Assignment 8

1. Fast Fourier transform. Show that, for $N = 3^n$, the coefficients $a_k^N$ can be computed with less than $6N \log_3 N$ operations. What about $N = 4^n, 5^n \ldots$?

2. Let $e$ be a character on the group $\mathbb{Z}(N)$. Show that there exists a unique $0 \leq \ell \leq N - 1$ such that 
   
   $e(k) = e^{2\pi i \ell k/N}$

   for all $k \in \mathbb{Z}(N)$. Conversely, every such function is a character. (Hint: Show that $e(1)$ is the $N$th root of 1.)

3. Infinite abelian groups $S^1$ and $\mathbb{R}$. A character here is a continuous function $e$ that satisfies $|e(x)| = 1$ and $e(x + y) = e(x)e(y)$.

   (a) Prove that all characters on $S^1$ are given by 
   
   $e_n(x) = e^{2\pi inx}$

   with $n \in \mathbb{Z}$.

   (Hint: If $f$ is continuous and $f(x + y) = f(x)f(y)$, then $f$ is differentiable. To see this, note that if $f(0) \neq 0$, then for small $\delta$, 
   
   $c = \int_0^\delta f(y)dy \neq 0$ and $cf(x) = \int_x^{x+\delta} f(y)dy$. Differentiate to conclude that $f(x) = e^{Ax}$ for some $A$.)

   (b) Prove that all characters on $\mathbb{R}$ are of the form 
   
   $e_k(x) = e^{2\pi ikx}$

   with $k \in \mathbb{R}$.

4. Let $G$ be a finite abelian group and $e : G \to \mathbb{C}$ a function that satisfies $e(a \cdot b) = e(a)e(b)$ for all $a, b \in G$. Prove that $e$ is either identically zero, or $e$ never vanishes. In the second case, show that, for each $a$, $e(a) = e^{2\pi ir}$ for some rational $r$ of the form $r = p/|G|, p \in \mathbb{N}$.

5. The convolution of two functions $f, g \in \ell^2(G)$ where $G$ is a finite abelian group, is defined by 
   
   $(f * g)(a) = \frac{1}{|G|} \sum_{b \in G} f(b)g(a \cdot b^{-1})$.

   Show that 
   
   $(\hat{f * g})(e) = \hat{f}(e)\hat{g}(e)$

   for all characters $e \in \hat{G}$. 