Assignment 1

1. Relations between symmetries and Fourier coefficients. Let $f \in L^1(\mathbb{T})$.
   
   (a) Assuming that the Fourier series converges absolutely, show that
   $$f(x) = \hat{f}(0) + \sum_{n \geq 1} [\hat{f}(n) + \hat{f}(-n)] \cos(2\pi nx) + i[\hat{f}(n) - \hat{f}(-n)] \sin(2\pi nx).$$
   
   (b) Show that $\hat{f}$ is even if $f$ is even (that is, $\hat{f}(k) = \hat{f}(-k)$). Then $f$ is given by a cosine series.
   
   (c) Show that $\hat{f}$ is odd if $f$ is odd. Then $f$ is given by a sine series.
   
   (d) Suppose that $f(x + \frac{1}{2}) = f(x)$ for all $x \in \mathbb{T}$. Show that $\hat{f}(k) = 0$ for odd $k$.
   
   (d) Show that $f$ is real-valued if and only if $\hat{f}(k) = \overline{\hat{f}(-k)}$.

2. Let $f$ be the odd function such that $f(x) = x(\frac{1}{2} - x)$ on $[0, \frac{1}{2}]$. Draw the graph of the function, and prove that
   $$f(x) = \frac{8}{\pi} \sum_{n \geq 1, \text{odd}} \frac{\sin(2\pi nx)}{n^3}.$$ 

3. (a) Let $f(x) = |x|$ on $[-\frac{1}{2}, \frac{1}{2}]$. Show that
   $$\hat{f}(k) = \begin{cases} 1/4 & \text{if } k = 0, \\ (-1)^k - 1 & \text{if } k \neq 0. \end{cases}$$
   
   (b) Let $f(x) = x^2$ on $[-\frac{1}{2}, \frac{1}{2}]$. Show that
   $$\hat{f}(k) = \begin{cases} 1/12 & \text{if } k = 0, \\ \frac{\cos(\pi k)}{2\pi^2 k^2} & \text{if } k \neq 0. \end{cases}$$
   
   (c) Prove the following relations. (The results above may help!)
   $$\sum_{n \geq 1, \text{odd}} \frac{1}{n^2} = \frac{\pi^2}{8}, \quad \sum_{n \geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}.$$
4. (a) Let \((a_n)_{n \geq 1}\) be a sequence of real numbers that decrease monotonically to 0, and \((b_n)_{n \geq 1}\) be a sequence of complex numbers such that all partial sums are bounded. Prove that \(\sum_n a_n b_n\) converges. (This is **Dirichlet’s test** for the convergence of series). You may want to prove and use the following identity:

\[
\sum_{n=M}^{N} a_n b_n = a_N \sum_{n=1}^{N} b_n - a_M \sum_{n=1}^{M-1} b_n - \sum_{n=M}^{N-1} (a_{n+1} - a_n) \sum_{m=1}^{n} b_m.
\]

(This is a discrete summation by parts.)

(b) Check that the Fourier series with coefficients \(\hat{f}(k) = 1/k\), for \(k \neq 0\), converges.

(c) Find the function \(f\), and plot it.