Assignment 4

1. Truncating $L^p(\mathbb{R}^d)$ functions.
   - Let $f \in L^2(\mathbb{R}^d)$, and define $f_n(x) = f(x) \chi_{\|x\|<n}$. Show that $f_n \in L^1(\mathbb{R}^d)$ and that
     \[
     \lim_{n \to \infty} \|f_n - f\|_2 = 0.
     \]
   - Let $f \in L^p(\mathbb{R}^d)$ with $1 \leq p \leq 2$. Let $f_1(x) = f(x) \chi_{|f|\geq 1}(x)$ and $f_2(x) = f(x) \chi_{|f|<1}(x)$. Show that $f_1 \in L^1$ and $f_2 \in L^2$.

2. Consistency of the definition of the $L^2$ Fourier transform.
   - Suppose that $(f_n)$ and $(g_n)$ are sequences in $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ that both converge to $f \in L^2$. Define $\hat{f}_n, \hat{g}_n$ using the $L^1$ transform. Show that $(\hat{f}_n)$ and $(\hat{g}_n)$ converge to the same limit. (This limit is $\hat{f}$ by definition).
   - Extend Plancherel’s identity from functions in $L^1 \cap L^2$, to any function in $L^2$.

3. Show that the Hermite functions, 
   \[ h_n(x) = \frac{(-1)^n}{n!} e^{-\pi x^2} \frac{d^n}{dx^n} e^{-2\pi x^2}, \]
   are eigenvectors of the $L^2(\mathbb{R})$ Fourier transform.
   **Hint:** Compute the generating function
   \[ \sum_{n \geq 0} t^n h_n(x). \]
   This gives a Gaussian, and one can get its Fourier transform. Equating the powers of $t$, one should obtain the result.

4. Show that the Hausdorff-Young inequality is false for $p > 2$.
   **Hint:** Extend Proposition 4.1 (Fourier transform of a Gaussian) to Gaussians with complex parameters $\lambda$ such that $\text{Re} \lambda > 0$. Then consider the Gaussian $g_{\lambda}$ with $\lambda = (a + ib)^{-1}$, $a > 0$, and show that $\|g_{\lambda}\|_p \sim |b|^{1/2}$ and $\|\hat{g}_{\lambda}\|_q \sim |b|^{1/q}$ as $|b| \to \infty$. 
