Assignment 7

Problem 1. (Shift operators) We consider the right and left shift operators on $\ell^2(\mathbb{N})$:

\[
S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots), \\
T(x_1, x_2, \ldots) = (x_2, x_3, \ldots).
\]

(a) Find $\|S\|$, $\|T\|$, $S^*$, $T^*$, $S^{-1}$, $T^{-1}$.
(b) Find $\text{ran } S$, $\text{ran } T$, $\ker S$, $\ker T$, and check that

\[
\text{ran } S = (\ker T)^\perp, \quad \text{ran } T = (\ker S)^\perp.
\]
(c) Find the spectrum of $S$ and $T$.

Problem 2. Let $T \in B(X)$, and $\alpha, \beta \in \rho(T)$. Let $R_\alpha = (T - \alpha \mathbb{1})^{-1}$ denote the resolvent.

(a) Show that it satisfies the Hilbert relation (or resolvent equation)

\[
R_\alpha - R_\beta = (\alpha - \beta)R_\alpha R_\beta.
\]
(b) Show that $R_\alpha R_\beta = R_\beta R_\alpha$.

Problem 3. Let $X$ be a separable Hilbert space. A bounded operator $T : X \to X$ is Hilbert-Schmidt if there exists an orthonormal basis $(e_n)$ such that

\[
\sum_n \|Te_n\|^2 < \infty.
\]

(a) Show that Hilbert-Schmidt operators are compact.

We define the norm of a Hilbert-Schmidt operator $T$ by

\[
\|T\|_{\text{HS}} = \left( \sum_{n \geq 1} \|Te_n\|^2 \right)^{1/2}.
\]

(b) Show that $\|\cdot\|_{\text{HS}}$ is a norm.
(c) Show that the Hilbert-Schmidt norm does not depend on the choice of the orthonormal basis.
Let us now remove the assumption that $T$ is bounded. We still suppose that $\sum_n \|Te_n\|^2 < \infty$ for some orthonormal basis $(e_n)$.

(d) Show that $T$ is not necessarily bounded. (*Hint: Construct a suitable operator using a Hamel basis that contains $(e_n)$. Thanks to Michael Doré for the hint!*)

**Problem 4.** Consider the integral operator $K : L^2([0,1]) \to L^2([0,1])$ with integral kernel $k(t,s)$, i.e.

$$Kf(t) = \int_0^1 k(t,s)f(s)ds.$$ 

Show that its Hilbert-Schmidt norm is

$$\|K\|_{\text{HS}} = \int_0^1 dt \int_0^1 ds |k(t,s)|^2.$$

We now study a compact operator that is not self-adjoint, and whose spectrum consists of \{0\} only. This suggests that the spectral decomposition for self-adjoint compact operators (Thm 5.15) does not have a simple extension to non self-adjoint operators.

**Problem 5.** Let $K : L^2([0,1]) \to L^2([0,1])$ be the integral operator defined by

$$Kf(t) = \int_0^t f(s) \, ds.$$ 

(a) Find the adjoint operator $K^*$.

(b) Use Problems 3 and 4 to show that $K$ is Hilbert-Schmidt with $\|K\|_{\text{HS}} = \frac{1}{\sqrt{2}}$. Then $K$ is compact.

(c) Show that $\|K\| = \frac{2}{\pi}$. (*Hint: Find the eigenvectors and eigenvalues of the bounded self-adjoint operator $K^*K$.*)

(d) Show that $0 \in \sigma_c(K)$.

(e) Show that $\sigma(K) = \sigma_c(K) = \{0\}$. 