n) \to a$.
   (b) Give the definitions of $\sup\{a_n \mid n \in \mathbb{N}\}$ and $\inf\{a_n \mid n \in \mathbb{N}\}$. (The *supremum* is also known as the least upper bound and the *infimum* as the greatest lower bound.)
   (c) Suppose that a set A has infimum l. Prove that for any $\epsilon > 0$ that there exists an element $a \in A$ satisfying $l \leq a < l + \epsilon$. (*Hint: You might want to argue by contradiction.*)
   (d) Prove that if the sequence (a_n) is decreasing and bounded below then $\inf\{a_n \mid n \in \mathbb{N}\} = \lim_{n\to\infty} a_n$.
   (e) Let $a_n = \dfrac{1000^n}{n!}$. Find $\lim_{n\to\infty} a_n$. (It is **not** enough to quote it as a limit.)
   (f) Find $\sup\{a_n \mid n \in \mathbb{N}\}$ and $\inf\{a_n \mid n \in \mathbb{N}\}$ where a_n is as in 5) above.

2. *January 2001 Question 2*
   (a) i. Give the definition of 'a sequence (a_n) of real numbers converges to a real number a'.
   ii. Give the definition of '(a_n) is a Cauchy sequence'.
   iii. Prove that if (a_n) converges to a real number a then (a_n) is a Cauchy sequence.
   (b) Which of the following sequences converges to a real number? Name any test or rule that you use. Find the values of the limits when they exist.
   i. $a_n = \left(\dfrac{n+2}{3n-1}\right)^3$
   ii. $a_n = \dfrac{\sqrt{n}+3}{2+\sin n}$
   iii. $a_n = \dfrac{1+(-1)^n}{n}$

(c) If a and b are rational numbers with $a < b$, write down an expression for a rational number c with $a < c < b$. Deduce that there are infinitely many rational numbers with $a < b < c$.

3. *January 2002 Question 2*

   (a) i. Give the definition of 'a sequence (a_n) of real numbers converges to a real number a'.

   ii. Give the definition of 'the sequence (a_n) of real numbers tends to infinity'.

   iii. Prove that if (a_n) is a sequence of non-zero real numbers for which $\left(\dfrac{1}{a_n}\right)$ tends to 0 then $(|a_n|)$ tends to infinity.

   (b) State, without proof, what the limit of the sequence (x^n) is when $|x| < 1$.

   (c) Investigate whether each of the following sequences converges to a real number. Name any test or rule that you use. Find the values of the limit where they exist.

   i. $a_n = \left(\dfrac{2^n + 4^n}{3^n}\right)$

   ii. $a_n = \left(\dfrac{\sin n + 5}{2 + \sqrt{n}}\right)$

   iii. $a_n = \left(\dfrac{\sqrt{n} + 2}{1 + 3\sqrt{n}}\right)$

   (d) Show that, for each real number a and natural number n, we have:

$$\frac{[a2^n]}{2^n} \leq a < \frac{[a2^n] + 1}{2^n}$$

   where $[y]$ denotes the integer part of y. Hence show that, if $a < b$, there is a rational number r with $a < r < b$.

4. *January 2003 Question 2*

   (a) i. Give the definition of 'a sequence (a_n) of real numbers converges to a real number a'.

   ii. Use the definition to show that $\left(\dfrac{1}{\sqrt{n}}\right)$ converges to 0.

   iii. Give the definition of 'a sequence (a_n) of real numbers is a Cauchy sequence'.

   (b) Prove that a convergent sequence is a Cauchy sequence.

   (c) In each of the following examples say if the sequence converges and find the value of the limit when the sequence converges. Name any test or rule that you use.

When the sequence doesn't converge find a convergent subsequence. (You may assume that $\left(\frac{1}{n^x}\right) \to 0$ when $x > 0$.)

i. $a_n = \dfrac{n^4 + n}{n(n+3)^2(n+5)}$

ii. $a_n = \dfrac{\cos n + 2}{\sqrt{n} + 7}$

iii. $a_n = (-1)^n + 1$

iv. $a_n = \dfrac{(-1)^n + 1}{n}$

5. *January 2005 Question 2*

(a) i. Give the definition of 'a sequence (a_n) of real numbers converges to a real number a'.

ii. Give the definition of 'the sequence (a_n) of real numbers tends to infinity'.

iii. Prove that if (a_n) is a sequence of non-zero real numbers and (a_n) tends to infinity, then $\left(\frac{1}{a_n}\right)$ converges to 0.

