(b) Show that there exists $\varphi \in \mathbb{R}$ such that $a + ib = e^{i\varphi}$.

(c) Conclude that if R is proper, then it can be expressed as $z \mapsto ze^{i\varphi}$, and if R is improper, then it takes the form $z \mapsto \overline{z}e^{i\varphi}$, where $\overline{z} = x - iy$.

2. Suppose that $R : \mathbb{R}^3 \to \mathbb{R}^3$ is a proper rotation.

(a) Show that $p(t) = \det(R - tI)$ is a polynomial of degree 3, and prove that there exists $\gamma \in S^2$ (where S^2 denotes the unit sphere in $\mathbb{R}^3$) with

$$R(\gamma) = \gamma.$$

[Hint: Use the fact that $p(0) > 0$ to see that there is $\lambda > 0$ with $p(\lambda) = 0$. Then $R - \lambda I$ is singular, so its kernel is non-trivial.]

(b) If $\mathcal{P}$ denotes the plane perpendicular to γ and passing through the origin, show that

$$R : \mathcal{P} \to \mathcal{P},$$

and that this linear map is a rotation.

3. Recall the formula

$$\int_{\mathbb{R}^d} F(x)\, dx = \int_{S^{d-1}} \int_0^\infty F(r\gamma) r^{d-1}\, dr\, d\sigma(\gamma).$$

Apply this to the special case when $F(x) = g(r)f(\gamma)$, where $x = r\gamma$, to prove that for any rotation R, one has

$$\int_{S^{d-1}} f(R(\gamma))\, d\sigma(\gamma) = \int_{S^{d-1}} f(\gamma)\, d\sigma(\gamma),$$

whenever f is a continuous function on the sphere S^{d-1}.

4. Let A_d and V_d denote the area and volume of the unit sphere and unit ball in $\mathbb{R}^d$, respectively.

(a) Prove the formula

$$A_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

so that $A_2 = 2\pi,\ A_3 = 4\pi,\ A_4 = 2\pi^2, \ldots$. Here $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt$ is the Gamma function. [Hint: Use polar coordinates and the fact that $\int_{\mathbb{R}^d} e^{-\pi|x|^2}\, dx = 1$.]

(b) Show that $dV_d = A_d$, hence

$$V_d = \frac{\pi^{d/2}}{\Gamma(d/2+1)}.$$

In particular $V_2 = \pi$, $V_3 = 4\pi/3, \ldots$.

5. Let A be a $d \times d$ positive definite symmetric matrix with real coefficients. Show that

$$\int_{\mathbb{R}^d} e^{-\pi(x, A(x))}\, dx = (\det(A))^{-1/2}.$$

This generalizes the fact that $\int_{\mathbb{R}^d} e^{-\pi|x|^2}\, dx = 1$, which corresponds to the case where A is the identity.

[Hint: Apply the spectral theorem to write $A = RDR^{-1}$ where R is a rotation and, D is diagonal with entries $\lambda_1, \ldots, \lambda_d$, where $\{\lambda_i\}$ are the eigenvalues of A.]

6. Suppose $\psi \in \mathcal{S}(\mathbb{R}^d)$ satisfies $\int |\psi(x)|^2\, dx = 1$. Show that

$$\left(\int_{\mathbb{R}^d} |x|^2 |\psi(x)|^2\, dx\right)\left(\int_{\mathbb{R}^d} |\xi|^2 |\hat{\psi}(\xi)|^2\, d\xi\right) \geq \frac{d^2}{16\pi^2}.$$

This is the statement of the Heisenberg uncertainty principle in d dimensions.

7. Consider the time-dependent heat equation in $\mathbb{R}^d$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_d^2}, \qquad \text{where } t > 0, \tag{15}$$

with boundary values $u(x,0) = f(x) \in \mathcal{S}(\mathbb{R}^d)$. If

$$\mathcal{H}_t^{(d)}(x) = \frac{1}{(4\pi t)^{d/2}} e^{-|x|^2/4t} = \int_{\mathbb{R}^d} e^{-4\pi^2 t|\xi|^2} e^{2\pi i x\cdot\xi}\, d\xi$$

is the d-dimensional **heat kernel**, show that the convolution

$$u(x,t) = (f * \mathcal{H}_t^{(d)})(x)$$

is indefinitely differentiable when $x \in \mathbb{R}^d$ and $t > 0$. Moreover, u solves (15), and is continuous up to the boundary $t = 0$ with $u(x,0) = f(x)$.

