Assignment 6

1. Suppose that \( f \neq 0 \) is a bounded function on \( \mathbb{R} \) with compact support. Prove that its Fourier transform does not have compact support.

   Hint: Show that \( \left| \frac{d^n}{dk^n} \hat{f}(k) \right| < C^n \) for some constant \( C \), so that \( \hat{f} \) has a Taylor series around 0 with infinite radius of convergence. Then \( \hat{f} \) is analytic and cannot have compact support.

2. Let \( d = 3 \). Find the Fourier transform of the function \( 1/\|x\| \). Since this function is not in any \( L^p \) space, explain the meaning of the Fourier transform.

   Hint: Think about distributions, and remember that the Fourier transform of the function \( \frac{1}{|x|} e^{-2\pi \mu |x|} \) is \( \frac{1}{\pi} \frac{1}{k^2 + \mu^2} \).

3. Prove that \( \frac{1}{x+i0} = \text{PV} \frac{1}{x} - i\pi \delta_0 \). Hint: use \( \frac{1}{x+i\varepsilon} = \frac{x}{x^2 + \varepsilon^2} - \frac{i\varepsilon}{x^2 + \varepsilon^2} \).

4. Compute rigorously the Fourier transforms of

   (a) \( \delta_{x_0} \).

   (b) \( x \).

   (c) \( x\delta_0 \).

5. Compute rigorously the Fourier transform of \( \frac{1}{x+i0} \). Hint: Obtain the following expression:

\[
\hat{\frac{1}{x+i0}}(\phi) = \lim_{\varepsilon \downarrow 0} \lim_{R \to \infty} \int_{-R}^{R} \frac{1}{x+i\varepsilon} \left[ \int_{-\infty}^{\infty} e^{-2\pi ikx} \phi(k) \, dk \right] \, dx.
\]

Then use Fubini and contour methods.