(b) State whether or not each of the following sequences converges to a real number, naming any test you use. For any that do converge to a real number, find the value of the limit.

i. $a_n = \left(\dfrac{n+2}{3n-1}\right)^3$

ii. $a_n = \dfrac{\sqrt{n} + 3}{2 + \sin n}$

iii. $a_n = \dfrac{1 + (-1)^n}{n}$

(c) If a and b are rational numbers with $a < b$, write down an expression for a rational number c with $a < c < b$. Deduce that there are infinitely many rational numbers with $a < b < c$.

6. *January 2006 Question 2*

(a) Give the definition of 'the sequence (a_n) of real numbers converges to the real number a'.

(b) Let (a_n) and (b_n) be two sequences such that $\lim_{n\to\infty} a_n = a$ where $a > 0$, and (b_n) diverges to $+\infty$. Prove that the sequence (c_n) defined by $c_n = a_n \cdot b_n$ diverges to $+\infty$.

(c) For each of the following sequences, say whether they converge, diverge to $+\infty$, diverge to $-\infty$ or none of these. For those that converge, compute their limit.

i. $a_n = \dfrac{n^2+4}{2n-3n^2}$

ii. $b_n = \dfrac{1}{n}(\sqrt{n^2+1}-1)$

iii. $c_n = \dfrac{\sqrt{n^3+\sin n}-5n}{2}$

(d) Give the definition of a Cauchy sequence.

(e) Prove that the sum of two Cauchy sequences is again a Cauchy sequence.

7. *January 2007 Question 2*

(a) State precisely what is meant by 'the sequence (a_n) of real numbers converges to the real number a'.

(b) State precisely what is meant by 'the sequence (a_n) of real numbers tends to infinity'.

(c) Prove that if (a_n) is a sequence of non-zero real numbers then $(|a_n|)$ tends to ∞ if and only if $\left(\dfrac{1}{a_n}\right)$ converges to 0.

(d) Investigate whether each of the following sequences converges to a real number. Name any test or rule that you use. Find the values of the limits when they exist.

i. $a_n = \dfrac{2^n+4^n}{3^n}$

ii. $a_n = \dfrac{\sin(8^n)+5}{2+7^n}$

iii. $a_n = \dfrac{8^n+2}{1+8^n+\cos(8^n)}$

8. *January 2008 Question 2*

(a) i. Give the definition of 'the sequence (a_n) of real numbers converges to the real number a'.

ii. Give the definition of 'a sequence (a_n) of real numbers is a Cauchy sequence'.

iii. Prove that if (a_n) converges to a real number a then (a_n) is a Cauchy sequence.

(b) Prove that if (a_n) converges to a real number a and $a_n \geq 0$ for all n, then $a \geq 0$.

(c) For each of the following examples say if the sequence converges to a real number, tends to infinity, tends to minus infinity, or none of these. You do not have to give proofs. Find the value of the limits when they exist.

i. $a_n = \left(-\frac{1}{\sqrt{n}}\right)$

ii. $a_n = \left(\frac{\sqrt{n}+7}{2+\cos n}\right)$

iii. $a_n = \left(\frac{3n+1}{4n+5}\right)^2$

iv. $a_n = (n+6^n)^{\frac{1}{n}}$

(d) Define a sequence (a_n) to be $a_n = 1,\ a_2 = 1$ and $a_{n+1} = (2a_n a_{n-1})^{\frac{1}{3}}$ for $n \geq 2$. Assuming (a_n) converges to a real number a, find the value of a.

9. *January 2009 Question 2*

(a) i. Define what it means for a sequence (a_n) to converge to a limit a.

ii. Define what it means for the sequence to be bounded.

iii. Prove that every convergent sequence is bounded.

