Assignment 2

1. Prove that the Fejér kernel “converges to Dirac”. Precisely, show that 
   \[ F_n(t) = \frac{1}{n} \frac{\sin^2(n\pi t)}{\sin^2(\pi t)} \] 
   satisfies
   
   (a) \( F_n(t) \geq 0 \). (Obvious!)

   (b) \( \int_{-\frac{1}{2}}^{\frac{1}{2}} F_n(t) \, dt = 1 \).

   (c) \( \lim_{n \to \infty} \int_{|t|>\delta} F_n(t) \, dt = 0 \) for any \( \delta > 0 \).

2. Properties of convolution. Suppose that \( f, g, h \in L^1([0,1]) \). Prove the following facts:
   
   (a) \( f \ast (g + h) = f \ast g + f \ast h \).

   (b) \( (cf) \ast g = c(f \ast g) = f \ast (cg) \) for any \( c \in \mathbb{C} \).

   (c) \( f \ast g = g \ast f \).

   (d) \( (f \ast g) \ast h = f \ast (g \ast h) \).

   (e) \( f \ast g \) is continuous.

   (f) \( \hat{f} \ast g(k) = \hat{f}(k) \hat{g}(k) \).

3. Abel summability. Let \( f \in L^1([0,1]) \), and with \( r \in (0,1) \), define
   \[ A_r f(x) = \sum_{k \in \mathbb{Z}} r^{|k|} \hat{f}(k) e^{2\pi i kx} \].
   
   (a) Prove that this series converges absolutely for all \( x \in [0,1] \), and that 
       \( A_r f(x) \) is a continuous function of \( x \).

   (b) Let \( P_r \) be the Poisson kernel such that \( A_r f = P_r \ast f \). Show that
       \[ P_r(x) = \sum_{k \in \mathbb{Z}} r^{|k|} e^{2\pi i kx} = \frac{1 - r^2}{1 - 2r \cos(2\pi x) + r^2} \].
(c) Prove that if \( f \in L^p([0,1]), 1 \leq p < \infty \), we have \( \lim_{r \to 1} \| A_r f - f \|_p = 0 \). (This should be similar to the proof of Theorem 3.2 (a).) What about \( p = \infty \)?

4. Show that the Fourier series of a periodic differentiable function \( f \in C^1(T) \) is absolutely convergent. (Hint: Use the Cauchy-Schwarz inequality and Parseval formula for \( f' \).)

5. Let \( f \in C^\alpha(T) \), with \( \alpha \in \mathbb{N} \). Show that \( \hat{f}(k) = o(1/|k|^\alpha) \).