Assignment 6

Problem 1. Let $X$ be the Banach space $C([0,1])$ with the sup norm, and $T : X \rightarrow X$ be the operator

$$(Tf)(x) = f(x) + \int_0^x f(y)dy.$$ 

Show that $\text{ran} \ T = X$ and $\text{ker} \ T = \{0\}$.

Hint: Replace the integral equation by a differential equation, and use appropriate theorems about existence and unicity of solutions.

Problem 2. (Projections)

(a) Let $M$ be a closed subspace of a Hilbert space $X$, and let $P$ be the orthogonal projector onto $M$. Show that $P^2 = P$, $\|P\| = 1$ (if $M \neq \{0\}$), and that

$$(Px, y) = (x, Py)$$

for any $x, y \in X$.

(b) Conversely, suppose that $P : X \rightarrow X$ is linear, and satisfies $P^2 = P$ and $(Px, y) = (x, Py)$ for all $x, y \in X$. Show that $P$ is the orthogonal projection onto some closed subspace.

Problem 3. Let $T$ be the multiplication operator by a function $g$. That is, we define $T : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ by

$$Tf(x) = g(x)f(x),$$

where $g$ is a fixed function, that we suppose to be continuous and bounded. Prove that

(a) $T$ is bounded;

(b) the spectrum is $\sigma(T) = \{g(x) : x \in \mathbb{R}\}$;

(c) give an example of a continuous, bounded function $g$ such that $T$ has eigenvalues;

(d) can you find a function $g \neq 0$ such that $T$ is compact?
Problem 4. (Unitary operators)

(a) Let \( U : X_1 \to X_2 \) be a unitary operator between Hilbert spaces. Show that \( \|U\| = 1 \) and \( U^* = U^{-1} \).

(b) Let \( \mathcal{S}_\mathbb{N} \) be the group of permutations (bijective) \( \mathbb{N} \to \mathbb{N} \). For \( \pi \in \mathcal{S}_\mathbb{N} \), define \( U_\pi : \ell^2(\mathbb{N}, \mathbb{C}) \to \ell^2(\mathbb{N}, \mathbb{C}) \) by
\[
U_\pi(x_1, x_2, \ldots) = (x_\pi(1), x_\pi(2), \ldots).
\]
Show that \( U_\pi \) is unitary. Show that this unitary representation of \( \mathcal{S}_\mathbb{N} \) preserves the group structure, in the sense that
\[
U_\pi U_{\pi'} = U_{\pi \circ \pi'}.
\]

Problem 5. Let \( T \) be a bounded operator on a Hilbert space \( X \).

(a) Suppose that \( (x, Tx) \) is real for all \( x \in X \). Show that \( T \) is symmetric.

We suppose in addition that \( T \) is positive, \( (x, Tx) > 0 \) for all \( x \in X \).

(b) Show that \( (x, Ty), x, y \in X \), is an inner product.

(c) Suppose there exists \( c > 0 \) such that \( (x, Tx) \geq c\|x\|^2 \) for all \( x \in X \). Show that \( X \) equipped with the new inner product is a Hilbert space.

(d) Suppose there exists no \( c > 0 \) such that \( (x, Tx) \geq c\|x\|^2 \) for all \( x \in X \). Show that \( X \) equipped with the new inner product is not complete.

This question is not easy, a hint may be necessary. Consider the inclusion map
\[
t : (X, (\cdot, \cdot)) \to (X, (\cdot, T\cdot))
\]
\[
x \mapsto x,
\]
and use the inverse mapping theorem.