(b) Which of the following sequences converge? Justify your answers. Find the limit for those sequences that converge to a real number.

i. $a_n = \frac{\sin n + \cos n}{\log n},\ n \geq 2$

ii. $b_n = \frac{\log n}{3 + \sin n + \cos n},\ n \geq 2$

iii. $c_n = \frac{\sin n + 2\log n}{\cos n + \log n},\ n \geq 2$

(c) Given a sequence (a_n), recall that the n^{th} term a_n is called a *floor term* if for every $m > n$, we have that $a_m \geq a_n$. Using this notion, show that any sequence which is bounded below has an increasing subsequence or a strictly decreasing subsequence.

10. *January 2010 Question 2*

(a) i. Define what it means for a sequence (a_n) to *converge to a limit* a.

ii. Give the limits of the following sequences:

A. $a_n = (3^n + 2^{2n})^{\frac{1}{n}}$

B. $a_n = \frac{4n^2 2^n + 2n^3 4^n}{8n^4 2^n + n^3 4^n}$

C. $a_n = (5n^3 + 6n^4)^{\frac{1}{n}}$

Justify your answers.

(b) i. Define what it means for a sequence (a_n) to *eventually* have some property.

ii. Show that if the limit b of a sequence (b_n) satisfies $b > 0$, then eventually $b_n > \frac{b}{2}$.

iii. Prove the quotient rule for sequences, i.e., if (a_n) converges to a and (b_n) converges to b, with $b_n > 0$ for all n and $b > 0$, then $\left(\frac{a_n}{b_n}\right)$ converges to $\frac{a}{b}$.

(c) State the Ratio Lemma for sequences (a_n) for which there exists a constant $0 \leq l < 1$ so that $a_{n+1} \leq l a_n$, for $n \geq 1$. Show how to prove the Ratio Lemma assuming the Sandwich Theorem.

11. *January 2011 Question 2*

(a) State and prove the Sandwich Theorem for sequences.

(b) Prove Bernoulli's inequality: If $x > -1$ and n is a natural number the

$$(1+x)^n \geq 1 + nx.$$

(c) Prove the following statement: If $x > 0$ then $(x^{\frac{1}{n}}) \to 1$.

(d) Prove that the sequence

$$(a_n) = \left(\left(\frac{1+\cos^2(n)}{2+\cos(n)}\right)^{\frac{1}{n}}\right)$$

tends to 1 as n tends to infinity.

2.3 Questions on Completeness

There will invariably be a long question on this topic in Analysis, and it is sometimes combined with part of a series or sequence question, so beware. There will seldom be an entire full question just on that topic, but will have familiar elements of sequences and series within. Expect this question to be split into three parts: One part asking for basic definitions, another part for simple proofs, and the last part on applications. You should at least know the definitions and how to apply the theorems learnt in Analysis I by now.

1. *January 2000 Question 3*

   (a) Suppose that $(a_n) \to a$ and $a_n \geq 0$ for all $n \in \mathbb{N}$. Use the definition of convergence to show that $a \geq 0$.

   (b) Consider the sequence (a_n) defined by $a_{n+1} = \sqrt{6 + a_n}$ and $a_1 = 0$ (where we take positive square root).

      i. Show that $a_n \leq 3$ for all n.

      ii. Show that (a_n) converges, quoting carefully any results that you use from the course.

      iii. Calculate the limit $\lim_{n\to\infty} a_n$.

   (c) Suppose that $x > 0$ has a representation as a recurring decimal, and $y > 0$ has a representation as a non-recurring decimal. Will the number xy have a recurring or non-recurring decimal? Justify your answer.

2. *January 2001 Question 3*

   (a) i. If A is a non-empty set of real numbers, and U is a real number, give the definitions of:

         A. U is an upper bound of A;

         B. A is bounded above, and;

         C. U is the least upper bound of A.

      ii. State the completeness property of the real numbers in terms of the above concepts.

      iii. From this completeness property deduce that an increasing sequence (a_n) of real numbers which is bounded above is convergent.