The reader may also wish to formulate the d-dimensional analogues of Theorem 2.1 and 2.3 in Chapter 5.

8. In Chapter 5, we found that a solution to the steady-state heat equation in the upper half-plane with boundary values f is given by the convolution $u = f * \mathcal{P}_y$ where the Poisson kernel is

$$\mathcal{P}_y(x) = \frac{1}{\pi}\frac{y}{x^2+y^2} \qquad \text{where } x \in \mathbb{R} \text{ and } y > 0.$$

More generally, one can calculate the d-dimensional Poisson kernel using the Fourier transform as follows.

(a) The **subordination principle** allows one to write expressions involving the function e^{-x} in terms of corresponding expressions involving the function e^{-x^2}. One form of this is the identity

$$e^{-\beta} = \int_0^\infty \frac{e^{-u}}{\sqrt{\pi u}} e^{-\beta^2/4u}\, du$$

when $\beta \geq 0$. Prove this identity with $\beta = 2\pi|x|$ by taking the Fourier transform of both sides.

(b) Consider the steady-state heat equation in the upper half-space $\{(x,y) : x \in \mathbb{R}^d,\ y > 0\}$

$$\sum_{j=1}^{d} \frac{\partial^2 u}{\partial x_j^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

with the Dirichlet boundary condition $u(x,0) = f(x)$. A solution to this problem is given by the convolution $u(x,y) = (f * P_y^{(d)})(x)$ where $P_y^{(d)}(x)$ is the d-dimensional Poisson kernel

$$P_y^{(d)}(x) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} e^{-2\pi|\xi|y}\, d\xi.$$

Compute $P_y^{(d)}(x)$ by using the subordination principle and the d-dimensional heat kernel. (See Exercise 7.) Show that

$$P_y^{(d)}(x) = \frac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}} \frac{y}{(|x|^2 + y^2)^{(d+1)/2}}.$$

9. A **spherical wave** is a solution $u(x,t)$ of the Cauchy problem for the wave equation in $\mathbb{R}^d$, which as a function of x is radial. Prove that u is a spherical wave if and only if the initial data $f, g \in \mathcal{S}$ are both radial.

10. Let $u(x,t)$ be a solution of the wave equation, and let $E(t)$ denote the energy of this wave

$$E(t) = \int_{\mathbb{R}^d} \left|\frac{\partial u}{\partial t}(x,t)\right|^2 + \sum_{j=1}^{d} \int_{\mathbb{R}^d} \left|\frac{\partial u}{\partial x_j}(x,t)\right|^2 dx.$$

We have seen that $E(t)$ is constant using Plancherel's formula. Give an alternate proof of this fact by differentiating the integral with respect to t and showing that

$$\frac{dE}{dt} = 0.$$

[Hint: Integrate by parts.]

11. Show that the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}$$

subject to $u(x,0) = f(x)$ and $\frac{\partial u}{\partial t}(x,0) = g(x)$, where $f, g \in \mathcal{S}(\mathbb{R}^3)$, is given by

$$u(x,t) = \frac{1}{|S(x,t)|} \int_{S(x,t)} [tg(y) + f(y) + \nabla f(y) \cdot (y - x)]\, d\sigma(y),$$

where $S(x,t)$ denotes the sphere of center x and radius t, and $|S(x,t)|$ its area. This is an alternate expression for the solution of the wave equation given in Theorem 3.6. It is sometimes called **Kirchhoff's** formula.

12. Establish the identity (14) about the dual transform given in the text. In other words, prove that

$$\int_{\mathbb{R}} \int_{S^2} \mathcal{R}(f)(t,\gamma) \overline{F(t,\gamma)} d\sigma(\gamma)\, dt = \int_{\mathbb{R}^3} f(x) \overline{\mathcal{R}^*(F)(x)}\, dx \tag{16}$$

where $f \in \mathcal{S}(\mathbb{R}^3)$, $F \in \mathcal{S}(\mathbb{R} \times S^2)$, and

$$\mathcal{R}(f) = \int_{\mathcal{P}_{t,\gamma}} f \quad \text{and} \quad \mathcal{R}^*(F)(x) = \int_{S^2} F(x \cdot \gamma, \gamma)\, d\sigma(\gamma).$$

[Hint: Consider the integral

$$\int\int\int f(t\gamma + u_1 e_2 + u_2 e_2) \overline{F(t,\gamma)}\, dt\, d\sigma(\gamma)\, du_1\, du_2.$$

Integrating first in u gives the left-hand side of (16), while integrating in u and t and setting $x = t\gamma + u_1 e_2 + u_2 e_2$ gives the right-hand side.]

