MA 131

THE MORSE SOCIETY, UNIVERSITY OF WARWICK

FIRST YEAR MOCK PAPER: Jan 2011

ANALYSIS (Prelimilary)

Time allowed: 1.5 hour

ANSWER QUESTION 1 AND 2 MORE QUESTIONS.

Question 1 is compulsory and is worth 35 marks. Each of the other 3 questions is worth 20 marks. This section is marked out of 75. If you have answered more than the required number of questions in this section, you will only be given credit for question 1 and for the best 2 of your other answers.

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1 . [Compulsory question] Consider the following statements and for each one state whether it is TRUE or FALSE. If you answer TRUE, no proof or explanation is required. If you answer FALSE, you should give a counter-example.

(a) Every bounded sequence has at least 2 convergent subsequences. [2]

(b) If (a_n) is null sequence then the series $\sum_{n=1}^{\infty} a_n$ converges. [2]

(c) An increasing sequence $(a_n)_{n=0}^{\infty}$ either diverges to ∞ or converges to a limit $a \geq a_0$. [2]

(d) The series $\sum_{n=1}^{\infty} a_n$ where $a_n = -1$ for n odd and $a_n = n$ for n even can be rearranged to sum to -53. [2]

(e) If $(a_n) \to \frac{1}{2}$ and $\exists b_n, c_n$ such that $a_n = \frac{b_n}{c_n}$ then $(b_n) \to 1$ and $(b_n) \to 2$. [2]

(f) For every real number a $\exists a_n$ of irrational numbers such that $\lim_{x\to\infty} a_n = a$. [2]

(g) If $a_{n+1} - a_n \to 0$ as $n \to \infty$ then a_n is convergent. [2]

(h) If a_n has the cauchy property then all subsequences of a_n are convergent. [2]

(i) If $\sum_{n=1}^{\infty} (a_n)^2$ is convergent then $\sum_{n=1}^{\infty} a_n$ is convergent. [2]

(j) If $\sum_{n=1}^{\infty} a_n$ does not converge then $a_n \nrightarrow 0$. [2]

(k) If $\sum_{n=1}^{\infty} |a_n|$ converges then $|\sum_{n=1}^{\infty} a_n|$ converges. [3]

(l) Every sequence that tends to $-\infty$ is eventually decreasing. [3]

(m) Every Cauchy sequence (a_n) in $\mathbb{Q}$ converges in $\mathbb{Q}$. [3]

(n) A sequence tends to infinity if it eventually exceeds any C we choose. [3]

(o) Every subset of real numbers which is bounded above has a least upper bound. [3]

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2 .

(a) Define what it means for a sequence (a_n) to tend to limit a.

Define a subsequence (a_{n_i}) of the sequence (a_n). Give 2 examples of subsequence (a_{n_i}) of the sequence $a_n = 1, 2, 0, -1, -2, 0, 1, 2, 0, ...$ [6]

(b) Which of the following sequences converge? Justify your answer. Can you find the limit for those that converge to a real number?

(i) $a_n = \frac{\log n}{\sin^2 n + \cos^2 n}$, for $n \geq 2$

(ii) $b_n = \frac{\log n^2 - 9n^2}{7n + 2^n}$, for $n \geq 2$

(iii) $c_n = \frac{\sqrt{n^2+1} + n \sin n \cos(n-1)}{(n-1)^2}$, for $n \geq 2$ [6]

(c) Define what it means for a sequence a_n to *eventually* have some property.

Show that if (b_n) is a sequence of positive real numbers and it converges to a limit b then eventually $b_n < b + 2$. State the range of possible values for b.

Now suppose $b > 0$ and the sequence a_n tends to a limit a. Prove that the sequence $c_n = \frac{a_n}{b_n}$ tends to limit $\frac{a}{b}$. [8]

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3 .

(a) If A is a non-empty set of real numbers, and L is a real number, define what it means by saying that "L is a lower bound of A".

In what conditions do we say L is the $infimum$ of A? [2]

(b) State the Completeness Axiom for real numbers in terms of sets bounded below.

Write down the infimums of the sets $\{x \in \mathbb{R} : x^2 < 3\}$ and $\{x \in \mathbb{R} : x^2 \leq 2\}$.

Hence briefly explain why Completeness Axiom does not hold in $\mathbb{Q}$. [5]

(c) Consider the sequence

$$a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}$$

(i) Write down the first 5 terms of the sequence when a_0 is 1.

(ii) Use induction to prove that $0 \leq a_n \leq 2$.

(iii) Assuming that a_n is convergent, find the limit of the sequence. [7]

(d) Define what it means for a sequence (c_n) to be Cauchy.

Prove that the sequence (c_n) is Cauchy if and only if (c_n) is convergent. [6]

4 .

(a) Define the partial sum of a series $\sum_{n=1}^{\infty} a_n$. State precisely what it means for a series $\sum_{n=1}^{\infty} a_n$ to converge. [3]

(b) State the integral test for the convergence of series. Use the integral test to show that the series $\sum_{n=1}^{\infty} \lambda e^{-\lambda n}$ converges. λ is a real constant. [5]

(c) Write down an example of a convergent series and a divergent series both of which satisfy the condition $|\frac{a_{n+1}}{a_n}| \to 1$. No explaination is required.

Show that for any sequence a_n if there exists N such that $0 \leq \frac{a_{n+1}}{a_n} \leq 1 - \epsilon < 1$ for all $n > N$ then $\sum_{n=1}^{\infty} a_n$ converges.

Reminder: $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N-1} a_n + \sum_{n=N}^{\infty} a_n$. [6]

(d) Determine whether the following series are conditionally convergent, absolutely convergent, or divergent. Justify your answers.

(i) $\sum_{n=2}^{\infty} (-1)^n \frac{n}{\log n + (n+1)^2}$

(ii) $\sum_{n=1}^{\infty} \frac{\sin n+1}{(n+1)^2}$ [6]

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