   (b) Define a sequence (a_n) by:

$$a_1 = 2 \text{ and } a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$$

   i. Show that $a_n^2 \geq 2$ for all n by calculating $2 - a_{n+1}^2$. Use this, and $a_n > 0$ for $n \geq 1$, to show that (a_n) is decreasing, by considering $a_{n+1} - a_n$.
   ii. Use the completeness property to deduce that (a_n) converges and calculate the limit.

3. *January 2002 Question 3*

   (a) If A is a non-empty set of real numbers, and U is a real number, give the definitions of:

      i. U is an upper bound of A;
      ii. A is bounded above, and;
      iii. U is the least upper bound of A.

   (b) State the completeness property of the real numbers in terms of the above concepts.

   (c) Prove that if A is a non-empty set of real numbers which is bounded above then for every $\epsilon > 0$ there exists some $a \in A$ with $\sup A - \epsilon < a \leq \sup A$. Deduce that there is a sequence (a_n) with $a_n \in A$ for every $n \geq N$ and $(a_n) \to \sup A$.

   (d) For each of the following sets find, without proof, the supremum.

      i. $A_1 = \left\{ \dfrac{\cos(n\pi)}{n} \;\middle|\; n \in \mathbb{N} \right\}$
      ii. $A_2 = \left\{ \left(1 + \dfrac{1}{n}\right)^n \;\middle|\; n \in \mathbb{N} \right\}$
      iii. $A_3 = \{x \mid x \in \mathbb{Q},\ x^2 \leq 2\}$

   (e) Give the definition of 'the sequence (a_n) is a Cauchy sequence'.

   (f) If (a_n) satisfies $|a_{n+1} - a_n| < \dfrac{1}{2^n}$ for every $n \geq 1$, show that (a_n) is a Cauchy sequence.

4. *January 2003 Question 3*

   (a) If A is a non-empty set of real numbers, and U is a real number, give the definitions of:

      i. U is an upper bound of A;
      ii. A is bounded above, and;
      iii. U is the least upper bound of A.

   (b) State the completeness property of the real numbers in terms of the above concepts.

(c) Find the least upper bound of the following sets:

i. $\mathbb{Q} \cap [0, 1]$, where $\mathbb{Q}$ is the set of rational numbers.

ii. $\{x \in \mathbb{R} \mid x^2 < 4\}$

(d) Prove Bernoulli's inequality:

$$(1+x)^n \geq 1 + nx \text{ for } n \geq 1 \text{ and } x \geq -1$$

(e) Use Bernoulli's inequality to show that $(\lambda^n) \to \infty$ when $\lambda > 1$.

(f) Write down the limits of the following sequences. You do not have to give proofs.

i. $a_n = \left(\dfrac{n^2}{2^n}\right)$

ii. $a_n = \left(\dfrac{2^n}{n!}\right)$

iii. $a_n = \left(\dfrac{n!}{n^n}\sqrt[n]{n}\right)$

iv. $a_n = (\sqrt[n]{2n})$

5. *January 2005 Question 3*

(a) i. If A is a non-empty set of real numbers, and U is a real number, give the definitions of:

A. U is an upper bound of A;

B. A is bounded above, and;

C. U is the least upper bound of A.

ii. State the completeness property of the real numbers in terms of the above concepts.

iii. From this completeness property deduce that an increasing sequence (a_n) of real numbers which is bounded above is convergent.

(b) Define a sequence (a_n) by:

$$a_n = 2 \text{ and } a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$$

Notice that $a_n > 0$ for all natural numbers n, by induction (you do not need to prove this). Calculate $a_{n+1}^2 - 2$ to show that $a_n^2 > 2$ for all natural numbers n. By considering $a_{n+1} - a_n$, show that (a_n) is strictly decreasing. Use the completeness property to show that (a_n) converges and calculate the limit.

6. *January 2006 Question 4*

   (a) In each of the following cases, give the value of the supremum of the set and state whether or not it lies in the set.

      i. $B_1 = \left\{ \dfrac{(-1)^n}{n} : n \geq 1 \right\}$

      ii. $B_2 = \left\{ 1 - \dfrac{1}{n^2} : n \geq 1 \right\}$

      iii. $B_3 = \{x : x \text{ is a rational number such that } x^2 \leq 2\}$

   (b) Let A, B be non empty sets of real numbers. Define the following terms:

      i. A is *bounded below*

      ii. a real number L is the *greatest lower bound* for A

      iii. B is *bounded above*

      iv. a real number U is the *least upper bound* for B

   (c) State the Completeness Axiom in terms of sets bounded above. State an equivalent version of the Completeness Axiom in terms of sets bounded below.