13. For each (t, θ) with $t \in \mathbb{R}$ and $|\theta| \leq \pi$, let $L = L_{t,\theta}$ denote the line in the (x,y)-plane given by

$$x \cos\theta + y \sin\theta = t.$$

This is the line perpendicular to the direction $(\cos\theta, \sin\theta)$ at "distance" t from the origin (we allow negative t). For $f \in \mathcal{S}(\mathbb{R}^2)$ the X-ray transform or two-dimensional Radon transform of f is defined by

$$X(f)(t,\theta) = \int_{L_{t,\theta}} f = \int_{-\infty}^{\infty} f(t\cos\theta + u\sin\theta, t\sin\theta - u\cos\theta)\, du.$$

Calculate the X-ray transform of the function $f(x, y) = e^{-\pi(x^2+y^2)}$.

14. Let X be the X-ray transform. Show that if $f \in \mathcal{S}$ and $X(f) = 0$, then $f = 0$, by taking the Fourier transform in one variable.

15. For $F \in \mathcal{S}(\mathbb{R} \times S^1)$, define the **dual X-ray transform** $X^*(F)$ by integrating F over all lines that pass through the point (x, y) (that is, those lines $L_{t,\theta}$ with $x\cos\theta + y\sin\theta = t$):

$$X^*(F)(x, y) = \int F(x\cos\theta + y\sin\theta, \theta)\, d\theta.$$

Check that in this case, if $f \in \mathcal{S}(\mathbb{R}^2)$ and $F \in \mathcal{S}(\mathbb{R} \times S^1)$, then

$$\int\!\!\int X(f)(t, \theta)\overline{F(t, \theta)}\, dt\, d\theta = \int\!\!\int f(x, y)\overline{X^*(F)(x, y)}\, dx\, dy.$$

7 Problems

1. Let J_n denote the n^{th} order Bessel function, for $n \in \mathbb{Z}$. Prove that

(a) $J_n(\rho)$ is real for all real ρ.

(b) $J_{-n}(\rho) = (-1)^n J_n(\rho)$.

(c) $2J_n'(\rho) = J_{n-1}(\rho) - J_{n+1}(\rho)$.

(d) $\left(\frac{2n}{\rho}\right) J_n(\rho) = J_{n-1}(\rho) + J_{n+1}(\rho)$.

(e) $(\rho^{-n}J_n(\rho))' = -\rho^{-n}J_{n+1}(\rho)$.

(f) $(\rho^n J_n(\rho))' = \rho^n J_{n-1}(\rho)$.

(g) $J_n(\rho)$ satisfies the second order differential equation

$$J_n''(\rho) + \rho^{-1}J_n'(\rho) + (1 - n^2/\rho^2)J_n(\rho) = 0.$$

(h) Show that

$$J_n(\rho) = \left(\frac{\rho}{2}\right)^n \sum_{m=0}^{\infty} (-1)^m \frac{\rho^{2m}}{2^{2m} m!(n+m)!}.$$

(i) Show that for all integers n and all real numbers a and b we have

$$J_n(a + b) = \sum_{\ell \in \mathbb{Z}} J_\ell(a) J_{n-\ell}(b).$$

2. Another formula for $J_n(\rho)$ that allows one to define Bessel functions for non-integral values of n, $(n > -1/2)$ is

$$J_n(\rho) = \frac{(\rho/2)^n}{\Gamma(n+1/2)\sqrt{\pi}} \int_{-1}^{1} e^{i\rho t}(1-t^2)^{n-(1/2)}\, dt.$$

(a) Check that the above formula agrees with the definition of $J_n(\rho)$ for integral $n \geq 0$. [Hint: Verify it for $n = 0$ and then check that both sides satisfy the recursion formula (e) in Problem 1.]

(b) Note that $J_{1/2}(\rho) = \sqrt{\frac{2}{\pi}}\rho^{-1/2}\sin\rho$.