   (d) Let A, B be non-empty sets of real numbers such that for each a in A and b in B we have that $a < b$. Show that A has a least upper bound and that B has a greatest lower bound.

7. *January 2007 Question 3*

   (a) i. If A is a non-empty set of real numbers, and U is a real number, give the definitions of:

         A. U is an upper bound of A;

         B. A is bounded above, and;

         C. U is the least upper bound of A.

      ii. State the completeness property of the real numbers in terms of the above concepts.

      iii. From this completeness property deduce that an increasing sequence (a_n) of real numbers which is bounded above is convergent.

   (b) For each of the following sets find, without proof, their supremum.

      i. $A = \left\{ \dfrac{\sin(\frac{n\pi}{2})}{n} \;\middle|\; n = 1, 2, \ldots \right\}$

      ii. $B = \left\{ \left(1 + \dfrac{1}{n}\right)^n \;\middle|\; n = 1, 2, \ldots \right\}$

      iii. $C = \{x \mid x \in \mathbb{Q} \text{ and } x^2 < 3\}$

iv. $D = \{x \mid x \neq \mathbb{Q} \text{ and } x^2 < 4\}$

8. *January 2008 Question 3*

(a) If A is a non-empty set of real numbers, and U is a real number, give the definitions of:

i. U is an upper bound of A;

ii. A is bounded above, and;

iii. U is the least upper bound of A.

(b) State the completeness property of the real numbers in terms of the above concepts.

(c) Find the least upper bound of the following sets.

i. $\mathbb{Q} \cap [0, 1]$, where $\mathbb{Q}$ is the set of rational numbers.

ii. $\{x \in \mathbb{R} : x^2 < 4\}$

iii. $\{x \in \mathbb{R} : x \neq \mathbb{Q},\ x^2 < 4\}$

(d) Prove Bernoulli's inequality:

$$(1 + x)^n \geq 1 + nx \text{ for } n \geq 1 \text{ and } x \geq -1$$

(e) Use Bernoulli's inequality to show that $(\lambda^n) \to \infty$ when $\lambda > 1$.

(f) Write down the limits of the following sequences. You do not have to give proofs.

i. $a_n = \left(\frac{n^2}{2^n}\right)$

ii. $a_n = \left(\frac{2^n}{n!}\right)$

iii. $a_n = \left(\frac{n!}{n^n}\sqrt[n]{n}\right)$

iv. $a_n = (\sqrt[n]{2n})$

9. *January 2009 Question 3*

(a) Define what it means for a non-empty set B of real numbers

i. to be bounded below.

ii. to have a greatest lower bound.

(b) Use this definition to state a version of the completeness axiom for real numbers.

(c) Guess the infimum and supremum of the set:

$$A = \left\{ 1 + \frac{1}{n^2} \;\middle|\; n \in \mathbb{N} \right\}$$

Now use the definition of infimum and supremum to prove your guesses.

i. Define what it means for a sequence c_n to be Cauchy.

ii. Use this definition to show that the sequence defined by:

$$c_n = 2 - \frac{\sin n}{n}$$

is Cauchy.

iii. Show, using the definition, that if (a_n) and (b_n) are Cauchy sequences, then the sequence given by $c_n = 3a_n - b_n$ is also Cauchy.

10. *January 2010 Question 3*

(a) One of the following is the Bolzano-Weierstrass theorem:

i. Every monotonic sequence has a convergent subsequence

ii. Every bounded sequence has a convergent subsequence

iii. Every bounded above sequence has a convergent subsequence

iv. Every increasing sequence has a convergent subsequence

Which one is the correct claim? Give a counterexample for each of the incorrect claims.

(b) Consider the sequence:

$$a_n = \frac{\sqrt{\frac{3}{2} + \frac{1}{2}(-1)^n}}{n}$$

and the set $A = \{a_1, a_2, \ldots\}$.

i. Write explicitly the first six elements of the sequence.

ii. Are all the elements of A rational? Are all the elements of A irrational?

iii. Show that A is bounded, and find the infimum and supremum.

iv. Prove that (a_n) is Cauchy.

(c) Suppose that (a_n) is Cauchy, and define the sequence (b_n) by:

$$b_n = a_{2n} - a_n$$

Show that b_n is Cauchy and find its limit.

11. *January 2011 Question 3*

   (a) Suppose that A is a non-empty set of real numbers. Define what it means for a real number u to be:

      i. an upper bound for A;

      ii. a least upper bound for A.

   (b) Suppose that (a_n) is any bounded increasing sequence. Prove that (a_n) tends to the least upper bound of the set $\{a_n : n \in \mathbb{N}\}$.

   (c) Let
$$b_n = (-1)^n \left(\frac{n}{n+1} + \frac{(-1)^n}{2(n+1)} \right).$$

      i. Calculate the first five terms of (b_n). Is (b_n) monotone?

      ii. Show that (b_n) is bounded. State, without proof, whether (b_n) converges.

      iii. Find an increasing subsequence of (b_n). Show that your subsequence is strictly increasing and identify the limit.

      iv. Write down a subsequence of (b_n) which converges to a limit different to that found in (iii). Write the limit explicitly.

   (d) Define what it means for a sequence (a_n) to be Cauchy.

   (e) Suppose that (a_n) satisfies $|a_{n+1} - a_n| \leq 2^{-n}$. Using the fact that $\displaystyle\sum_{k=n}^{\infty} 2^{-k} \to 0$ as $n \to \infty$, show that (a_n) is Cauchy.

2.4 Questions on Series

There will invariably be a long question on series. Expect the question to be split into three parts: One part asking for basic definitions, another part for simple proofs, and the last part on applications. You should at least know the definitions and how to apply the theorems learnt in Analysis I by now.

1. *January 2000 Question 4*

   (a) i. Give the definition of the n^{th} partial sum for the series $\sum a_n$.

   ii. Suppose that $a_n \geq 0$ for all $n \in \mathbb{N}$. Prove from the definition of convergence for a series that $\sum a_n$ converges if and only if the sequence of partial sums is bounded.

   (b) Determine whether each of the following series diverges or converges. If you use a test you should show that the appropriate conditions are satisfied.

   i. $\displaystyle\sum_{n=1}^{\infty} \frac{(n^{10} + 1000n)}{(n^2 + 10)^5}$

   ii. $\displaystyle\sum_{n=1}^{\infty} \frac{(n^{10} + 1000n)}{(n^3 + 10)^5}$

   iii. $\displaystyle\sum_{n=1}^{\infty} n^{100} e^{-n^2}$

   iv. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n + \frac{1}{n}}$

2. *January 2001 Question 4*

   (a) Define the sequence of partial sums of the series $\sum a_n$. Give the definition of 'the series $\sum a_n$ converges'.

   (b) i. Let a and r be real numbers and let n be a natural number. Derive the formula:

   $$a + ar + \ldots + ar^n = a\left(\frac{1 - r^{n+1}}{1 - r}\right) \quad \text{when } r \neq 1$$

   ii. Prove that the series $\displaystyle\sum_{n=0}^{\infty} ax^n$ converges when $x \in (-1, 1)$ and obtain a formula for the sum. (You may assume that $y^n \to 0$ when $|y| < 1$).

iii. Find N so that $\left|\sum_{k=1}^{N} \frac{1}{2^k} - 1\right| < 10^{-6}$.

(c) Express the repeating decimal $7.7777\ldots$ as a fraction $\frac{m}{n}$ where m and n are integers with no common factors.

(d) State, without proof, for which values of $p > 0$ the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

(e) Determine whether each of the following series converges or not. If you use a convergence test, state the name of the test and show the appropriate conditions are satisfied.

i. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^8+3}}$

ii. $\sum_{n=1}^{\infty} \frac{3^n}{n!}$

iii. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\frac{1}{2}}}$

iv. Is the series in part c) absolutely convergent?