(c) Prove that

$$\lim_{n\to -1/2} J_n(\rho) = \sqrt{\frac{2}{\pi}}\rho^{-1/2}\cos\rho.$$

(d) Observe that the formulas we have proved in the text giving F_0 in terms of f_0 (when describing the Fourier transform of a radial function) take the form

$$F_0(\rho) = 2\pi\rho^{-(d/2)+1}\int_0^\infty J_{(d/2)-1}(2\pi\rho r) f_0(r) r^{d/2}\, dr, \tag{17}$$

for $d = 1, 2$, and 3, if one uses the formulas above with the understanding that $J_{-1/2}(\rho) = \lim_{n\to -1/2} J_n(\rho)$. It turns out that the relation between F_0 and f_0 given by (17) is valid in all dimensions d.

3. We observed that the solution $u(x,t)$ of the Cauchy problem for the wave equation given by formula (3) depends only on the initial data on the base on the backward light cone. It is natural to ask if this property is shared by *any* solution of the wave equation. An affirmative answer would imply uniqueness of the solution.

Let $B(x_0, r_0)$ denote the closed ball in the hyperplane $t = 0$ centered at x_0 and of radius r_0. The **backward light cone** with base $B(x_0, r_0)$ is defined by

$$\mathcal{L}_{B(x_0,r_0)} = \{(x,t) \in \mathbb{R}^d \times \mathbb{R} : |x - x_0| \leq r_0 - t,\quad 0 \leq t \leq r_0\}.$$

Theorem *Suppose that $u(x,t)$ is a C^2 function on the closed upper half-plane $\{(x,t):\ x \in \mathbb{R}^d, t \geq 0\}$ that solves the wave equation*

$$\frac{\partial^2 u}{\partial t^2} = \triangle u.$$

If $u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0$ for all $x \in B(x_0,r_0)$, then $u(x,t) = 0$ for all $(x,t) \in \mathcal{L}_{B(x_0,r_0)}$.

In words, if the initial data of the Cauchy problem for the wave equation vanishes on a ball B, then *any* solution u of the problem vanishes in the backward light cone with base B. The following steps outline a proof of the theorem.

(a) Assume that u is real. For $0 \leq t \leq r_0$ let $B_t(x_0, r_0) = \{x : |x - x_0| \leq r_0 - t\}$, and also define

$$\nabla u(x,t) = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_d}, \frac{\partial u}{\partial t}\right).$$

Now consider the energy integral

$$\begin{aligned} E(t) &= \frac{1}{2}\int_{B_t(x_0,r_0)} |\nabla u|^2\, dx \\ &= \frac{1}{2}\int_{B_t(x_0,r_0)} \left(\frac{\partial u}{\partial t}\right)^2 + \sum_{j=1}^d \left(\frac{\partial u}{\partial x_j}\right)^2 dx. \end{aligned}$$

Observe that $E(t) \geq 0$ and $E(0) = 0$. Prove that

$$E'(t) = \int_{B_t(x_0,r_0)} \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + \sum_{j=1}^d \frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_j \partial t}\, dx - \frac{1}{2}\int_{\partial B_t(x_0,r_0)} |\nabla u|^2\, d\sigma(\gamma).$$

(b) Show that

$$\frac{\partial}{\partial x_j}\left[\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial t}\right] = \frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_j \partial t} + \frac{\partial^2 u}{\partial x_j^2}\frac{\partial u}{\partial t}.$$

(c) Use the last identity, the divergence theorem, and the fact that u solves the wave equation to prove that

$$E'(t) = \int_{\partial B_t(x_0,r_0)} \sum_{j=1}^d \frac{\partial u}{\partial x_j}\frac{\partial u}{\partial t}\nu_j\, d\sigma(\gamma) - \frac{1}{2}\int_{\partial B_t(x_0,r_0)} |\nabla u|^2\, d\sigma(\gamma),$$

where ν_j denotes the j^{th} coordinate of the outward normal to $B_t(x_0, r_0)$.

(d) Use the Cauchy-Schwarz inequality to conclude that

$$\sum_{j=1}^d \frac{\partial u}{\partial x_j}\frac{\partial u}{\partial t}\nu_j \leq \frac{1}{2}|\nabla u|^2,$$

and as a result, $E'(t) \leq 0$. Deduce from this that $E(t) = 0$ and $u = 0$.