3. *January 2002 Question 4*

(a) Define what it means to say that the series $\sum_{n=1}^{\infty} a_n$:

i. converges

ii. converges absolutely

iii. converges conditionally

(b) Suppose that $\sum_{n=1}^{\infty} a_n$ converges absolutely. Prove that $\sum_{n=1}^{\infty} a_n$ converges. You may assume that a sequence converges if and only if it is Cauchy.

(c) Give an example of a series which is conditionally convergent. (You do not have to justify your answer.)

(d) State, with reasons, whether the following series converge or diverge. You may assume any results/tests about sequences or series.

i. $\sum_{n=1}^{\infty} \frac{2n^2+15n}{n^3+7}$

ii. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n^2+1)^{\frac{1}{3}}}$

iii. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

4. *January 2003 Question 4*

(a) Define what it means to say that the series of real numbers, $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

(b) State with brief reasons whether each of the following series converges.

i. $\displaystyle\sum_{n=1}^{\infty} (-1)^n n$

ii. $\displaystyle\sum_{n=1}^{\infty} [(-1) + (-1)^n]$

iii. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$

iv. $\displaystyle\sum_{n=1}^{\infty} \left|\sin\left(\frac{n\pi}{2}\right)\right|$

(c) Suppose that $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$. Prove that if $\displaystyle\sum_{n=1}^{\infty} b_n$ converges then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges. (You may assume any results about sequences.)

(d) For which values of $p \in \mathbb{R}$ does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge? (You do not need to give reasons for your answer.)

(e) State, with reasons, whether following series converges or diverges.

i. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3 - 3n - 4}{n^5 + 2n^4 + n^2 + 1}$

ii. $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2 - n}{4n^3 - 2n^2 + n - 1}$

iii. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 7^n}{3^n + 8^n}$

5. *January 2005 Question 4*

(a) For each of the following series, say whether they converge (and if so whether they converge absolutely or conditionally) or diverge (and if so whether they diverge to plus infinity, minus infinity, or neither).

i. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(2n)}{1+n+n^2}$

ii. $\displaystyle\sum_{n=1}^{\infty} \frac{2n+3}{1+n^2}$

iii. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{1+2n}}$

iv. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$

(b) State the alternating series test and give an example of a series to which it applies.

(c) It has been known since the 17$^{\text{th}}$ century that π can be represented as:

$$\pi = 4\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$

In 1995, the following formula was discovered:

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right)$$

Estimate the number of terms required to approximate π to within 10^{-10} for both formulas. **Hint:** Use the fact (which you do not need to prove) that the term in brackets in the second formula lies between 0 and 4 for every value of k.

(d) What are the values of x such that the series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges? State the different cases you consider and the name of the test you are using in each case. Don't forget to justify your answer even for those values of x for which the series doesn't converge.

6. *January 2006 Question 3*

(a) Give the definition of the sequence of partial sums of the series $\displaystyle\sum_{n=1}^{\infty} a_n$. State precisely what is meant by 'the series $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges to ∞'.

(b) Give (without proof) an example of a convergent series which is not absolutely convergent.

(c) Determine whether each of the following series converges or not. If you use a convergence test state the name of the test and show the appropriate conditions are satisfied.

i. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\log n}$

ii. $\displaystyle\sum_{n=1}^{\infty} n^{77} e^{-n}$

iii. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$

(d) Prove the following *sum rule for series*:

Let $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ be convergent. Then $\displaystyle\sum_{n=1}^{\infty}(a_n + b_n)$ is convergent, and:

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

(e) Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!} x^n$ converges for all x with $|x| < e^{-1}$.

7. *January 2007 Question 4*

(a) State the ratio test for the convergence of series.

(b) State the integral test for the convergence of series.

(c) Show in two different ways that the series:

$$\sum_{i=1}^{\infty} \frac{1}{i^2}$$

converges. Can the ratio test be used in this situation? Explain your answer.

(d) Let $S(x)$ be defined by:

$$S(x) = 1 + \sum_{i=1}^{\infty} \frac{(-1)^i}{i} x^i$$

i. Show that for all $0 \leq x \leq 1$, $S(x)$ is convergent.

ii. Show that for all $0 \le x < 1$, $S(x)$ is absolutely convergent.

8. *January 2008 Question 4*

(a) State the Alternating test for the convergence of series.

(b) Which geometric series $\sum_{n=0}^{\infty} x^n$ converge and which of them diverge? Give a proof.

(c) State the integral test for the convergence of series. Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by using the integral test.

(d) For each of the following series determine whether it converges and whether it converges absolutely. Prove your answer.

i. $\sum_{n=1}^{\infty} \frac{7^n}{n^7}$

ii. $\sum_{n=1}^{\infty} (-1)^n \frac{(n+1)}{n^2}$

9. *January 2009 Question 4*

(a) Explain what is meant by a *partial sum* of a series $\sum_{n=1}^{\infty} a_n$. Explain what it means for the series $\sum_{n=1}^{\infty} a_n$ to be convergent.

(b) Using the definition of a convergent sequence, show that $\sum_{n=1}^{\infty} a^n$ is convergent if and only if $|a| < 1$.

(c) Is it true that $\sum_{n=1}^{\infty} a_n^2$ is convergent if $\sum_{n=1}^{\infty} a_n$ is convergent? Justify your answer.

(d) Write down an example of a convergent series and a of a divergent series both of which satisfy the condition $\frac{a_{n+1}}{a_n} \to 1$ and $a_n \ge 0$.

(e) Determine whether the series are conditionally convergent, absolutely convergent, or divergent. Justify your answers.

i. $\sum_{n=1}^{\infty} \left(\frac{\sin n}{2^n + 1} + \frac{1}{n^2} \right)$

ii. $\displaystyle\sum_{n=1}^{\infty} 1+(-1)^n$

iii. $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{1}{2\sqrt{n}+3}$

iv. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n!}$

10. *January 2010 Question 4*

(a) Explain what it meant by the *partial sums* of a series $\displaystyle\sum_{n=1}^{\infty} a_n$. State precisely what it means for the series $\displaystyle\sum_{n=1}^{\infty} a_n$ to *diverge to infinity*.

(b) State what it means for a series $\displaystyle\sum_{n=1}^{\infty} a_n$ to be *absolutely convergent*. Give an example (without proof) of a series which is convergent but not absolutely convergent.

(c) State the version of the Ratio Test for series that does not require every term to be positive.

(d) Determine whether each of the following series converges or not. If you use a convergence test, state the name of the test and show that the appropriate conditions are satisfied.

i. $\displaystyle\sum_{n=1}^{\infty} \frac{\pi^n+e^n+3^n}{4^n+\pi^n}$

ii. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\frac{1}{3}}-\frac{1}{2}}$

iii. $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{2^n n!}$

(e) Prove that the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to infinity. Give the full statement of any results you use in your proof.

11. *January 2011 Question 4*

(a) i. State the Alternating Test for the convergence of series.

ii. Suppose that (a_n) is decreasing and null. Show that

$$0 \leq \sum_{k=N}^{\infty} (-1)^k a_k \leq a_N$$

for all even numbers N.

iii. Let $b_n = \dfrac{(-1)^n}{(n+1)^{\frac{1}{3}}}$. Find an N such that the difference between the sum of the first N terms, $\sum_{k=1}^{N} b_k$, and the whole series is guaranteed to be less than 10^{-2}. Give justifications.

(b) Either prove the statements below, or give a counter-example.

i. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

ii. If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

iii. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^2$ converges.

(c) Consider $\sum_{n=1}^{\infty} \frac{1}{n} x^n$, where x is a real number (positive or negative). Find all numbers x such that (i) the series converges; (ii) the series converges absolutely; (iii) the series diverges to $+\infty$; (iv) the series does not converge. (No justifications needed).

(d) Consider the series $\sum_{n=1}^{\infty} \left(\sqrt{n+2} - \sqrt{n}\right)$.

i. State whether it converges or diverges.

ii. Give a full proof of whether it converges or diverges.

3 Conclusion

If you can do ALL of the questions listed above, as well as successfully writing out the proofs, you shouldn't have a problem passing the exam, and might *just* might, achieve your First. Good luck! The best is yet to be :)